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Deep Learning Application in Rainbow Options

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ABSTRACT

Due to the rapid advancements in computer technology, researchers are attracted to solving challenging problems in many different fields. The price of rainbow options is an interesting problem in financial fields and risk management. When there is no closed-form solution to some options, numerical methods must be used. Choosing a suitable numerical method involves the most appropriate combination of criteria for speed, accuracy, simplicity and generality. Monte Carlo simulation methods and traditional numerical methods have expensive repetitive computations and unrealistic assumptions on the model. Deep learning provides an effective and efficient method for options pricing. In this paper, the closed-form formula or Monte-Carlo simulation are used to generate data in European and Asian rainbow option prices for the deep learning model. The results confirm that the deep learning model can price the rainbow options more accurately with less computation time than Monte-Carlo simulation.

1 Introduction

An option is a kind of financial derivative i.e., its value depends on the performance of one or more underlying assets. More precisely, an option is a contract between two parties that gives the holder the right (but not obligation) to buy or sell some of the underlying asset in the future. In general, options can be divided into path-independent and path-dependent categories. For instance, European options are path-independent and their payoffs depend on the price of the underlying asset at the maturity whereas Asian options are path-dependent that their payoffs depend on the average price of the underlying asset over part or all of its life. Both European and Asian options are traded in the financial markets, both in exchanges and over-the-counter. The buyer of an option gives the seller an amount called premium when setting up the contract. The premium value depends on the underlying assets prices on the current time and the kind of option [4, 21, 15, 18].

Uncertainty and randomness in financial markets have made analysing and identifying the behaviour of these markets attractive and challenging. Financial mathematics tries to model and analyse these markets with a combination of stochastic processes, economics, financial engineering, numerical analysis and random calculations. Artificial intelligence is in its second new era, while the research on this context that how machines can display levels of human intelligence began in the 1950s

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[16]. Nowadays, the artificial intelligence is often the implementation of machines on very large neural networks with very many layers. The purpose of these neural networks is to mimic the human brain function, i.e., by receiving extensive stimuli and then decomposing them through the layers and neurons which learn to relate the input to the output. With enough data, we can train an artificial neural network to learn the best relation between inputs and outputs and then it can be used to do it frequently [2, 20]. The input and output sets can be very large, but it turns out that the large-scale artificial neural network are skilled in relating big data to multiple outcomes through training. Actually, machines can be trained much faster than humans and perform this analysis quickly.

For large neural networks also known as deep learning, there may be millions of parameters [13]. To find these parameters, it is computationally impossible to obtain numerical gradients for optimization. Fortunately, we can do this analytically with linear calculations using the backpropagation algorithm. In summary, learning is an optimization problem and large-scale learning is much easier when done analytically than numerically. With the development of artificial intelligence technology and the machine learning, many issues, including financial issues that do not have analytical solutions, have been investigated through the machine learning. In addition to the computational power required for parametric calibration, the classic parametric models make unrealistic economic and statistical assumptions. The use of data-oriented approaches based on non-parametric models may be a suitable alternative. Recently, in [8] the performance of option pricing based on the most popular machine learning algorithms is investigated. In [12] a new forward-backward stochastic differential solver is introduced based on deep learning and least square regression for American options. However, the multi-asset option pricing has been investigated by various methods such as alternating direction implicit methods, finite difference methods, mesh-free methods, and Monte-Carlo simulation [3, 10, 11, 14], but in the best of our knowledge, the machine learning model has just been implemented for single-asset option pricing [6]. In this paper we use the machine learning and artificial neural networks, which is an effective and practical method in calculating and solving challenges in various sciences, to price the European and Asian rainbow options. This method eliminates the need for unrealistic economic and statistical assumptions in practice.

In Section 2 we describe four types of two-asset European options, including a call option on the maximum of two assets, a call option on the minimum of two assets, a relative outperformance option and a product option. Furthermore, the arithmetic and geometric two-asset Asian option which have been priced by implementing the Monte-Carlo method. In Section 3 some concepts and computational methods in the machine learning and thee artificial neural networks are reviewed in summary. Moreover, a brief description of the deep learning and the TensorFlow software library is provided. Section 4 implements the concepts and computational methods presented in Sections 2 and 3 for a deep learning model to estimate pricing of options which are generated according to four data sets from four mentioned European options and then the results are reported and concluded. Thereupon according to the standard Monte-Carlo simulation method and Monte-Carlo simulation with variance reduction technique for two-asset Asian option with arithmetic and geometric average, we train the four sets of generated prices data, and then the results are represented numerically and intuitively. Finally, we investigate the efficiency of deep learning in two-asset Asian option pricing in terms of computational time. Finally, at the last section we conclude the efficiency of the methods and compare the obtained results.

2 Multi-Asset Options

The price of a multi-asset option is the solution of the multi-dimensional Black-Scholes (B-S) equation which is the following d-dimensional partial differential equation (PDE):

$$\frac{\partial V(\boldsymbol{S},t)}{\partial t}+\frac{1}{2}\sum_{i,j=1}^{d}\sigma_i\,\sigma_j\rho_{ij}S_iS_j\frac{\partial^2 V(\boldsymbol{S},t)}{\partial S_i\partial S_j}+r\sum_{i=1}^{d}S_i\frac{\partial V(\boldsymbol{S},t)}{\partial S_i}-rV(\boldsymbol{S},t)=0, \tag{1}$$

for $(\boldsymbol{S},t)\in(0,\infty)^d\times[0,T)$, with the final condition $V(\boldsymbol{S},T)=\ V_T(\boldsymbol{S})$, where $V(\boldsymbol{S},T)$ is the value of the option in the multi-asset $\mathbf{S}\ =\ (S_1,S_2,\ldots,S_d)$ at time t. T is the maturity time, σ_i is the volatility of underlying asset S_{i}, ρ_{ij} is the correlation between the i-th and the j-th assets, and r is the risk-free interest rate. Rainbow Options refer to all multi-asset options whose payoff depends on more than one underlying risky asset. For now, we restrict the discussion to the two-asset case, i.e., $d=2$. Two-dimensional B-S equation with two assets $x=S_1$ and $x=S_2$ is the following PDE:

$$\frac{\partial V}{\partial\tau}=\frac{1}{2}\sigma_x^2x^2\frac{\partial^2V}{\partial x^2}+\frac{1}{2}\sigma_y^2y^2\frac{\partial^2V}{\partial y^2}+\sigma_x\sigma_y\rho\frac{\partial^2V}{\partial x\partial y}+rx\frac{\partial V}{\partial x}+ry\frac{\partial V}{\partial y}-rV, \tag{2}$$

for $(x,y,\tau)\in\Omega\times[0,T)$, with the initial condition $V(x,y,0)=V_0(x,y)$which is the equation (1) with variable changing $\tau=T-t$ in the truncated domain $\Omega=(0,x_{max})\times(0,y_{max})$. Now we consider some various European and Asian rainbow options.

2.1 European Rainbow Options

2.1.1 Call Option on the Maximum of Two Assets

The European call option on the maximum of two assets with the strike price K has the payoff $V(x,y,\tau)=\max\{\max\{\mathrm{x}-\mathrm{K},\ \mathrm{y}-\mathrm{K}\},0\}$. The closed form solutions of the option is as follows [17]:

$$\begin{aligned}V(x,y,\tau)=\ &xM(d_1,d;\rho_1)+yM\big(d_2,-d+\sigma\sqrt{\tau};\rho_2\big)\\&-Ke^{-r\tau}\big[1-M\big(-d_1+\sigma_x\sqrt{\tau},-d_2+\sigma_y\sqrt{\tau};\rho\big)\big],\end{aligned} \tag{3}$$

where M is the cumulative bivariate normal distribution function and defined as

$$M(a,b;\ \rho)=\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{b}\int_{-\infty}^{a}\exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right)dx\,dy, \tag{4}$$

and other parameters are as follows:

$$d_1=\frac{ln\left(\frac{x}{K}\right)+\left(r+\frac{\sigma_x^2}{2}\right)T}{\sigma_1\sqrt{T}},\qquad d_2=\frac{ln\left(\frac{y}{K}\right)+\left(r+\frac{\sigma_y^2}{2}\right)T}{\sigma_2\sqrt{T}},\qquad d=\frac{ln\left(\frac{x}{y}\right)+\frac{\sigma^2}{2}\tau}{\sigma\sqrt{T}}, \tag{5}$$

$$\sigma=\sqrt{\sigma_x^2+\sigma_y^2-2\rho\sigma_x\sigma_y},\qquad \rho_1=\frac{(\sigma_x-\rho\sigma_y)}{\sigma},\qquad \rho_2=\frac{(\sigma_y-\rho\sigma_x)}{\sigma}, \tag{6}$$

2.1.2 Call Option on the Minimum of Two Assets

The call option on the minimum of two assets with the strike price K has the payoff $V(x,y,0)=\max\{\min\{\mathrm{x}-\mathrm{K},\ \mathrm{y}-\mathrm{K}\},0\}$. The closed form solution for this option is given by [17]

$$\begin{aligned}V(x,y,\tau)=\ &xM(d_1,-d;-\rho_1)+yM\big(d_2,d-\sigma\sqrt{\tau};-\rho_2\big)\\&-Ke^{-r\tau}\big[1-M\big(d_1-\sigma_x\sqrt{\tau},\ d_2-\sigma_y\sqrt{\tau};\rho\big)\big],\end{aligned} \tag{7}$$

where its parameters are the same as the parameters in Section 2.1.1.

2.1.3 Relative Outperformance Options

The relative outperformance options have been introduced by Derman [5] and Zhang [19]. The payoff of a relative outperformance call option with the strike price K is $V(x,y,0) = \max\{\frac{x}{y} - K, 0\}$, which has the following closed form solution

$$V(x,y,\tau) = e^{-r\tau}[FN(d_2) - KN(d_1)], \tag{8}$$

where

$$F = \frac{x}{y}\exp\left((\sigma_y^2 - \rho\sigma_x\sigma_y)\tau\right), \qquad \sigma = \sqrt{\sigma_x^2 + \sigma_y^2 - 2\rho\sigma_x\sigma_y}, \tag{9}$$

$$d_1 = \frac{ln\left(\frac{F}{K}\right) + \left(\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}, \tag{10}$$

and N is the cumulative standard normal distribution function.

2.1.4 Product Options

The two-asset product call option has the payoff $V(x,y,0) = \max\{x \times y - K, 0\}$ and the closed form solution as [19]

$$V(x,y,\tau) = e^{-r\tau}[FN(d_2) - KN(d_1)], \tag{11}$$

where

$$F = xy\, e^{(\rho\sigma_x\sigma_x)\tau}, \qquad \sigma = \sqrt{\sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y}. \tag{12}$$

The parameters d_1 and d_2 are the same as (10).

2.1.4 Asian Rainbow Options

The payoff of a multi-asset Asian option is determined by an average of underlying asset prices [9]. Suppose $S_i(t_j)$, $i = 1,2,\dots,d$ shows the asset price S_i at time t_j, then the arithmetic payoff V in the continuous case is

$$V = \max\left\{\frac{1}{T}\int_0^T \left(\sum_{i=1}^{d} \alpha_i S_i(\tau)\right) d\tau - K, 0\right\}, \tag{13}$$

where α_i are the weights such that $\sum_{i=1}^{d}\alpha_i = 1$. Since the problem of arithmetic Asian option pricing cannot be reduced to a one-dimensional problem, therefore there is no simple pricing formula for this problem [7]. Now we consider a two-asset Asian call option with the payoff V and the continuous arithmetic average with identical weights, i.e.,

$$V = \max\left\{\frac{1}{T}\int_0^T \left(\frac{1}{2}S_1(t) + \frac{1}{2}S_2(t)\right) dt - K, 0\right\}. \tag{14}$$

By choosing N sufficiently large, this option can be approximated with the corresponding discrete arithmetic average with the following payoff

$$V = \max\left\{\frac{1}{N+1}\sum_{j=0}^{N} \left(\frac{1}{2}S_1(t_j) + \frac{1}{2}S_2(t_j)\right) - K, 0\right\}.$$

Furthermore, we suppose the assets prices follow a geometric Brownian motion with the risk-free interest rate as follow

$$\begin{aligned} S_1(t) &= S_1(0)\, exp\left\{\left(r - \frac{\sigma_1^2}{2}\right)t + \sigma_1 W_1(t)\right\}, \\ S_2(t) &= S_2(0)\, exp\left\{\left(r - \frac{\sigma_2^2}{2}\right)t + \sigma_2 W_2(t)\right\}, \end{aligned} \tag{15}$$

where $Cov\big(dW_1(t), dW_2(t)\big) = \rho dt$. Inspired by the above approach, the continuous geometric two-asset Asian option payoff is defined as

$$V = \max\left\{\exp\left\{\frac{1}{T}\int_0^T \log\left(S_1(t)^{\frac{1}{2}} S_2(t)^{\frac{1}{2}}\right) dt\right\} - K, 0\right\}. \tag{16}$$

With implementing the standard Monte-Carlo (MC) method, an estimation of an Asian option price can be achieved. In order to improve the efficiency of the standard Monte-Carlo method, we will apply a variance reduction technique based on the antithetic variate (MC-AV).

3 Deep Learning

In the data age, for storage the various forms of information, powerful sources are provided. In the most area, analysing plenty of these data and making decision based on them is practically out of the human ability. Therefore, the deep learning algorithms try to use the data for improving human utilizing services. The deep learning is a branch of the machine learning and artificial intelligence which tries to model the abstract concepts with learning in various levels. The machine learning makes a mathematical model based on sample data which is known as training data for prediction or decision without explicit instruction. In this study, the supervised learning has been used for the model learning. In the supervised learning, the model is trained with the labelled training data in order to the machine can decide properly about the labels of data which has not seen yet and confront by them in the future. Deep learning actually is learning with the neural networks which has many hidden layers. In deep learning a library which is called TensorFlow is used for training the neural networks with more than two layers. TensorFlow is a powerful open source software library to show all computations and samples in a machine learning algorithm such as: mathematical operations, parameters and updates rules in the large scale [1]. In the first glance, the computations which is done in TensorFlow seems complicated, but this complexity makes it possible to implement complex and difficult models more easily. TensorFlow uses data flow diagram to display all possible calculations in a specific application.

4 Two-Asset Rainbow Options Pricing with Deep Learning

In this section, pricing of the two-asset rainbow options is determined by using the deep learning method. To use neural networks and deep learning for pricing options, we need to generate large enough data of these options.

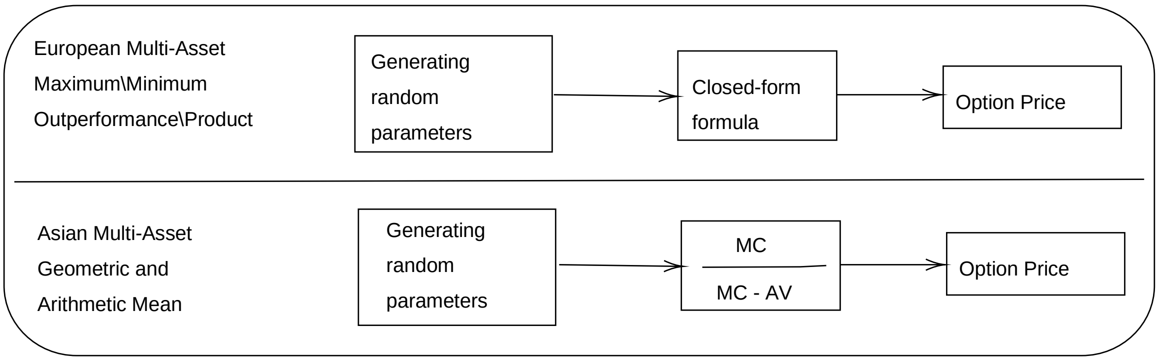

Fig. 1: Flowchart of the Data Generating Process for the Two-Asset Options.

Fig. 1 shows the flowchart of the data generating process for the two-asset options in Sections 2.1 and 2.2. The top first flowchart is for the European options which first generates the parameters randomly and then we give them to the four closed-form solutions in Section 2.1, afterwards we get the European option price. As a result, we have generated data related to the machine learning. As the top flowchart, the bottom one generates the data according to the Asian options with geometric and arithmetic mean using the MC simulation and the MC-AV method.

Table 1: Parameters Range for 100,000 Simulation of Two-Asset European and Asian Option Price.

Parameters	Range
Initial price of the first stock (S_1)	\$10 − \$300
Initial price of the second stock (S_2)	\$10 − \$300
Strike price	1% − 5%
Maturity time	1 − 3 years
Correlation coefficient	−0.9 − 0.9
Volatility of S_1	10% − 90%
Volatility of S_2	10% − 90%

The required values of the parameters are randomly selected according to the range of parameters in table 1. Considering the above-mentioned options, eight data sets with a size of 100000 for eight different options have been generated. The option pricing theory implies that the option price V is linearly correlated with the underlying asset price S and the strike price K. Therefore, we can standardize the generated data by dividing both the underlying asset price and the option price by the strike price. We then use the modified data as input to the deep learning network [6]. To use the neural networks and deep learning in pricing of the two-asset options, we first need to train a model. For this purpose, we divide the eight modified data sets with 100,000 observations in each group into subsets containing 80,000 for the training data and 20,000 for the test data. The details of the deep learning network for the European and Asian options pricing are as follows: in this deep learning network, there are 8 input hyper-parameters. These hyper-parameters through 4 hidden layers which each layer has 300 neurons, are adjusted by activation functions. The activation functions in each layer are respectively called as follows: SELU activation function in the first layer, ELU activation function in the second layer and ReLU activation function in the third and fourth layers. In the final output layer which has unique neurons, the activation function SoftPlus performs computational operations on that layer. In this learning algorithm, the mean square error (MSE) as the cost function and Adam's optimization as optimization algorithm have been used.

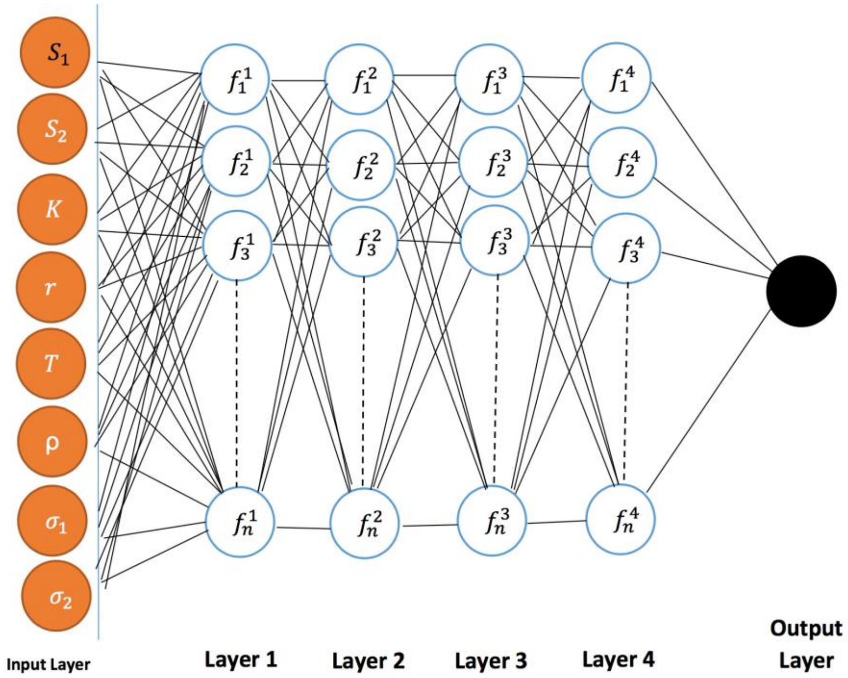

Fig. 2: European Neural Network Pricing Model with a Maximum of Two Assets.

Figure 2 shows the neural network model of the option pricing, where the input layer includes the existing hyper-parameters, 4 hidden layers $f_j^i, i = 1, 2, 3, 4, j = 1, 2, \dots, 300$ in which f_j^i indicates the i-th function and j-th neurone. The outer layer shows the option price according to the activation function SoftPlus.

5 Accuracy analysis of numerical results

By considering the eight generated data sets for European and Asian Options, we divide the data into eight groups with sizes 500, 1000, 5000, 10000, 20000, and 50000 where in each data set, we use 80% of the data as the training set and the rest as the test set. We teach the model for the option pricing by using the generated data. For each data set, we implemented 100 runs of deep learning using 10 different training sets and test sets. In these 100 runs of deep learning that has been done, we consider the following five criteria for the efficiency of the deep learning:

- The median of bias (Indicates the median of relative error of the predicted option price).
- The 95% bias mean (Indicates the mean error of the predicted option price less than 95th percentile).
- Mean squared error (MSE).
- Correlation coefficient between original and predicted data.
- The value of R^2 (the proportion of variance in the dependent variable that can be explained by

the independent variable).

Bias shows the relative prediction error. To calculate it, we first compute the absolute value of the difference between the predicted option price and the option price based on the original data and divide it by the original data option price. We now examine the numerical results in the generated data.

Table 2: Results of the Deep Learning Efficiency in European Options Pricing for the Data Generated by the Closed-Form Formulas.

	Sample Size	500	1000	5000	10000	20000	50000
Max of 2 assets	Bias Median	0.01471	0.01470	0.01498	0.01518	0.01541	0.01553
	Bias Median 95%	0.03271	0.03396	0.03394	0.03468	0.03396	0.03352
	MSE	$6.34e{-}5$	$8.10e{-}5$	$1.23e{-}5$	$2.35e{-}5$	$8.81e{-}5$	$8.33e{-}5$
	ρ	0.99893	0.99897	0.99904	0.99927	0.99963	0.99960
	R^2	0.99889	0.99891	0.99927	0.99931	0.99947	0.99953
Min of 2 assets	Bias Median	0.01435	0.01444	0.01482	0.01491	0.01504	0.01517
	Bias Median 95%	0.03811	0.03521	0.03639	0.03697	0.03701	0.03698
	MSE	$2.38e{-}4$	$1.94e{-}5$	$3.61e{-}5$	$1.22e{-}5$	$2.47e{-}5$	$1.89e{-}5$
	ρ	0.99943	0.99949	0.99952	0.99951	0.99963	0.99977
	R^2	0.99911	0.99932	0.99931	0.99935	0.99941	0.99963
Outperformance	Bias Median	0.01364	0.01363	0.01405	0.01427	0.01454	0.01467
	Bias Median 95%	0.02270	0.02443	0.02436	0.02430	0.02453	0.02513
	MSE	$5.29e{-}5$	$1.93e{-}6$	$3.31e{-}6$	$4.17e{-}6$	$4.46e{-}6$	$4.71e{-}6$
	ρ	0.99881	0.99886	0.99893	0.99915	0.99952	0.99971
	R^2	0.99899	0.99899	0.99991	0.99993	0.99969	0.99961
Product option	Bias Median	0.01556	0.01555	0.01596	0.01619	0.01644	0.01658
	Bias Median 95%	0.04462	0.04634	0.04628	0.04622	0.04623	0.04766
	MSE	$6.16e{-}5$	$2.73e{-}5$	$4.41e{-}6$	$5.33e{-}6$	$5.63e{-}6$	$6.29e{-}6$
	ρ	0.99803	0.99868	0.99871	0.99882	0.99854	0.99863
	R^2	0.99793	0.99796	0.99886	0.99891	0.99969	0.99899

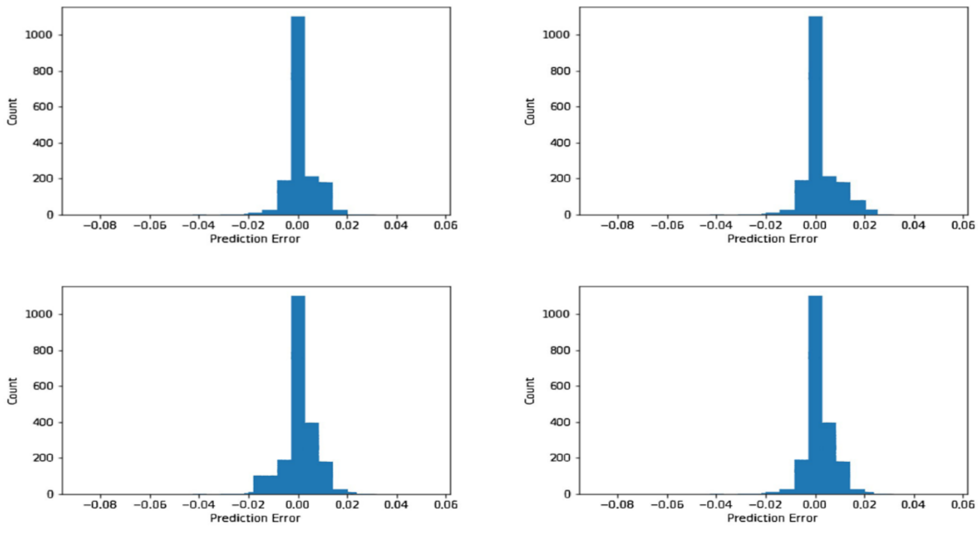

Fig. 3: Prediction Error Density of Two-Asset European Call Options in the Test Sets of the Option on the Maximum Two Assets (Top Left), the Option on the Minimum Two Assets (Top Right), the Relative Outperformance Option (Bottom Left) and the Product Option (Bottom Right)

Table 2 illustrates the explained criteria for the efficiency of deep learning in the European option pricing for the generated data by the simulation of the closed-form option pricing formula for different options and different sample size. The results obtained from the generated data show that the model has been trained properly and with comparison the results for these four different European options, we can see that the relative outperformance option has been trained better than other options. Furthermore, however our model for the generated data by the closed-form formula for pricing the product option is well trained but its accuracy is weaker than other three options. After presenting the results of deep learning for different sample sizes and different European options in Table 2, we now demonstrate the efficiency of deep learning intuitively. Fig. 3 shows the density of prediction error for four European options in the test sets which their error density is located in the interval $(-2\%, 2\%)$.

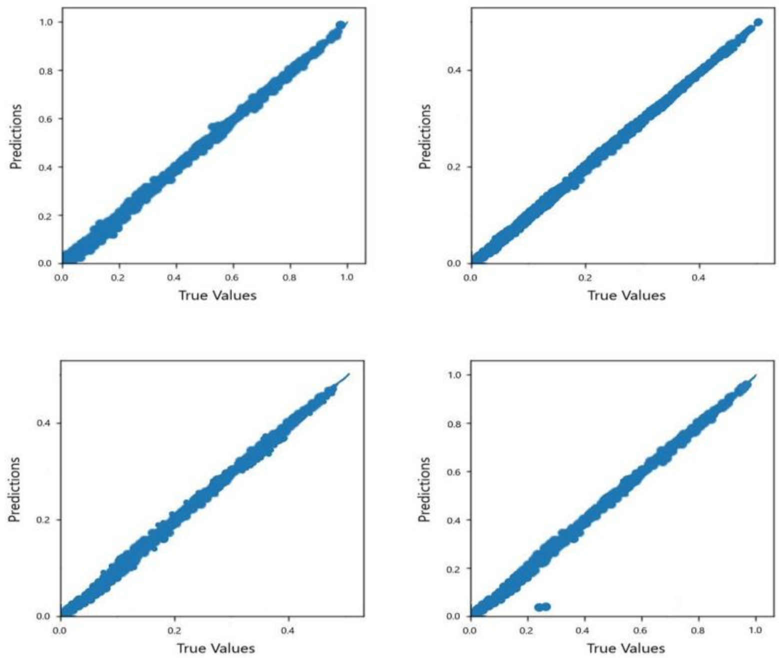

Fig. 4: The True Value Versus Predicted Value of Two-Asset European Call Options in the Test Sets of the Option on the Maximum of Two Assets (Top Left), the Option on the Minimum of Two Assets (Top Right), the Relative Outperformance Option (Bottom Left) and the Product Option (Bottom Right).

Figure 4 shows the true value versus predicted value for these four mentioned European options. we can see that the generated data in the product option deviates more than the other three options, which means the model for the generated data in this option is less accurate. Table 3 demonstrates the above-mentioned criteria for deep learning efficiency in two-asset Asian options pricing with geometric and arithmetic mean for the data generated by the MC and MC-AV methods. The results show that the model for two-asset Asian options pricing is well trained. Furthermore, the data generated by MC-AV has higher accuracy. Now we demonstrate the efficiency of deep learning for two-asset Asian options intuitively.

Table 3: Results of the Deep Learning Efficiency in Two-Assets Geometric and Arithmetic Asian Options Pricing for the Data Generated by MC and MC-AV.

	Sample Size	500	1000	5000	10000	20000	50000
MC Geometric	Bias Median	0.06495	0.06499	0.06535	0.06558	0.06593	0.06597
	Bias Median 95%	0.09648	0.09731	0.09757	0.09751	0.09783	0.09769
	MSE	$4.89e{-}3$	$1.29e{-}4$	$2.25e{-}4$	$3.14e{-}4$	$3.46e{-}4$	$8.19e{-}5$
	ρ	0.99305	0.99423	0.99471	0.99445	0.99419	0.99427
	R^2	0.99576	0.99579	0.99668	0.99674	0.99743	0.99775
MC-AV Geometric	Bias Median	0.04312	0.04316	0.04354	0.04371	0.04432	0.04414
	Bias Median 95%	0.06328	0.06490	0.06574	0.06579	0.06611	0.06626
	MSE	$7.97e{-}4$	$4.38 * e{-}4$	$5.34e{-}4$	$6.22e{-}4$	$6.54e{-}4$	$7.16e{-}6$
	ρ	0.99522	0.99640	0.99688	0.99663	0.99637	0.99631
	R^2	0.99681	0.99693	0.99668	0.99788	0.99759	0.99839
MC Arithmeetic	Bias Median	0.05243	0.05237	0.05275	0.05293	0.05353	0.05325
	Bias Median 95%	0.07237	0.07400	0.07416	0.07521	0.07532	0.07539
	MSE	$6.13e{-}4$	$7.49e{-}4$	$6.21e{-}5$	$7.15e{-}5$	$7.37e{-}5$	$9.42e{-}5$
	ρ	0.99643	0.99666	0.99666	0.99692	0.99736	0.99751
	R^2	0.99522	0.99596	0.99603	0.99612	0.99728	0.99744
MC-AV Arithmeetic	Bias Median	0.03160	0.03154	0.03182	0.03211	0.03273	0.03242
	Bias Median 95%	0.04247	0.04317	0.04333	0.04349	0.04453	0.04456
	MSE	$3.24e{-}4$	$4.38e{-}4$	$2.39e{-}5$	$5.04e{-}5$	$6.49e{-}5$	$1.64e{-}6$
	ρ	0.99734	0.99757	0.99775	0.99783	0.99745	0.99842
	R^2	0.99541	0.99587	0.99622	0.99635	0.99746	0.99763

Figure 5 shows the prediction error density graphs of four generated data sets from Asian options in the test sets which their error density is located in the interval (−4%,4%).

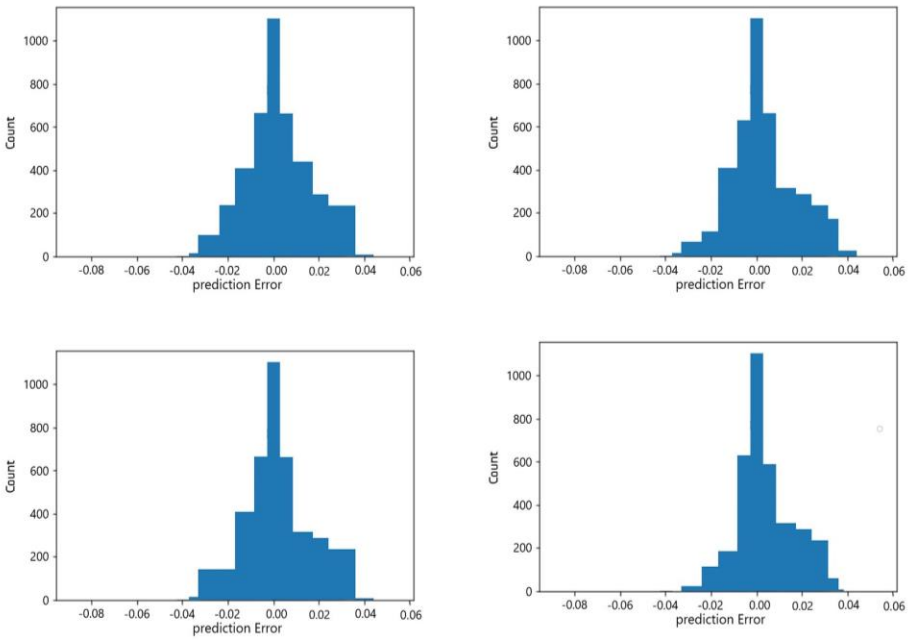

Fig. 5: Prediction Error Density of Two-Asset Asian Call Options in the Test Sets of MC for Geometric Asian Option (Top Left), MC-AV for Geometric Asian Option (Top Right), MC for Arithmetic Asian Option (Bottom Left) and MC-AV for Arithmetic Asian Option (Bottom Right).

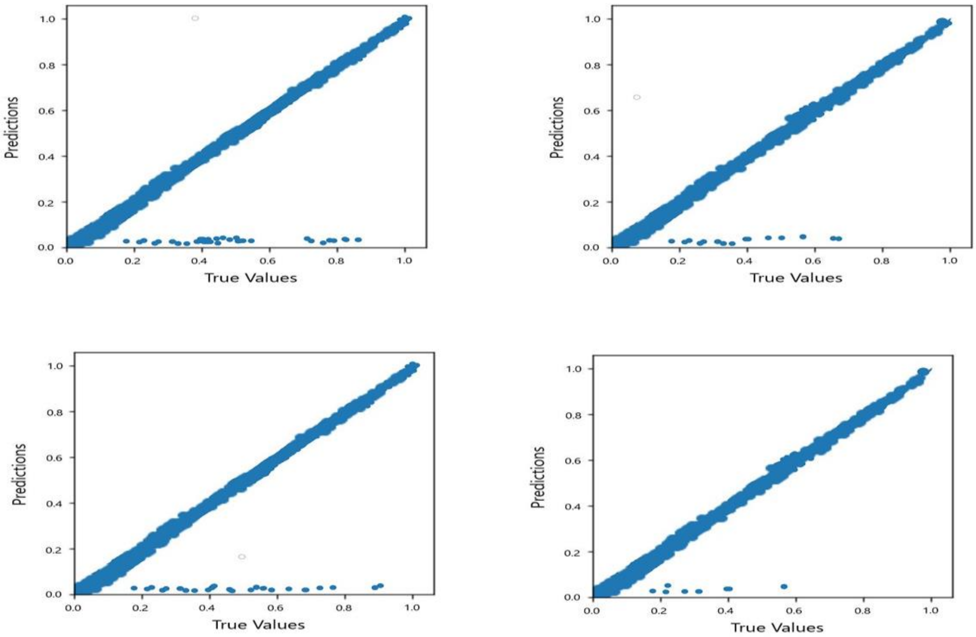

Fig. 6: The True Value Versus Predicted Value of Two-Asset Asian Call Options in the Test Sets of MC for Geometric Asian Option (Top Left), MC-AV for Geometric Asian Option (Top Right), MC for Arithmetic Asian Option (Bottom Left) and MC-AV for Arithmetic Asian Option (Bottom Right).

Fig. 6 shows the true value versus predicted value of two-asset Asian options with geometric and arithmetic mean for the generated data by the MC and MC-AV simulations. As we can see that the generated data with the MC simulation has more deviation than the generated data by MC-AV in both geometric and arithmetic Asian options. Therefore, this model for the generated data by MC-AV is more accurate. Table 4 compares the computation time for 1000 and 100000 arithmetic and geometric Asian option prices in three methods deep learning (DL), MC and MC-AV. These computation have been done by a computer with specifications include a CPU of Intel(R) Core(TM) i5-8250U CPU@1.60GHz 1.80 GHz, a GPU of NVIDIA GeForce 840M, a RAM of 8GB, and an HDD of 1TB. Therefore, the deep learning methods can also be used by individual investors.

Table 4: The Computation Time.

Asian Options	Geometric			Arithmetic		
No. of Prices	DL	MC	MC-AV	DL	MC	MC-AV
1000	0.31s	1.42s	2.82s	0.49s	1.93s	3.11s
100000	0.54s	19.24s	26.18s	0.92s	20.14s	35.42s

The results presented in Table 4 indicate that the computation time by the DL for 100000 geometric and arithmetic Asian option prices is less than one second while, for MC and MC-AV simulations take 2 to 35 seconds. Hence, the deep learning method has a good performance in the computation time.

6 Conclusion

In this study, the pricing of the European and Asian rainbow options using the deep learning method a data driven approach - has been investigated. For this purpose, random pricing data are generated by the closed-form formula of two-asset European options and MC and MC-AV simulations for two-asset geometric and arithmetic Asian options according to the range of the required parameters. Furthermore, we showed that how the Monte-Carlo simulation with a variance reduction technique such as the antithetic variance increases the accuracy of the results. After normalization the generated data, we divide the data into training and test data sets. The training data has been used as input to the deep learning network. For the accuracy analysis, five criterion including the median of bias, the 95% bias mean, MSE, correlation coefficient between the original data and the predicted data, and the value of R2 have been computed for different scenarios. In addition, the computation time for the Monte-Carlo-based methods and the deep learning method have been computed. The results demonstrate that from both accuracy and computational time points of view, the deep learning method is reliable to price the rainbow options. The computational time reduction for the deep learning method in comparison with the Monte-Carlo-based methods is significant. Especially for the large data set, at least a factor of 20 reductions in computational time is achieved with the proposed method.

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