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Triangular fuzzy numbers multiplication: QKB method

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A B S T R A C T

Triangular Fuzzy numbers (TFNs) are vast and common representation of fuzzy data in applied sciences. Multiplication is a very important operation for fuzzy numbers. It is needed to decompose fuzzy systems such as fully triangular fuzzy regression models where the unknown and unrestricted triangular fuzzy coefficients multiplied by known TFNs as data input. Tens of research works and application of triangular fuzzy regression have dealt with degenerated existing multiplication expressions. This paper highlighted the drawbacks of such expressions and propounded a simple method (named as QKB method). The method is a straightforward method where there is no exaggeration for multiplying two or more TFNs. It respects the trinity-order condition of a TFN where the number without it cannot be considered as a TFN. Besides, it is suitable for known and unknown multiplied TFNs with conserving homogeneity principle such that the resultant of two symmetric TFNs has to be symmetric either, to prove that a proposed new membership function for a TFN (named quantified membership function) has been used. Illustrative examples have shown the soundness of the proposed method and the drawbacks of existing expressions. Furthermore, its expression of multiplication is more efficient than other expression in the sake of computation and computerization.

1. Introduction

TFNs are the most popular type of fuzzy numbers and they are widely used in representing uncertainty in applied sciences because of their ability of expressing the perception of experts. TFN has been revealed after introducing fuzzy trigonometry [3]. The experts usually think in uncertainty as one central number surrounded with fuzzy extent. In general fuzzy numbers as special case of fuzzy set introduced by Zadeh [9] in 1965 who has proposed extension principle [10] such that what is suitable for real numbers is suitable for fuzzy numbers as well. The multiplication is an important operation for TFNs in systems where its parameters and coefficients

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such as fuzzy mathematical programming problems and fuzzy regression analysis. Unfortunately, there is no guarantee for multiplication of two TNFs to get TFN as resultant in the consideration of extension principle [5] (i.e., for any two TFNs $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ where $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$ we have $\tilde{A} \times \tilde{B} = \tilde{C} = (c_1, c_2, c_3)$ however, the condition of TNF which is $c_1 \leq c_2 \leq c_3$ is not necessary conserved where c_1, c_2, and c_3 are the left point of TFN $\tilde{C}$, the central point and right point respectively). We can call the aforementioned condition of TFN as trinity-order. On overcoming this problem and to conserve the keystone of TFN, i.e., trinity-order, there are some formulas of multiplication of two TFNs in literature and all can be categorized into three main expressions [5, 6, 10]. Although those expressions conserve the trinity-order condition, they have drawbacks which make them practically inefficient where all of them are nonlinear to express the multiplication when one of TFNs is unknown and unrestricted TFNs and both of them yield irrational deviation of the central point of the TFN resultant. Moreover, one of them cannot conserve the homogeneity principle such that the multiplication of two symmetric TFNs produces symmetrical TFN resultant. In another meaning, if $\tilde{A}$ and $\tilde{B}$ are symmetric TFN which means the shape of TFN trigonometrically is isosceles triangle, $\tilde{A} \times \tilde{B} = \tilde{C} = (c_1, c_2, c_3)$ where $\tilde{C}$ must be symmetrical TFN, i.e., $c_1 - c_2 = c_3 - c_2$. Keeping in mind the homogeneity principle, Chen and Nien [4] on proposing a mathematical programming approach to construct a regression model for intuitionistic triangular fuzzy data overcoming the limitation of Arefi and Taheri [2] intuitionistic approach who has used the approximation of Dubois and Prade [5], and in order to avoid the effect of the unknown model coefficients, Chen and Nien [4] incorporate two dummy intuitionistic triangular fuzzy variables. One of these variables is nonnegative and another is non-positive to mitigate the effect of multiplying unknown coefficient into a known input value. However, this method to parametrize the constructed model's slopes proposed by Chen and Nien [4], is not always satisfied although the related constraints are involved in their mathematical programming problem. The contribution of this paper after proposing a new form of membership function of a TFN, is introducing a multiplication operation which with its direct application, it is general for different type of TFNs (symmetric or asymmetric, and negative or positive TFNs), sound such that trinity-order of a resultant is mintained, and rational while the deviation of the TFN resultant components is logical without approximations. Moreover, with the simplicity of the proposed multiplication, no of such multiplication has been considered in existing methods. Furthermore, the proposed multiplication is linear and easy to be computed with efficient time complexity. The rest of this paper is organized as below. Some digressive preliminaries, definition of TFNs and Quantified TFN membership function is introduced in Section 2. Section 3 allocated to discuss the existing multiplication methods of two TFNs. without either exaggeration or degeneration, a straightforward method of TFNs multiplication proposed in Section 4. In Section 5, proposed method is illustrated with different numerical examples and compared with other existing methods. Section 6 concludes the paper's work.

2. Preliminaries

In this section some indispensable definitions are presented. Throughput this paper, we use $S = \{x_1, x_2, \dots, x_n\}$ to denote the discourse set. $\tilde{A}$ and $\tilde{A}(x)$ are fuzzy set and its membership function in S.

Definition 1. Fuzzy set: Let S be a nonempty set. A fuzzy set $\tilde{A}$ in S is characterized by the membership function $\mu_{\tilde{A}}: S \to [0,1]$. The membership function of an element s in $\tilde{A}$ for each $x \in S$, $\mu_{\tilde{A}}(x)$ is interpreted as the degree of membership of s in $\tilde{A}$. A one can obviously notice that $\mu_{\tilde{A}}(x) = 0$ when s does not exactly belong to $\tilde{A}$ and $\mu_{\tilde{A}}(x) = 1$ if x perfectly belongs to $\tilde{A}$. It explicit that $\tilde{A}$ can be determined by the set of tuples

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in S\}.$$

Frequently, we will denote $\tilde{A}(x)$ instead of $\mu_{\tilde{A}}(x)$.

Definition 2. $\widetilde{A}$ is a fuzzy set perceived by:

a. $\widetilde{A}$ is defined on a real line.
b. There is only and only on value, $m \in S$ exists such that $\mu_{\widetilde{A}}(m) = 1$ is called mean pointed value of $\widetilde{A}$.
c. $\widetilde{A}$ is convex, i.e., $\mu_{\widetilde{A}}(.)$ is defined as:

$$\mu_{\widetilde{A}}(\rho x_1 + (1-\rho)x_2) \geq \min\left(\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\right), x_1, x_2 \in S \text{ and } \rho \in [0,1].$$

Definition 3. Fuzzy number: A fuzzy number denotes a coherent set of possible values. Hence, every fuzzy number is fuzzy set with convexity and normality of fuzzy set of the real line.

2.1 Membership function of TFN

$\widetilde{A}$ is TFN if it can be written as a triple of real numbers l, m and r satisfying the trinity-order such that we can write $\widetilde{A} = (l, m, r)$ and its membership function is given by

$$\widetilde{A}(x) = \begin{cases} \frac{x-l}{m-l}, l \leq x \leq m; \\ \frac{r-x}{r-m}, m \leq x \leq r; \\ 0,\ l > x > r. \end{cases}$$

2.2 Quantified membership function TFN

Based on TFN in the previous section (i.e., Section 2.1) and behind the philosophy of expressing uncertainty in trigonometry [3], a TFN $\tilde{A}$ is represented as $\tilde{A} = (m, \alpha, \beta)$ where m is the main value of TFN and α and β represent left and right margins around m, respectively. In this representation of TFN α and $\beta \geq 0$. Depends upon this representation and taking in the consideration that for any element of x in $\tilde{A}$, such element can be either

The main pointed value m;

A value in the left right angle triangle of TFN so, $x = m - (\alpha - q)$ for some $0 \leq q \leq \alpha$ such that when $q = 0$ we get the lower-pointed value of TFN (i.e., $m - \alpha$) and when $q = \alpha$ we get m or;

A value in the right right angle triangle of TFN so, $x = m + (\beta - q)$ for some $0 \leq q \leq \beta$ such that when $q = 0$ we get the upper-pointed value of TFN (i.e., $m + \beta$) and its membership value at then is 0, and when $q = \beta$ we get m where its membership is 1.

Plugging the possible value of 2 and 3 we can obtain what we can call it quantified membership function depends on the value of the quantifier q. The quantified membership function $\tilde{A}(q)$ is given below.

$$\tilde{A}(q) = \begin{cases} \frac{q}{\alpha}, & 0 \leq q \leq \alpha; \\ \frac{q}{\beta}, & 0 \leq q \leq \beta. \end{cases}$$

The linear relation between $\tilde{A}(q)$ and $\tilde{A}(x)$ is straightforward with considering the following.

The lower-pointed value l and left margin α in TFN are linked to each other where $l = m - \alpha$.

The upper-pointed value n and right margin β in TFN are linked to each other where $n = m + \beta$. The trigonometric representation of a TAIFN is shown in Figure 1.

2.3 Symmetric and asymmetric TFN

A TFN $\tilde{A} = (m, \alpha, \beta)$ is called symmetric when the mean-pointed value m divides the number into two congruent sections, i.e., two equal right angled triangles in trigonometry where $\alpha = \beta$ otherwise TFN $\tilde{A}$ is asymmetric (i.e., $\alpha \neq \beta$).

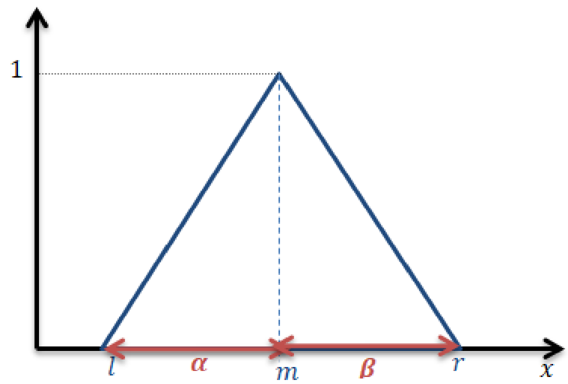

Figure 1. Trigonometric representation of a TFN.

2.4 Positive and negative TFN

A TFN $\tilde{A} = (m, \alpha, \beta)$ is positive (i.e., $\tilde{A} > 0$) iff its lower-pointed value is positive (i.e., $(m - \alpha) \geq 0$) and $\tilde{A}$ is negative (i.e., $\tilde{A} < 0$) iff its upper-pointed value is negative (i.e., $(m + \beta) \leq 0$). Otherwise $\tilde{A}$ is neither positive nor negative.

3. Existing methods of two TFNs multiplication

After they had pointed out that multiplication of two TFNs and according to extension principle does not necessarily conserve the trinity-order condition, Dobois and Prade [5] proposed an approximation of multiplication of two TFNs $\tilde{A} = (m_1, \alpha_1, \beta_1)$ and $\tilde{B} = (m_2, \alpha_2, \beta_2)$ as follows:

$$\tilde{A} \times \tilde{B} \approx \begin{cases} (m_1 m_2, m_1\alpha_2 + m_2\alpha_1, m_1\beta_2 + m_2\beta_1), & \tilde{A} > 0, \tilde{B} > 0; \\ (m_1 m_2, m_2\alpha_1 - m_1\alpha_2, m_2\beta_1 - m_1\beta_2), & \tilde{A} < 0, \tilde{B} > 0; \\ (m_1 m_2, -m_1\beta_2 - m_2\beta_1, -m_1\alpha_2 - m_2\alpha_1), & \tilde{A} < 0, \tilde{B} < 0. \end{cases} \quad (1)$$

As a modification of expression 1 Zeng et al. [11] rectify the cases above and proposed the following multiplication.

$$\tilde{A} \times \tilde{B} = (m_1 m_2, |m_1|\alpha_2 + |m_2|\alpha_1, |m_1|\beta_2 + |m_2|\beta_1). \quad (2)$$

Based on extension principle another formula of multiplying a TFN $\tilde{A} = (l_1, m_1, n_1)$ by another TFN $\tilde{B} = (l_2, m_2, n_2)$ where (l_1, l_2), (m_1, m_2) and $(n_1,\ n_2)$ are the lower-pointed , main-pointed and upper-pointed values of $\tilde{A}$ and $\tilde{B}$, respectively [6] is given as:

$$\tilde{A} \times \tilde{B} \approx (\min\{l_1 l_2, l_1 n_2, n_1 l_2, n_1 n_2\}, m_1\, m_2\, , \max\{l_1 l_2, l_1 n_2, n_1 l_2, n_1 n_2\}) \quad (3)$$

The drawbacks of expressions 1, 2 and 3 can be outlined as follows:

- For two TFNs, if both of their main-pointed values are 0 their product according to expressions 1 and 2 resultant is always (0,0,0) and that is logically incorrect while that mean zero without margin , i.e., crisp zero. Moreover, if $\tilde{A}$ and/or $\tilde{B}$ neither positive nor negative expression 1 is failed.
- In expression 1, Giving cases of the product of two TFNs in the sake of positivity and negativity make this method practically failed when we have fully fuzzy regression or mathematical programming problems where the coefficient(s) of equation(s) are unknown multiplied by some TFN of input data whereas |.| in expression 2 cause complication for least squares approach may be used in estimating the coefficients of fully fuzzy regression analysis. Furthermore, expression 3 is not suitable for unknown coefficients of regression although Al-Qudaimi and Kumar [1] overcome

the flaws related to multiplication of unknown with known triangular interval-valued fuzzy number of Rabiei et al. [8] Regression approach, the proposed multiplication method of Al-Qudaimi and Kumar [1] is confounding and it is computationally complicated in estimating the regression coefficients using manual least squares approach.

- Expression 1, 2 and 3 all cause irrational deviation of margin s which drop more and unsubstantial uncertainty onto the TFN resultant.
- The product of Expression 3 doesn't respect the homogeneity in multiplication where for two symmetric TFNs; their product must give symmetric TFN as resultant.

These drawbacks will be shown with the help of examples in Section 5.

The proof of homogeneity in multiplication of two symmetric TFNs, $\tilde{A}=(l_1,m_1,n_1)$ and $\tilde{B}=(l_2,m_2,n_2)$ is straightforward. knowing that $\tilde{A}$ and $\tilde{B}$ are symmetric if their right and left margin $(\alpha_1=m_1-l_1, \alpha_2=m_2-l_2$ and $\beta_1=n_1-m_1, \beta_2=n_2-m_2)$ respectively, are mutually equal , $\alpha_1=\beta_1$ and $\alpha_2=\beta_2$. Hence, using the quantified membership function for $\tilde{A}$ and $\tilde{B}$ we have $\frac{q}{\alpha_1}\cdot\frac{q}{\alpha_2}=\frac{q^2}{\alpha_1\alpha_2}$ and while $\alpha_1\alpha_2=\beta_1\beta_2$ the left and right margin s of TFN resultant (say $\tilde{C}=(l,m,n)$) are equal. Thus, and we can safely write $\alpha_{\tilde{C}}=\beta_{\tilde{C}}=\alpha_1\alpha_2=\beta_1\beta_2$.

4. Proposed QKB method

A straightforward QKB method for multiplying two TFNs is propounded in this section as stepwise into 3 following steps.

Step 1: For two TFNs $\tilde{A}$ and $\tilde{B}$, Unify their formation to be represented as $(m,\alpha,\beta)$where m, α,β are the main-pointed value, lift margin and right margin of a TFN respectively. If $\tilde{A}$ and/or $\tilde{B}$ in three trinity-ordered representation, i.e., (l,m,n) where l, m and n are lower-pointed, main-pointed and upper-pointed values respectively then

$$\alpha=m-l \tag{3}$$

and

$$\beta=n-m. \tag{4}$$

Step 2: Perform direct multiplication such that for Two TFNs, $\tilde{A}_1=(m_1,\alpha_1,\beta_1)$ and$\tilde{A}_2=(m_2,\alpha_2,\beta_2)$,

$$\tilde{C}_{\tilde{A}_1\times\tilde{A}_2}=\tilde{A}_1\times\tilde{A}_2=(m_1m_2,\alpha_1\alpha_2,\beta_1\beta_2)=(m,\alpha,\beta).$$

Moreover, in case of multiplication of n TFNs (say $\tilde{A}_1,\tilde{A}_2,\dots\tilde{A}_n$) we have

$$m=\prod_{i=1}^{n} m_i,$$

$$\alpha=\prod_{i=1}^{n} \alpha_i,$$

and

$$\beta=\prod_{i=1}^{n} \beta_i$$

so, we have

$$\tilde{C}_{\tilde{A}_1\times\tilde{A}_2\times\dots\times\tilde{A}_n}=(m,\alpha,\beta)$$

Step 3: If it is necessary represent the resultant as three lower, main and upper pointed values, i.e., as trinity-order representation, (l,m,n), use equations 3 and 2 of Step 1.

5. Illustrative examples

To establish the easiness and soundness of QKB method propounded in Section 4, critical examples of TFNs are provided in Table 1.

Table 1. Examples of multiplication of two TFNs using the existing expressions and QKB method

$\tilde{A} * \tilde{B}$ $(m_1, \alpha_1, \beta_1)(m_2, \alpha_2, \beta_2)$	**Expression (1)**	**Expression (2)**	**Expression (3)**	**QKB method**
$(20, 3, 3)(30, 2, 2)$	$(600, 130, 130)$	$(600, 130, 130)$	$(600, 124, 136)$	$(600, 6, 6)$
$(0, 2, 3)(0, 3, 2)$	No answer	$(0, 0, 0)$	$(0, 9, 6)$	$(0, 6, 6)$
$(0, 0, 1)(0, 0, 2)$	$(0, 0, 0)$	$(0, 0, 0)$	$(0, 0, 2)$	$(0, 0, 2)$
$(-3, 0, 1)(0, 0, 5)$	$(0, 0, 15)$	$(0, 0, 15)$	$(0, 6, 0)$	$(0, 0, 5)$
$(-3, 2, 2)(3, 1, 1)$	$(-9, 9, 9)$	$(-9, 9, 9)$	$(-9, 11, 7)$	$(-9, 2, 2)$

The given examples show the difference between QKB method and other existing expressions. QKB method over and above respecting homogeneity principle, it maintains rational margins around the main-pointed value. Besides, it is simple for any two unknown TFNs, i.e., $(m_1, \alpha_1, \beta_1)(m_2, \alpha_2, \beta_2) = (m_1 m_2, \alpha_1 \alpha_1, \beta_1\, \beta_2)$ and for multiplication of any n numbers of TFNs, the result is $(\prod_{i=1}^{n} m_i\,, \prod_{i=1}^{n} \alpha_i\,, \prod_{i=1}^{n} \beta_i)$ without complication and degeneration. Whereas, that is not applicable using expression 1 and expression 3 and for expression 3, we have $(m_1 m_2, |m_1|\alpha_2 + |m_2|\alpha_1, |m_1|\beta_2 + |m_2|\beta_1)$ which is affected by the of main-pointed in obtaining the margins such that for any two zeros of main-pointed values of two TFNs, we will get always zero-TFN, i.e., $(0, 0, 0)$. Moreover, expression 2 causes high deviation in the margins of nonzero-TFN resultants.

Another important factor to compare sound methods against efficiency is time of calculation or computation, QKB method for two TFNs needs only multiplication operation, 3 times. Whereas, multiplication of:

- Expression 1 needs 5 times and 2 times for addition and moreover, 2 times comparison of each TFN in three cases, i.e., 12 times for total comparison of known two TFNs.
- Expression 2 needs 5 times and 2 times for addition as well as 4 times for absolute, i.e., $(|.|)$.
- Expression 3 needs 10 times and 3 times comparison to find the minimum for the lower-pointed value and 3 for maximum for the upper-pointed value.

In computation with long data size need multiplication operation it is necessary to consider the time as time complexity which navigates the gross time required for a method or an algorithm completely and it is usually express in O-notation follows the size of the input [7].

6. Conclusions

Multiplication is very important operation in fuzzy zone. With TFNs which is the wider used fuzzy numbers where essentially they represent uncertainty as a main-pointed value surrounded with margins, multiplication is the fundamental operation for decomposing fuzzy systems of equations in mathematical programming and regression models. Hence, it is very important to use straightforward unified method to multiply two TFNs and in general n TFNs. Such a method has to maintain trinity-order condition as well as homogeneity principle. In this paper, a TFN quantified membership function and simple unified method (named as QKB method) are propounded, the quantified membership function used simply to proof the homogeneity principle such that two terms cannot be equal if they are not dually homogeneous. The simplicity of QKB method overcomes the drawbacks of existing expressions. Illustrative examples show the straightforwardness and soundness of QKB method. Amazingly existing expressions have widely been used for many applications of triangular fuzzy regression models. Researcher may find QKB method as simple and sound as they can use in future TFNs multiplication applications.

Conflict of interest: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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