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Ranking method for efficient units by RPA and TOPSIS in DEA

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This paper considers the rank of set efficient units in Data envelopment analysis (DEA). DEA measures the efficiency of decision making units (DMUs) within the range of less than or equal to one. The corresponding efficiencies are referred to as relative efficiencies, which describe the best performances of DMUs, and these efficient units determine efficiency frontier. This research proposes an extended on a current research by a technique for order preference by similarity to an ideal solution (TOPSIS) method. Therefore, in this paper, we first introduce two methods namely regular polygon area (RPA) and TOPSIS. Then using common set of weights in order to all efficient units obtained from DEA models, they are projected into two-dimensional plane. Finally, the units are ranked by RPA and TOPSIS methods. Also, with the numerical example, our method is compared with other methods. The obtained results of numerical example show that they are almost close to each of several methods.

1. Introduction

Data envelopment analysis technique, which is developed based on the mathematical programming, evaluates the relative efficiency of a set of homogeneous decision making units. The efficiency of each DMU is a function of the amount and number of inputs and outputs DMUs. To measure the technical efficiency of any observed input-output bundle, one needs to know the maximum quantity of output that can be produced from the relevant input bundle. However, in DEA we a benchmark technology from the observed input-output bundles of the firms in the sample, such as a production frontier is constructed. Justifying of each unit on frontier is interpreted as efficiency and any deviation from this frontier is interpreted as inefficiency.

Efficient units can be ranked by several methods. Andersen and Petersen [2] suggested a criterion that permits one to rank order of the units that have all been found to be entirely technical efficiency by DEA. It is worth to note that the AP model can be infeasible, sometimes. A potential problem of feasibility with these supper efficiency models has been studied by Seiford and Zhu [10], Alder et al. [1]. For some efficient observations, there are may not be any input-oriented or output-oriented projection onto a frontier, constructed from the remaining observations, in the data set. Balf [9] proposed RPA method for ranking efficient units.

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There are many Multi-criteria decision-making (MCDM) methods in the literature, as TOPSIS [15] or the TOPSIS-ELECTRE [16] or the new Reference Ideal Method (RIM) [17, 18].

In this study we extend the Balf approach [9] for ranking efficient units in present of TOPSIS method. Although, we know that the TOPSIS method is a way to rank units, but in this paper, we use a combination of regular polygon area (RPA) and TOPSIS (RPA-TOPSIS method) to rank DMUs. There are two important points to this approach. One, the use of optimal common weights, second, an application of images the initial units in two-dimensional space under optimal common weights. In the latter case, working in two-dimensional space is much easier than in higher dimensional space. The particular case of this method is in accordance with the TOPSIS method.

The remainder of this paper is organized as follows. The next section represents background. In Section 3 introduces a brief discussion about supper-efficiency ranking techniques. Section 4, gives a complete ranking of DMUs by RPA-TOPSIS method. Numerical example is in Section 5 and conclusion of the method is presented in the last Section.

2. Background

This section gives information about RPA and TOPSIS which are combined together in order to illustrate the proposed approach in the next section.

2.1 RPA method

This section reviews a rule for calculating of RPA presented by Balf [9]. For this purpose, we relate that area of a triangle can be written as determinant form for each of the following quaternion cases. The proof all the Theorems are given [9].

First case: The origin is one of the vertexes,

Second case: The origin lies inside triangle,

Third case: The origin lies outside triangle,

Forth case: The origin lies on one of the triangle edges.

Theorem 1: If points $O(0,0)$, $A(x_1, y_1)$ and $B(x_2, y_2)$ be coordinates of triangle vertexes then the area of triangle ΔOAB is as follows:

$$S_{\Delta OAB} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \tag{1}$$

Theorem 2: If points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be arbitrarily coordinates of triangle vertexes in anti-clock wise sense, then the area of triangle ΔABC is calculated as follows:

$$S_{\Delta ABC} = \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right] \tag{2}$$

Theorem 3: The area of any regular polygon with n vertex $p_j(x_j, y_j), j = 1,2,...,n$ in anti-clock wise sense is calculating as follows:

$$S_{p_1 p_2 \dots p_n} = \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \dots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right] \tag{3}$$

2.2 A method for finding common set of weights (CSW)

In DEA for calculating the efficiency of different DMUs, different sets of weights are obtained. It seems to be unacceptable in reality. So the following model is used to find common set of weights. This model has some advantages that will be discussed later. This idea is formulated simultaneously maximizing the ratio of outputs and inputs for all decision making units (DMUs). So, we present the following programming problem:

$$\begin{aligned} \max \quad & z \\ s.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - z\sum_{i=1}^{m} v_i x_{ij} \ge 0, \quad j=1,\dots,n, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j=1,\dots,n, \\ & z \ge 0,\ u_r, v_i \ge \varepsilon, \quad r=1,\dots,s,\ i=1,\dots,m. \end{aligned} \tag{4}$$

Note that instead of solving n linear programming DEA models, only one non-linear programming problem is solved and the efficiency for all DMUs are obtained.

2.3 TOPSIS

TOPSIS (technique for order preference by similarity to an ideal solution) method is presented in Chen and Hwang [5]. Consider a MCDM problem that can be concisely expressed in matrix format as Table 1 together weights vector $w=(w_1, w_2, \dots, w_n)$, $\sum_{i=1}^{n} w_i = 1$.

Table 1: Decision matrix

	C_1	C_2	...	C_n
A_1	x_{11}	x_{12}	...	x_{1n}
A_2	x_{21}	x_{22}	...	x_{2n}
...	...	...	...	...
A_m	x_{m1}	x_{m2}	...	x_{mn}

where $A_1, A_2, \dots, A_m$ are possible alternatives among which decision makers have to choose, $C_1, C_2, \dots, C_n$ are criteria with which alternative performance are measured, x_{ij} is the rating of alternative A_i with respect to criterion C_j.

The procedure of TOPSIS can be related in a series of steps:

***Step 1*:** Calculate the normalized decision matrix. The normalized value n_{ij} is calculated as

$$n_{ij} = x_{ij} \Big/ \sqrt{\sum_{j=1}^{m} x_{ij}^2}, \quad i=1,\dots,n, \quad j=1,\dots,m.$$

Then we set $\alpha_{ij} = w_i n_{ij}$, $j=1,\dots,m$, $i=1,\dots,n.$

***Step* 2:** Determine the positive ideal and negative ideal solution.

$$A^{+}=(\alpha_1^{+},\ldots,\alpha_n^{+})=\left\{(\max_j \alpha_{ij} \mid i\in I),(\min_j \alpha_{ij} \mid i\in J)\right\}$$

$$A^{-}=(\alpha_1^{-},\ldots,\alpha_n^{-})=\left\{(\min_j \alpha_{ij} \mid i\in I),(\max_j \alpha_{ij} \mid i\in J)\right\}$$

where I is associated with benefit criteria and J is associated with cost criteria.

***Step* 3:** Calculate the separation measures, using the $n-$dimensional Euclidean distance. The separation of each alternative from the positive ideal solution is given as

$$d_j^{+}=\left\{\sum_{i=1}^{n}(\alpha_{ij}-\alpha_i^{+})^2\right\}^{\frac{1}{2}}, \quad j=1,\ldots,m,$$

Similarly, the separation from the negative ideal solution is given as

$$d_j^{-}=\left\{\sum_{i=1}^{n}(\alpha_{ij}-\alpha_i^{-})^2\right\}^{\frac{1}{2}}, \quad j=1,\ldots,m$$

***Step* 4:** Calculate the relative closeness to the ideal solution. The relative closeness of the alternative A_j with respect to A^{+} is defined as $R_j=\dfrac{d_j^{-}}{d_j^{-}+d_j^{+}},\ j=1,\ldots,m$.

Since $d_j^{+}\geq 0$ and $d_j^{-}\geq 0$, then, clearly, $R_j\in[0,1]$.

***Step* 5:** Rank the preference order. For ranking DMUs using this index, we can rank DMUs in decreasing order.

3. Ranking Technique

Suppose we have a set of n productive units, DMUs. Each $DMU_j, (j=1,\ldots,n)$ consumes m different inputs to produce s different outputs. Two types of orientation DEA models are often used to evaluate DMUs' relative efficiency, CRS (constant return to scale) models, such as CCR model [4], and VRS (variable return to scale) models, such as BCC model [3]. For example, CCR model in multiplier form is defined as a linear programming model as follows:

$$\begin{aligned}
\max \quad & \sum_{r=1}^{s} u_r y_{ro} \\
s.t. \quad & \sum_{r=1}^{s} u_r y_{rj}-\sum_{i=1}^{m} v_i x_{ij}\leq 0, \quad j=1,\ldots,n, \\
& \sum_{i=1}^{m} v_i x_{io}=1, \\
& u_r\geq\varepsilon,\ v_i\geq\varepsilon, \quad r=1,\ldots,s\ , i=1,\ldots,m.
\end{aligned} \tag{5}$$

where $\varepsilon>0$ is a "non-Archimedean element" defined to be smaller than any positive real number. The BCC model adds an additional constant variable, u_o, in order to permit variable return-to-scale:

$$\begin{aligned} \max \quad & \sum_{r=1}^{s} u_r y_{ro} + u_o \\ S.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \le 0, \quad j = 1,\dots,n, \\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & u_r \ge \varepsilon,\ v_i \ge \varepsilon, \qquad r = 1,\dots,s\ , i = 1,\dots,m. \end{aligned} \tag{6}$$

In most models of DEA the best performers have efficiency score unity, and these units lie on frontier efficiency. Several authors have proposed methods for ranking these efficient units [2, 6, 7, 10-14]. The methodology enables an extreme efficient unit "o" to obtain an efficiency score greater than one by eliminating the $o-$th constraint in the model (5), as shown in model (7).

$$\begin{aligned} \max \quad & \sum_{r=1}^{s} u_r y_{ro} \\ S.t. \quad & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,\dots,n, \quad j \ne o \\ & \sum_{i=1}^{m} v_i x_{io} = 1, \\ & u_r \ge \varepsilon,\ v_i \ge \varepsilon, \quad r = 1,\dots,s\ , i = 1,\dots,m. \end{aligned} \tag{7}$$

The next section presents a new method that ranks the importance of the efficient units. Our goal based on translating the basic idea of combining RPA and DEA.

4. Ranking of DMUs by RPA and TOPSIS method

This section deals with ranking of DMUs by RPA and TOPSIS method. Suppose that we have n DMUs each with m inputs and s outputs. The vectors v and u are the weight vectors for input and output, respectively. Set $E_0 = \left\{ j \,\middle|\, DMU_j \text{ is efficient} \right\}$ and (v^*, u^*) be optimal common set of weights by model (4). Let us define function f as $f : R^{m+s} \to R^2, \quad f(x, y) = (v^* x, u^* y)$.

Define the set B as $B = \left\{ z_j \,\middle|\, z_j = (v^* x_j, u^* y_j), j \in E_O, x_j \in R^m, y_j \in R^s \right\}$ and let us $A^+ = (w_1^+, w_2^+)$ and $A^- = (w_1^-, w_2^-)$ be positive and negative ideal vectors respectively, where $w_1^+ = \min v^* x_j$, $w_2^+ = \max u^* y_j$, $w_1^- = \max v^* x_j$ and $w_2^- = \min u^* y_j$ for $j \in E_o$. However, we rank the members B instead of $DMU_j,\ j \in E_o$.

This section describes the ranking approach. Suppose that T be convex hall of $\left\{ z_j \,\middle|\, z_j \in B \right\}$, i.e. $T = convex(B)$. It is trivial that T is a convex polygonal in R_+^2. Also suppose S be regular polygon area (RPA) of T, that is, $S = RPA(T)$. For ranking $z_p \in B$ we first remove it from the set T. Set $T_p = convex(B - \{z_p\})$ and $S_P = RPA(T_p)$. Obviously, $T_p \subseteq T$ and $S_p \le S$ or $\frac{S}{S_P} \ge 1$, (see Fig.1). Then we suggest a rank criteria of z_p as $\theta_p = (\frac{S}{S_p})(\frac{d_P^-}{d_P^- + d_P^+})$, where d_P^+ and d_P^- are distance values z_p of positive ideal A^+ and negative ideal A^-

introduced in subsection 2.3, respectively. It is worthwhile that if $d_P^- = 0$ then $\theta_p = 0$. Also, if $S = S_p$ then we have $\theta_p = \frac{d_P^-}{d_P^- + d_P^+}$, namely we only use TOPSIS method for rank z_p.

Figure 1 (a) shows a convex hall of seven points together with its area S and positive ideal and negative ideal A^+ and A^-, respectively. Meanwhile, Figure 1(b) shows a convex hall after removing z_1, as we gain S_1 and $S - S_1$.

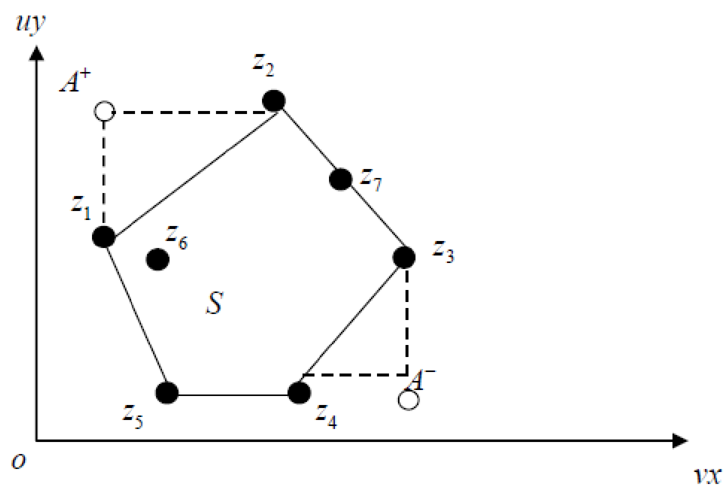

Figure 1 (a): Convex hall of $\{z_j\}_{j=1,\ldots,7}$

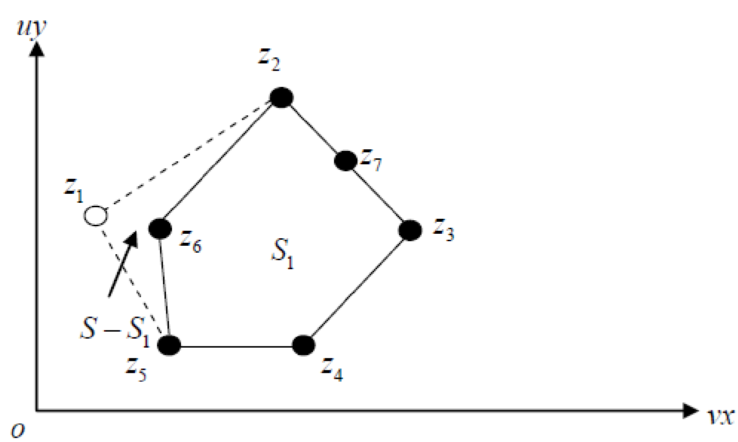

Figure 1 (b): Convex hall of $\{z_j\}_{j=2,\ldots,7}$

5. Numerical Example

Consider 19 DMUs with two inputs and two outputs (Table 2) [11]. In Table 2, DMUs 1, 2, 5, 9, 15 and 19 are CCR efficient. The operations for ranking efficient units 1, 2, 5, 9, 15 and 19 according to model (5), the common set of weights have obtained as follows:

$$u_1^* = 0.01000, \quad u_2^* = 0.089560, \; v_1^* = 0.351901, \; v_2^* = 0.498316, z^* = 0.44$$

Therefore we acquire:

$$z_1 = (72.16,\ 70.27), z_2 = (36.29,\ 36.29), z_5 = (111.89,\ 110.91),\ z_9 = (65.66,\ 28.65), z_{15} = (121.22,\ 97.05),\ z_{19} = (45.5, 45.5)$$

Now we rank the units of the set $B = \{z_1, z_2, z_5, z_9, z_{15}, z_{19}\}$ where $A^+ = (36.29, 110.91)$ and $A^- = (121.22, 28.65)$.

Table 2: Inputs and outputs values

DMU	Input 1	Input 2	Output 1	Output 2
1	81	87.6	5191	205
2	85	12.8	3629	0
3	56.7	55.2	3302	0
4	91	78.8	3379	0
5	216	72	5368	639
6	58	25.6	1674	0
7	112.2	8.8	2350	0
8	293.2	52	6315	414
9	186.6	0	2865	0
10	143.4	105.2	7689	66
11	108.7	127	2165	266
12	105.7	134.4	3963	315
13	235	236.8	6643	236
14	146.3	124	4611	128
15	57	203	4869	540
16	118.7	48.2	3313	16
17	58	47.4	1853	230
18	146	50.8	4578	217
19	0	91.3	0	508

The Figure 2 shows a convex hall of the set B in (vx, uy) space. Let $T = convex(B)$. For ranking units of the set B, first we compute $RPA(T) = S$. According to the Figure 2, the point z_1 is an interior point. Therefore, it is not effective in measuring $RPA(T) = S$. Hence, we have:

$$S = \frac{1}{2}\left[\begin{vmatrix} 65.66 & 28.65 \\ 121.22 & 97.05 \end{vmatrix} + \begin{vmatrix} 121.22 & 97.05 \\ 111.89 & 110.91 \end{vmatrix} + \begin{vmatrix} 111.89 & 110.91 \\ 45.5 & 45.5 \end{vmatrix} + \begin{vmatrix} 45.5 & 45.5 \\ 36.29 & 36.29 \end{vmatrix} + \begin{vmatrix} 36.29 & 36.29 \\ 65.66 & 28.65 \end{vmatrix}\right] = 4186.43$$

Hence,

$$\begin{cases} d_1^+ = 54.21 \\ d_1^- = 64.34 \end{cases}, \quad \begin{cases} d_2^+ = 74.62 \\ d_2^- = 85.27 \end{cases}, \quad \begin{cases} d_5^+ = 75.6 \\ d_5^- = 82.79 \end{cases}, \quad \begin{cases} d_9^+ = 87.35 \\ d_9^- = 55.56 \end{cases}, \quad \begin{cases} d_{15}^+ = 86.05 \\ d_{15}^- = 68.4 \end{cases}, \quad \begin{cases} d_{19}^+ = 66.06 \\ d_{19}^- = 77.57 \end{cases}$$

It follows:

$$\theta_1 = 0.5427, \quad \theta_2 = 0.58, \quad \theta_5 = 0.69, \quad \theta_9 = 0.93, \quad \theta_{15} = 0.67, \quad \theta_{19} = 0.5412$$

However, rank of units 1, 2, 5, 9, 15 and 19 is given according to values of $\theta_j,\ j = 1,2,5,9,15,19$ in Table 3.

The results of ranking using RPA-TOPSIS method are compared with Tchebycheff norm, AP, MAJ and RPA methods in Table 2. As shown in Table 2, DMU19 and DMU9 have highest and lowest rank in MAJ and Tchebycheff models, respectively. Meanwhile, both of them (DMU19 and DMU9) are infeasible in AP model. Also, notice that all DMUs are ranked very close to each other in MAJ and Tchebycheff models, while this is not happened in AP model. In model AP, DMU2 and DMU1 have first and last rank, respectively. In RPA method

DMU9 and DMU1 have highest and lowest rank. Also, it is observed that RPA-TOPSIS method has closest rank to RPA.

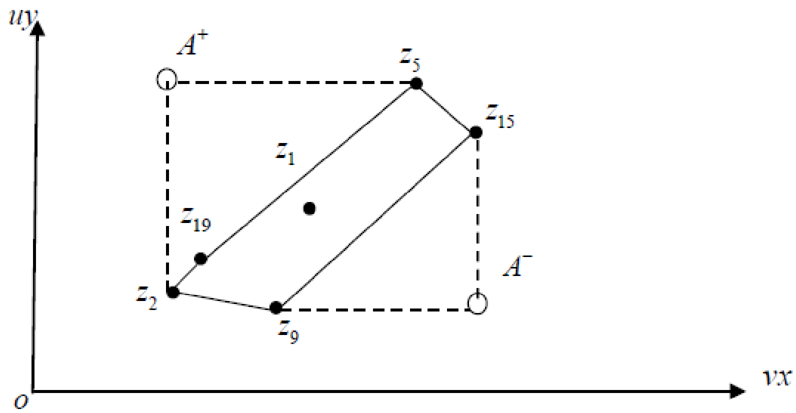

Figure 2: Convex hall of $\{z_j\}_{j=1,...,7}$

Table 3: The results of using different models for ranking of DMUs

Model/DMU	1	2	5	9	15	19
AP	4	1	3	-	2	-
MAJ	5	3	2	6	4	1
Tch. Norm	5	2	3	6	4	1
RPA	6	4	3	1	2	5
RPA-TOPSIS	5	4	2	1	3	6

6. Conclusions

This paper shows a simple notion and important for ranking kind of efficient units. The TOPSIS technique and the RPA method are two separate ways to rank units. In this paper, we used the combined TOPSIS and RPA approach to rank the DMUs. The combination of these two approaches has been based on two simple and important notions. One is to use common optimal weights and the other is to use images of primary units in two-dimensional space using the same optimal common weights. We have also shown that in a particular case our method will be coincidence with a TOPSIS method. Finally, one may be interest to work on fuzzy programming with this approach or ordinal data in present inhomogeneous units.

Conflict of interest: The author declares that he have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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