Adv. Math. Fin. App., 2023, **8**(2), P. 667-685

**Advances in Mathematical Finance & Applications**

**www.amfa.iau-arak.ac.ir**
**Print ISSN**: 2538-5569
**Online ISSN**: 2645-4610
**Doi:** 10.22034/AMFA.2022.1952682.1705

Applied-Research Paper

# Multi-Objective Possibility Model for Selecting the Optimal Stock Portfolio

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ARTICLE INFO
*Article history:*
Received 2022-02-19
Accepted 2022-07-25

Keywords:
Mean-variance model
Optimal portfolio
Possibility space
Objective functions

ABSTRACT

In this paper, we utilize fuzzy numbers and possibility theory to model possibility. The purpose of this work is to determine the optimal investment model based on the neural network method for fuzzy LR, trapezoidal, and triangular numbers in an optimal portfolio listed on the Tehran Stock Exchange. The aim is to maximize "returns" and minimize "risk" in order to find the optimal portfolio. Therefore, to achieve this goal, the problem of multi-objective nonlinear programming is addressed. Additionally, the mean-variance model and the standard mean deviation are substituted instead of the Markowitz mean-variance model to examine the selection of the optimal portfolio in the possible space. Finally, by calculating the possibility model of fuzzy numbers, we obtain the optimal stock portfolio, which can be used to construct the stock portfolio with the highest returns and the lowest risk.

## 1 Introduction

Investment decision-making is one of the main issues in financial management. The selection of appropriate tools and techniques that can create optimal portfolios is a key objective in the investment world. The primary purpose of stock portfolio selection is to allocate investments optimally among different assets, based on the assessment of two criteria: risk and return. Consequently, the most suitable stocks are determined along with their respective ratios. In essence, investors strive to maximize profits or returns while minimizing risk. The study of portfolio management is crucial as it pertains to profitability, and an improved model for portfolio selection can lead to higher profits. The first model for the stock portfolio problem was proposed by Markowitz [20], with the key parameters being the expected return and risk, following a normal distribution. Furthermore, investors exhibit rational behavior by investing in multiple shares with the goal of forming an optimal portfolio.

An optimal stock portfolio is one that achieves a balance between risk and return on investment [24]. By considering the variables of risk and return, various optimization techniques can be employed to create an optimal portfolio [3, 17, 18, 19], with the aim of achieving a balance between risk and return

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in the financial market. Fuzzy numbers are utilized in several optimization problems to model the uncertainty arising from factors such as arbitrariness, imprecision, and poor determination of parameter values [2]. Financial markets are influenced by numerous unpredictable factors, resulting in fuzzy uncertainty regarding the return on risky assets. Uncertainties in events, systems, and processes are typically random and can thus be addressed through probability-based approaches. While probability theory has proven useful and successful in many cases, its applicability is limited to a specific type of uncertainty. In many situations, our lack of complete knowledge about a process or system is not solely due to the random aspects governing them; rather, it may stem from insufficient, vague, inaccurate, contradictory, and other similar factors. Therefore, one type of uncertainty can be expressed through probability theory, which pertains to the uncertainty arising from random aspects. The theory of possibility provides a suitable framework for modeling and describing numerous processes and systems involving uncertainty, as many forms of uncertainty encountered in different contexts fall under the category of possible aspects. The mathematical foundation of possibility theory is based on fuzzy set theory. In other words, fuzzy set theory plays a role for possibility theory similar to the role of set theory for probability theory [22]. By replacing the probabilistic model with the possibility model, this study explores the achievement of desired goals and examines selected data from the Tehran Stock Exchange. The objective of this research is to identify the optimal stock portfolio within the realm of fuzzy numbers using two groups of shares from the Tehran Stock Exchange.

In this context, the possibility space replaces the probabilistic space to explore potential developments. Additionally, a neural network is utilized to determine the best possible investment and ultimately construct the optimal portfolio. Stock portfolio optimization encompasses various securities, including debt and equity mutual funds [20]. Optimization is a subject that aims to find the best solution to a problem while considering goals and constraints [29]. Given that there are infinite possibilities for composing a stock portfolio, the objective is to answer the following questions: How can the optimal portfolio be determined? Which stocks, when combined, can form the optimal portfolio? Can the model presented in this study demonstrate the optimal stock portfolio? This paper first introduces fuzzy numbers and a multi-objective optimization model, followed by an expression of the optimal multi-objective model for three types of fuzzy numbers: L-R, triangular, and trapezoidal, within the framework of possibility. The purpose of this model is to address ambiguity by utilizing fuzzy numbers. Moreover, the theory of possibility is suitable for describing uncertainty, and that is why the possibility space is utilized. Space creates probability.

## 2 Theoretical Foundations and Research Background

Inaccurate and provide the basis for reasoning, control, and decision making in conditions of uncertainty, inaccurate and ambiguity. Therefore, the types of fuzzy numbers are introduced, the type of which is determined by their membership function, and according to the drawing, their membership function will be as follows. Fuzzy numbers refer to a possible value, which is a weight between 0 and 1. This weight is called the membership function [7]. It, also, requires α-cutting to perform all mathematical calculations of fuzzy numbers. More information on the fuzzy method can be found [12]. The types of fuzzy numbers are plotted according to the rules of their membership function and are given in Table 1, and when these rules are linearized, triangular and trapezoidal fuzzy numbers are created. The flowchart performs all the steps of the article as Fig. 1.

In a study entitled Stock Portfolio Optimization with Linear and Fixed Trading Cost considered the Markowitz mean-variance model as the core of research which is mathematically a simple Markowitz model [16]. But its main advantage is the ability to add constraints to check the real market conditions.

According to the study of stock portfolio optimization algorithms with mean-variance and difficult real-world constraints [1], this model assumes that the return on assets has a normal distribution and there is no transaction cost, while investors may want to limit the number of assets in their portfolio or set a high and low limit for investing in each asset because of some reasons such as transaction costs, minimum order volume, and difficulty in managing the portfolio.

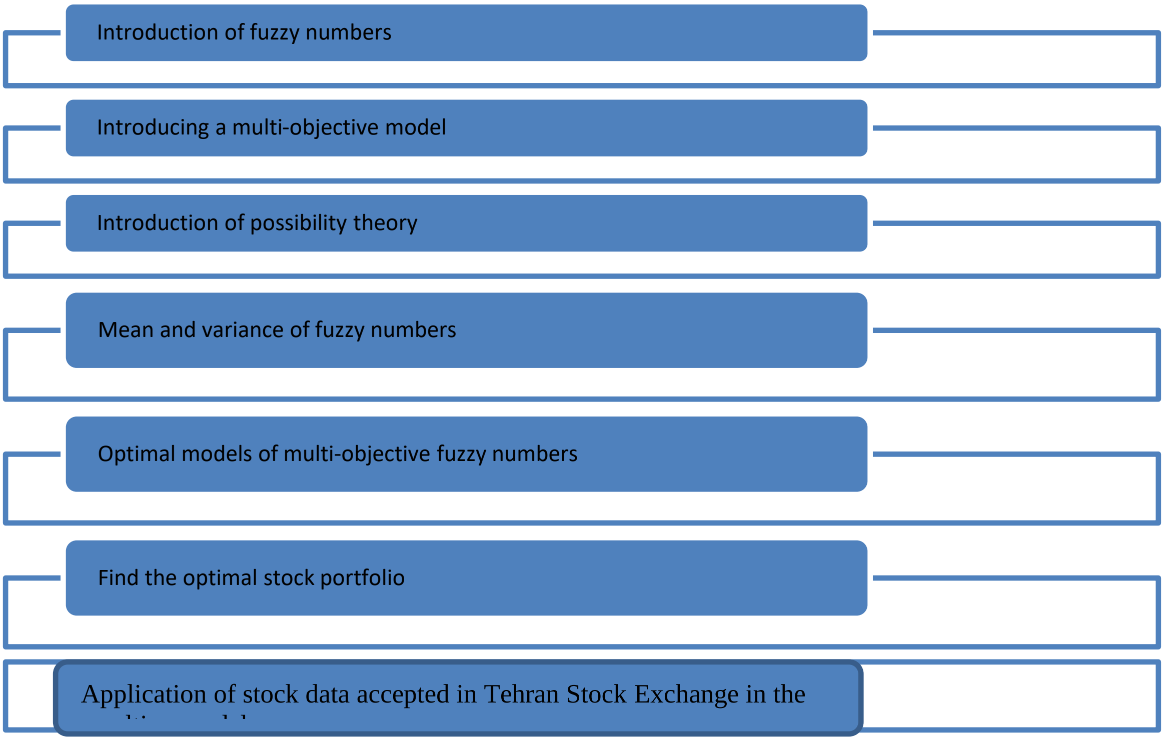
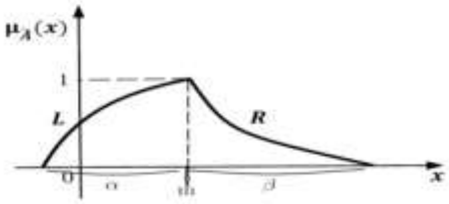
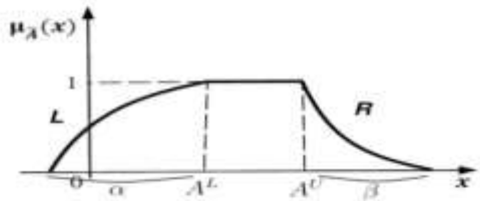
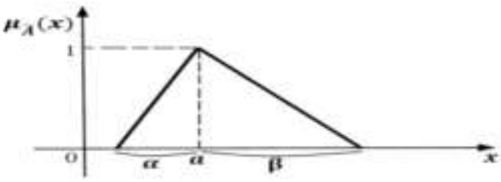
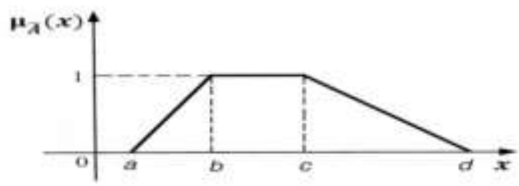



**Fig. 1:** Step-by-Step Article Flowchart

Based on Markowitz's portfolio selection study another important reason for this is its computational complexity, which occurs in solving large-scale quadratic planning problems. [8] used the standard deviation function instead of the risk function in the Markowitz model. Also, [12] in a study entitled "Decision Making: Descriptive, Normative, and Prescriptive Interactions", criticized the Markowitz model for inefficiency with conventional models of risk-taking preferences. Therefore, to resolve the ambiguity in the atmosphere of uncertainty, fuzzy set theory was first proposed by [31] in the study of the outline of a new approach in the analysis of complex systems and decision making. Also, in the study of fuzzy stock portfolio optimization model under real constraints, [14] considered a portfolio optimization model in a fuzzy environment, in which stock returns and trading volume were considered as fuzzy variables. And, in the study of [33] entitled Mean - Multi-period variance of fuzzy stock portfolio selection model with risk control and its size limitations, a fuzzy multi-objective stock portfolio model has been studied that seeks to maximize ultimate wealth by controlling risk.

Recently, several researchers have considered the issue of fuzzy stock portfolio selection. Researchers such as Watada in a study entitled Fuzzy Portfolio Model for Investment Decision Making, [21] in the study of portfolio selection using fuzzy decision theory, and [15] in the study of portfolio selection issues impractical of a fuzzy approach, examined portfolio selection using fuzzy decision theory. [25] in stock portfolio selection based on high and low possibility distributions, presented two types of stock portfolio selection models based on fuzzy probability and probabilistic distribution. [10] in the study of stock portfolio selection under independent possibility information, introduced a possibility planning for the stock portfolio selection problem based on the minimum-maximum criterion.

**Table 1**. Types of Fuzzy Numbers

| | |
|---|---|
| $\mu_{\tilde{A}}(x)=\begin{cases} L\left(\frac{m-x}{\alpha}\right). & x \le m. \\ R\left(\frac{x-m}{\beta}\right). & x \ge m. \end{cases}$ (1) |  **Fig. 1**: Figure 1 Fuzzy number membership function L-R |
| $\mu_{\tilde{A}}(x)=\begin{cases} L\left(\frac{A^L-x}{\alpha}\right). & x \le A^L \\ R\left(\frac{x-A^U}{\beta}\right). & x \ge A^U. \\ 1. & b \le x \le c. \end{cases}$ (2) |  **Fig.2:** Figure 2 L-R fuzzy interval membership function |
| With the linearization of the L and R functions, the fuzzy number and the fuzzy interval L-R become triangular fuzzy number and trapezoidal fuzzy number, respectively, which are expressed as follows. | |
| $\mu_{\tilde{A}}(x)=\begin{cases} 1-\left(\frac{a-x}{\alpha}\right). & a-\alpha \le x \le a. \\ 1-\left(\frac{x-a}{\beta}\right). & a \le x \le \beta + a. \end{cases}$ (3) |  **Fig.3:** Figure 3 Triangular fuzzy number membership function |
| $\mu_{\tilde{A}}(x)=\begin{cases} \left(\frac{x-a}{b-a}\right). & a \le x \le b. \\ 1. & b \le x \le c. \\ \left(\frac{d-x}{d-c}\right). & c \le x \le d. \end{cases}$ (4) |  **Fig.4:** Figure 4 Fuzzy number membership of a trapezoid |

Researchers such as [27] in a study of fuzzy portfolio selection problems, fuzzy optimization, and decision making, [13] in a study of a range of linear distance programming problems and its application in portfolio selection Stocks and [6] in the study of stock portfolio selection based on fuzzy probabilities and probability distributions, introduced range planning models quantitative portfolio selection. In addition, [32] on the mean and possible variance of fuzzy numbers and [33] on the assumption that borrowing is not allowed, express the optimal portfolio of risky assets based on the mean-variance.

## 3 Rresearch Methodology

Investors form a portfolio to reduce investment risk. This portfolio contains stocks that aim to divide the investment risk between several shares. In such a way the profit of one share compensates for the loss of the other share. Based on the concept of possibility and fuzzy numbers, decision-making in a fuzzy environment is introduced with a set of decisions in which the fuzzy goal and fuzzy constraint are integrated. Therefore, model (5) is expressed as a two-objective model under constraints.

**Model 5.** The two-objective mean-variance model with transaction costs, which are the costs of an economic transaction, can be expressed as follows [10].

$$\begin{aligned} &\text{maximize} \quad M(x)=E(R)-\sum_{i=1}^{n} c_i\left|x_i-x_i^0\right|. \quad R=\sum_{i=1}^{n} R_i x_i. \\ &\text{minimize} \quad V(x)=\sum_{i=1}^{n}\sum_{j=1}^{n} Cov(R_i\,.\,R_j)x_i x_j. \\ &\text{Subject to} \quad \sum_{i=1}^{n} x_i = 1. \\ &l_i \le x_i \le u_i. \quad x_i \ge 0. \quad i=1.2.\ldots, n. \end{aligned} \tag{5}$$

Therefore, in the study of stocks selected in model (5) with the aim of maximizing returns and minimizing risk, which according to the types of fuzzy numbers and their membership function, an example is given that is a report of 12 months of a year stock Are Tehran Stock Exchange, the selection of which is completely random and are examined according to the calculated models, then its convergence to the equilibrium point is shown through the neural network [10, 21].

## 4 Introductions and Definitions

**Definition 1 (α-cut ):** α-cut A fuzzy set $\tilde{A}$ is a non-fuzzy set $A_\alpha$ that includes all members of X that have values greater than or equal to α, ie:

$$A_\alpha=\left\{x\in X \mid \mu_{A_\alpha}(x)\ge\alpha\right\}. \quad \alpha\in[0.1]\ , \tag{6}$$

α-slices provide a description of fuzzy sets using definite sets [21].

**Definition 2 (Possibility Size):** Suppose X is a reference set and P (x) is a power set. Function [21]:

$$\pi: P(x) \to [0.1]$$

Is called a measure of possibility whenever:

1) $\pi(\emptyset)=0.\pi(x)=1.$
2) For each A and B of P(x), $A\subseteq B \implies \pi(A)\le\pi(B)$.
3) For each sequence $\{A_i\}$ of the elements P (x) that $A_1\subseteq A_2\subseteq\ldots$ , $\lim_{i\to\infty}\pi(A_i)=\pi(\lim_{i\to\infty}A_i)$
4) 1) For each A and B of P (x), $\pi(A\cup B)=\max[\pi(A).\pi(B)]$.

The purpose of optimization is to find the best acceptable solution according to the constraints and needs of the problem, so neural networks are used to find the optimal stock portfolio.

### 4.1 Proposed neural network model

Consider the following linear programming problem in general [10].

$$\begin{aligned} &\text{Minimize} \quad c^T x. \\ &\text{Subject to} \quad \begin{cases} Ax=b. \\ Bx\le d. \end{cases} \end{aligned} \tag{7}$$

Where in $x=(x_1.x_2.\ldots.x_n)^T\in R^n$،$c\in R^n$ ، $A\in R^{m\times n}$, rank(A)=m،(0<m<n), $B\in R^{m\times n}$ ،$b\in R^m$ and $d\in R^p$ Is. The following neural network model can be used to obtain the optimal solution to problem (8) as follows. Zadeh [10]

$$\begin{cases} \frac{dx}{dt} = -(I-P)[c + B^T(y+Bx-d)^+] - Q(Ax-b). \\ \frac{dy}{dt} = -\frac{1}{2}y + \frac{1}{2}(y+Bx-d)^+. \end{cases} \quad (8)$$

Where in

$P=A^T(AA^T)^{-1}A$, $Q=A^T(AA^T)^{-1}$,

$(y+Bx-d)^+ = ([y+Bx-d]_1^+ . [y+Bx-d]_2^+ . \dots . [y+Bx-d]_m^+)$.

$[y+Bx-d]_k^+ = \max\{(y+Bx-d)_k, 0\}$, k=1,2, … ,m.

**Lemma1:** $x^*$ is the optimal solution (7) if and only if $y^* \geq 0$ exists so that $(x^*. y^*)^T$ holds in the following condition[27].

$(I-P)[c+B^TY^*]+Q(Ax^*-b)=0$,

$(y^*+Bx^*-d)^+-y^*=0$

Using Lemma1, it is easy to see that $x^*$ is the optimal solution to problem (7) if and only if it exists $y^* \geq 0$ so that $(x^*. y^*)^T$ is the equilibrium point of the neural network presented in (7). Therefore, when the neural network converges to the equilibrium point, the trajectory x (t) converges to the optimal solution of problem (7) [27].

## 4.2 Mean and Possible Variance of Fuzzy Numbers

Assuming that $\tilde{A}$ are a fuzzy number with α-cut $=[a_1(\alpha). a_2(\alpha)]. \alpha \in [0.1]\ A_\alpha$, then:

$$M_*(\tilde{A}) = 2\int_0^1 \alpha a_1(\alpha)\, d\alpha = \frac{\int_0^1 \alpha a_1(\alpha)\, d\alpha}{\int_0^1 \alpha\, d\alpha} = \frac{\int_0^1 Pos[\tilde{A} \leq a_1(\alpha)] a_1(\alpha) d\alpha}{\int_0^1 Pos[\tilde{A} \leq a_1(\alpha)] d\alpha} = \frac{\int_0^1 Pos[\tilde{A} \leq a_1(\alpha)] \times \min A_\alpha d\alpha}{\int_0^1 Pos[\tilde{A} \leq a_1(\alpha)] d\alpha}, \quad (9)$$

Where Pos indicates the possibility, i.e:

$Pos[\tilde{A} \leq a_1(\alpha)] = \pi((-\infty. a_1(\alpha)]) = \sup_{u \leq a_1(\alpha)} \tilde{A}(u) = \alpha.$

(Because $\tilde{A}$ is continuous) Therefore $M_*(\tilde{A})$ is called the low optimal average $\tilde{A}$ .

Similarly, $M^*(\tilde{A})$ defines the above average probability value as follows.

$$M^*(\tilde{A}) = 2\int_0^1 \alpha a_2(\alpha)\, d\alpha = \frac{\int_0^1 \alpha a_2(\alpha)\, d\alpha}{\int_0^1 \alpha\, d\alpha} = \frac{\int_0^1 Pos[\tilde{A} \geq a_2(\alpha)] a_2(\alpha) d\alpha}{\int_0^1 Pos[\tilde{A} \geq a_2(\alpha)] d\alpha} = \frac{\int_0^1 Pos[\tilde{A} \geq a_2(\alpha)] \times \max A_\alpha d\alpha}{\int_0^1 Pos[\tilde{A} \geq a_2(\alpha)] d\alpha}, \quad (10)$$

Where they have used the following draw.

$Pos[\tilde{A} \geq a_2(\alpha)] = \pi([a_2(\alpha). \infty)) = \sup_{u \geq a_2(\alpha)} \tilde{A}(u) = \alpha.$

Therefore, the following symbol is introduced.

$$M(\tilde{A}) = [M_*(\tilde{A}). M^*(\tilde{A})] = \int_0^1 \alpha [a_1(\alpha). a_2(\alpha)]\, d\alpha, \quad (11)$$

That is, M $(\tilde{A})$ is a closed interval bounded by the mean and low probability mean values of $\tilde{A}$.

## 4.3 Variance of Fuzzy Numbers

The possible variance of $\tilde{A} \in f$ is introduced as follows.

$$Var(\tilde{A}) = \int_0^1 Pos[\tilde{A} \leq a_1(\alpha)] \left( \left[\frac{a_1(\alpha)+a_2(\alpha)}{2} - a_1(\alpha)\right]^2 \right) d\alpha + \int_0^1 Pos[\tilde{A} \geq a_2(\alpha)] \left( \left[\frac{a_1(\alpha)+a_2(\alpha)}{2} - a_2(\alpha)\right]^2 \right) d\alpha = \int_0^1 \alpha \left( \left[\frac{a_1(\alpha)+a_2(\alpha)}{2} - a_1(\alpha)\right]^2 + \left[\frac{a_1(\alpha)+a_2(\alpha)}{2} - a_2(\alpha)\right]^2 \right) d\alpha = \frac{1}{2}\int_0^1 \alpha \left( a_2(\alpha) + a_1(\alpha) \right) d\alpha. \quad (12)$$

Also, the standard deviation from $\tilde{A}$ is defined as follows.

$\sigma_{\tilde{A}} = \sqrt{Var(\tilde{A})}$,

The covariance between fuzzy numbers $\tilde{A}$ with α-cut $A_\alpha = [a_1(\alpha). a_2(\alpha)]$ and $\tilde{B}$ with α-cut

$B_\alpha = [b_1(\alpha).\, b_2(\alpha)]$ is defined as follows.

$$\text{Cov}(\tilde{A}.\tilde{B}) = \frac{1}{2}\int_0^1 \alpha\big(a_2(\alpha) - a_1(\alpha)\big)\big(b_2(\alpha) - b_1(\alpha)\big)\, d\alpha \quad (13)$$

Mean-variance model for stock portfolio selection

The Markowitz mean-variance model is formulated as follows, assuming the stock portfolio X=($x_1$, $x_2$, … , $x_n$)$^T$ and returns r=($r_1$, $r_2$, … , $r_n$)$^T$ and F=(1, 1, … , 1)$^T$.

$$E(r) = \bar{r}X, \quad D(x) = X^T VX, \quad (14)$$

Therefore, the mean-variance model of stock portfolio selection is as follows.

$$\text{Minimize} \quad X^T VX,$$
$$\text{Subject to} \begin{cases} \bar{r}X = \mu. \\ F^T X = 1. \\ X \geq 0\,. \end{cases} \quad (15)$$

If L and R are strictly descending functions, then the α-cut sets of $r_i$ are expressed as follows.

$$[r_i]_\alpha = [a_i - \alpha_i L^{-1}(\alpha).\, b_i + \beta_i R^{-1}(\alpha)]. \quad \forall\alpha \in [0.1]\,.\, i = 1.2.\dots.n. \quad (16)$$

Where α is interpreted as the degree of possibility of the i-th asset in the stock portfolio selection problem. According to (11), (12) and (13) the following values can be easily obtained.

$$M(r_i) = \int_0^1 \alpha[a_i - \alpha_i L^{-1}(\alpha) + b_i + \beta_i R^{-1}(\alpha)]\, d\alpha = \frac{a_i + b_i}{2} - \alpha_i E_L + \beta_i E_R. \quad (17)$$

$$\text{Var}(r_i) = \frac{1}{2}\int_0^1 \alpha[\beta_i R^{-1}(\alpha) + \alpha_i L^{-1}(\alpha) + b_i - a_i]^2\, d\alpha = \frac{1}{2}(\beta_i^2 F_{RR} + 2\alpha_i\beta_i F_{RL} + \alpha_i^2 F_{LL}) + (b_i - a_i)(\beta_i E_R + \alpha_i E_L) + \frac{1}{4}(b_i - a_i)^2, \quad (18)$$

And

$$\text{Cov}(r_i.r_j) = \frac{1}{2}\int_0^1 \alpha[\beta_i R^{-1}(\alpha) + \alpha_i L^{-1}(\alpha) + b_i - a_i][\beta_j R^{-1}(\alpha) + \alpha_j L^{-1}(\alpha) + b_j - a_j]\, d\alpha = \frac{1}{2}\big[\beta_i\beta_j F_{RR} + (\alpha_i\beta_j + \alpha_j\beta_i)F_{RL} + \alpha_i\alpha_j F_{LL} + (b_j - a_j)(\beta_i E_R + \alpha_i E_L) + (b_i - a_i) \times (\beta_j E_R + \alpha_j E_L)\big] + \frac{1}{4}(b_i - a_i)(b_j - a_j). \quad (19)$$

where in

$$E_R = \int_0^1 \alpha R^{-1}(\alpha) d\alpha\,.\, E_L = \int_0^1 \alpha L^{-1}(\alpha) d\alpha.$$
$$F_{RR} = \int_0^1 \alpha(R^{-1}(\alpha))^2 d\alpha\,.\, F_{LL} = \int_0^1 \alpha(L^{-1}(\alpha))^2 d\alpha.$$
$$F_{RL} = \int_0^1 \alpha R^{-1}(\alpha) L^{-1}(\alpha) d\alpha. \quad (20)$$

Then the mean value and variance of the possible returns associated with the X portfolio are as follows.

$$M(r) = \sum_{i=1}^n \left[\frac{1}{2}(a_i + b_i) + \beta_i E_R - \alpha_i E_L\right] x_i\,. \quad (21)$$

$$\text{Var}(r) = \frac{1}{2}F_{LL}[\textstyle\sum_{i=1}^n \alpha_i x_i]^2 + \frac{1}{2}F_{RR}[\sum_{i=1}^n \beta_i x_i]^2 + F_{RL}[\sum_{i=1}^n \alpha_i x_i][\sum_{i=1}^n \beta_i x_i] + [\sum_{i=1}^n (b_i - a_i)x_i][\sum_{i=1}^n (\beta_i E_R + \alpha_i E_L)x_i] + \frac{1}{4}[\sum_{i=1}^n (b_i - a_i)x_i]^2. \quad (22)$$

Similar to the Markowitz average-variance method for stock portfolio selection, the average probability value is the return on investment and the probability variance is called the investment risk. Mean and probability variance of the model (15) can be replaced by mean and probability variance by [1, 32]. Therefore, the possible mean-variance model of the stock portfolio selection problem can be expressed as follows.

Minimize Var(r)=$\frac{1}{2}F_{LL}[\sum_{i=1}^{n}\alpha_i x_i]^2+\frac{1}{2}F_{RR}[\sum_{i=1}^{n}\beta_i x_i]^2+F_{RL}[\sum_{i=1}^{n}\alpha_i x_i][\sum_{i=1}^{n}\beta_i x_i]+[\sum_{i=1}^{n}(b_i-a_i)x_i][\sum_{i=1}^{n}(\beta_i E_R+\ \alpha_i E_L)x_i]+\frac{1}{4}[\sum_{i=1}^{n}(b_i-a_i)x_i]^2$,

Subject to

$$\begin{cases}\sum_{i=1}^{n}\left[\frac{1}{2}(a_i+b_i)+\beta_i E_R-\alpha_i E_L\right]x_i\ \geq \mu.\\ u_i\geq x_i\geq l_i.\, i=1.2.\dots.\ n.\\ \sum_{i=1}^{n}x_i=1.\end{cases} \tag{23}$$

Especially if $r_i\ .\ i=1.2.\dots.n$ L-R type fuzzy numbers are symmetric with center $a_i=b_i$ and $L^{-1}=R^{-1}$ then $F_{RR}=F_{LL}=F_{RL}\ .E_R=E_L$ , And also with the square root of the quadratic, the mean-variance possibility model (23) is defined as follows.

Minimize $\frac{\sqrt{2}}{2}\sqrt{F_{RR}}\sum_{i=1}^{n}(\alpha_i+\beta_i)\,x_i$.

Subject to

$$\begin{cases}\sum_{i=1}^{n}[a_i+E_R(\beta_i-\alpha_i)]x_i\ \geq \mu.\\ u_i\geq x_i\geq l_i.\, i=1.2.\dots.\ n.\\ \sum_{i=1}^{n}x_i=1.\end{cases} \tag{24}$$

## 5 Special modes of fuzzy numbers L-R

When the fuzzy number has a membership function (1), then the fuzzy number will be of type L-R and in cases where the fuzzy intervals L-R are linear functions and the differences AL, AR are negligible, then the resulting interval of the fuzzy number will be a trapezoid like Equation (2). And if the fuzzy numbers L-R are linear functions, such as the membership function of Equation (3), the resulting fuzzy number is called triangular. Triangular fuzzy numbers are trapezoids of special states of L-R fuzzy numbers, which are examined.

### 5.1 Triangular fuzzy number [4]

Suppose $r_i=(a_i.\gamma_i).\, i=1.2.\dots.n$ fuzzy numbers are triangles symmetric with center $a_i$ and width $>0\ \ \gamma_i$, An α-cut of $r_i$ can be represented by:

$[r_i]_\alpha$=$[a_i-(1-\alpha)\,\gamma_i, a_i+(1-\alpha)\,\gamma_i]$ $\quad \forall\ \alpha\in[0,1]$ , i=1, … , n.
(25)

To be shown. From (11), (12) and (13) the following relations are easily obtained.

$\overline{M}(r_i)$= $a_i$. i=1, …, n.
(26)

Var($r_i$) =$2\int_0^1\alpha(\gamma_i-\alpha\gamma_i)^2\,d\alpha$=$\frac{{\gamma_i}^2}{6}$, i=1, … , n,
(27)

Cov($r_i.r_j$)= $2\int_0^1\alpha((\gamma_i-\alpha\gamma_i)(\gamma_j-\alpha\gamma_j))\,d\alpha$=$\frac{\gamma_i\gamma_j}{6}$, i,j=1, … , n,
(28)

So the average value and the possible variance of the returns related to the stock portfolio ($x_1$, $x_2$, … , $x_n$) are obtained as follows[4].

$\overline{\mathrm{M}}(\mathrm{r_i})= \overline{M}(\sum_{i=1}^{n} r_i\, x_i)= \sum_{i=1}^{n} \overline{M}(r_i)\, x_i=\sum_{i=1}^{n} a_i\, x_i.$

(29)

$\mathrm{Var}(r)=\frac{(\sum_{i=1}^{n} \gamma_i x_i)^2}{6}.$

(30)

Therefore, the mean-variance model of the possibility of selecting a stock portfolio with a root of risk for first place is expressed as follows [4].

Minimize $\quad \mathrm{Var(r)}=\sum_{i=1}^{n} \gamma_i x_i.$

(31)

$$\text{Subject to} \quad \begin{cases} \sum_{i=1}^{n} a_i x_i \ \geq \mu. \\ u_i \geq x_i \geq l_i.\, i = 1.2. \dots.\ n. \\ \sum_{i=1}^{n} x_i = 1. \end{cases}$$

## 5.2 Trapezoidal fuzzy number

In fuzzy numbers, the mean-variance of the stock portfolio model is expressed as follows [4].

Maximize $\quad \bar{r}^T \mathrm{X},$

(32)

$$\text{Subject to} \quad \begin{cases} \sqrt{X^T V X} \ \leq \sigma \\ F^T X \ \leq 1. \\ U \geq X \geq L. \end{cases}$$

Suppose $r_j = (a_j.\, b_j.\, \gamma_i.\, \beta_i).\, j = 1.2. \dots.\, n$; n fuzzy numbers are trapezoidal. An α-cut of $r_j$ is expressed as follows[4].

$[r_j]_\alpha=[a_j - (1-\alpha)\,\gamma_j, b_j + (1-\alpha)\,\beta_j] \qquad \forall\, \alpha \in [0,1]$ .

(33)

Using the definitions of low and high mean averages and the variance of fuzzy numbers can be easily shown [4].

$M_*(r_j)=a_j - \frac{\gamma_j}{3}, \quad M^*(r_j)=b_j + \frac{\beta_j}{3},$

(34)

$$Var_*(r_j) = \frac{1}{18}\gamma_j^2. \qquad Var^*(r_j) = \frac{1}{18}\beta_j^2.$$

$$Cov_*(r_i.r_j) = 2\int_0^1 \alpha[\,(1-\alpha)\gamma_i - \frac{\gamma_i}{3}][(1-\alpha)\gamma_j - \frac{\gamma_j}{3}]d\alpha = \frac{1}{18}\gamma_i\gamma_j$$

$$Cov^*(r_i.r_j) = 2\int_0^1 \alpha[\,(1-\alpha)\beta_i - \frac{\beta_i}{3}][(1-\alpha)\beta_j - \frac{\beta_j}{3}]d\alpha = \frac{1}{18}\beta_i\beta_j.$$

The average and variance of low and high probability of return on assets with portfolio $(x_1, x_2, \dots, x_n)$ are as follows, respectively[4].

$M_*(r)=\sum_{i=1}^{n} M_*\,(x_i r_i) = \sum_{i=1}^{n}(\,a_i - \frac{\gamma_j}{3})x_i\,,$

(35)

$M^*(r)=\sum_{i=1}^{n} M^*(x_i r_i) = \sum_{i=1}^{n}(\,b_j + \frac{\beta_j}{3}\,)\,x_i,$

$Var_*(r) = \sum_{i=1}^{n} x_i^2 \frac{1}{18}\gamma_i^2 =$
$\frac{1}{18}(\sum_{i=1}^{n} \gamma_i x_i)^2.$

$Var^*(r_j) = \sum_{i=1}^{n} x_i^2 \frac{1}{18}\beta_i^2 = \frac{1}{18}(\sum_{i=1}^{n}\beta_i x_i)^2.$

(36)

For all $x_i \geq 0. i = 1.2. \ldots . n.$

The average and low probability standard deviation model of stock portfolio selection can be formulated as follows [4].

Maximize $\sum_{i=1}^{n}( a_i - \frac{\gamma_i}{3})x_i$

(37)

Subject to $\begin{cases} \frac{\sqrt{2}}{6}(\sum_{i=1}^{n}\gamma_i x_i) \leq \sigma \\ \sum_{i=1}^{n} x_i = 1. \\ u_i \geq x_i \geq l_i. i = 1.2. \ldots . n. \end{cases}$

Maximize $\sum_{i=1}^{n}( b_i - \frac{\beta_i}{3})x_i,$

(38)

Subject to $\begin{cases} \frac{\sqrt{2}}{6}(\sum_{i=1}^{n}\beta_i x_i) \leq \sigma. \\ \sum_{i=1}^{n} x_i = 1. \\ u_i \geq x_i \geq l_i. i = 1.2. \ldots . n. \\ 1 \geq u_i \geq x_i \geq l_i \geq 0. \sum_{i=1}^{n} l_i < 1. i = 1.2. \ldots . n. \end{cases}$

From the definition of efficient stock portfolio, the concepts of optimal low and high potential stock portfolio are defined as follow [4].

**Definition 7.** The optimal solution (37) is called a low potential optimization stock portfolio.

**Definition 8.** The optimal solution (38) is called an optimal potential stock portfolio.

Model (32) in linear space becomes linear programming (37) and (38) which is much better than model (32). In particular, if $r_i = (a_i. b_i. \gamma_i). i = 1.2. \ldots . n$ the fuzzy numbers are symmetrical trapezoids, which $\gamma_i = \beta_i$ then the constraints of the model of the mean standard deviation of low probability (37) are equal to the constraints of the model of the mean of the standard deviation of high probability (38). If $r_i = [a_i. b_i]. i = 1.2. \ldots . n.$ n are the interval of fuzzy numbers, then $\gamma_i = \beta_i. i = 1.2. \ldots . n.$، then the model (37) means the mean standard deviation of the low possible as follows.

Maximize $\sum_{i=1}^{n} a_i x_i,$

(39)

Subject to $\begin{cases} \sum_{i=1}^{n} x_i = 1. \\ u_i \geq x_i \geq l_i. i = 1.2. \ldots . n. \end{cases}$

And model (38) the average standard deviation of the above is as follows.

Maximize $\sum_{i=1}^{n} b_i x_i,$

(40)

Subject to $\begin{cases} \sum_{i=1}^{n} x_i = 1. \\ u_i \geq x_i \geq l_i. i = 1.2. \ldots . n. \end{cases}$

The optimal solution for model (39) is only the low optimal potential portfolio with maximum return, where the average low potential standard deviation is based on the potential weight of the minimum α-cut in asset return. And a so-called investment is pessimistic. Similarly, the optimal solution for model (40) is only the optimal high-yield portfolio with maximum returns, the average deviation of the high probability criterion is made on average based on the probability weight of the maximum α-cut in the return on assets, and the so-called investment is of the optimistic type. In these examples, neural networks are, also, used to obtain the equilibrium point to find the optimal possible portfolio.

The optimal solution of the model is exactly where the graph is drawn that has a velocity line of zero. That is, where a straight line is obtained creating the optimal stock portfolio. We introduced the two-objective model in (5) and then the single-objective models for fuzzy triangular, trapezoidal and L-R fuzzy numbers.

## 6 Research Findings

For the expressed types of fuzzy numbers, a numerical example is given. Sample data were selected from the monthly report during one year of Tehran Stock Exchange stocks, which is a random selection of stocks and their returns and risk values are examined. Therefore, the multi-objective model (5) becomes the final model of fuzzy numbers L-R and trapezoidal and triangular in relations (31) and (32), (37) and (38), (39) and (40) that We will solve numerical examples through them.

### 6.1 Fuzzy number L-R

The first example includes five shares of S*Mobarakeh Steel, Shazand Petr, Siman Fars Noe, Chadormalu, and Piazar Agro from Iran Stock Exchange and the second example includes five shares of Loghman Pharm, Maskan Invest, E.Kh Shargh, Azarab Ind-R and Palayesh Tehran. Is.

**Table 2**: Average and Risk of Both Groups of Stocks on a Monthly Basis

| *The first group* | | | | | |
|---|---|---|---|---|---|
| take stock | S*Mobarakeh Steel | Shazand Petr | Siman Fars Noe | Chadormalu | Piazar Agro |
| Monthly returns | 0.016 | 0.04 | 0.07 | 0.06 | 0.03 |
| Monthly risk | 0.15 | 0.17 | 0.16 | 0.2 | 0.21 |
| The second group | | | | | |
| take stock | Loghman Pharm | Maskan Invest | E.Kh Shargh | Azarab Ind-R | Palayesh Tehran |
| Monthly returns | 0.015 | 0.021 | 0.07 | 0.03 | 0.031 |
| Monthly risk | 0.17 | 0.13 | 0.17 | 0.22 | 0.22 |

Thus, the mean mean-variance model (24) for the first group is obtained as follows.

$$\text{Minimize } \frac{\sqrt{2}}{2}(0/21x_1 + 0/15x_2 + 0/17x_3 + 0/16x_4 + 0/2x_5. \tag{41}$$

$$\text{Subject to } \begin{cases} 0/016x_1 + 0/04x_2 + 0/07x_3 + 0/06x_4 + 0/03x_5 \geq \mu. \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1. \\ u_i \geq x_i \geq l_i. i = 1.2.3.4.5. \end{cases}$$

Also, the mean-variance model of possibility (24) for the second group is as follows.

$$\text{Minimize } \frac{\sqrt{2}}{2}(0/17x_1 + 0/13x_2 + 0/17x_3 + 0/22x_4 + 0/22x_5). \tag{42}$$

$$\text{Subject to } \begin{cases} 0/015x_1 + 0/021x_2 + 0/07x_3 + 0/03x_4 + 0/031x_5 \geq \mu. \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1. \\ u_i \geq x_i \geq l_i. i = 1.2.3.4.5. \end{cases}$$

How do investors choose the value of μ? Investors must first determine the maximum and minimum expected returns μ, ie:

$\mu_{max}$= $max\{0/25x_1 + 0/22x_2 + 0/2x_3 + 0/15x_4 + 0/05x_5\}$

$\mu_{min}$= $min\{0/25x_1 + 0/22x_2 + 0/2x_3 + 0/15x_4 + 0/05x_5\}$

With the limits of $x_i \geq 0. i = 1.2.3.4.5$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 1$ then investors choose the appropriate μ value from $\mu_{min} = 0/112$ to $\mu_{max} = 0/212$ the solution to problem (41) and (42) is not possible as long as $\mu > 0.112$.

It is calculated with the expected return values $\mu = 0/112 . 0/12 . 0/14. \ 0/16 . 0/18 . 0/2 . 0/212$ the relevant computational results are mentioned in Model (7). Fig.5 shows the neural network modeling through the MATLAB program. Problems (41) and (42) are modeled by considering the required value of μ in MATLAB and where x = 0 means the neural network has reached the equilibrium point of the optimal stock portfolio which are given in Table 3. The same method is used for the rest of the fuzzy numbers.

**Table 3.** Some of the Optimal Potential Portfolios of the First and Second Groups

| *The first group* | | | | | | | |
|---|---|---|---|---|---|---|---|
| μ | 0/112 | 0/12 | 0/14 | 0/16 | 0/18 | 0/2 | 0/212 |
| *Minimize* $\sum_{i=1}^{5} \sqrt{F_{RR}} (\alpha_i + \beta_i) x_i$ | 0/0919 | 0/0942 | 0/1001 | 0/1059 | 0/1123 | 0/1206 | 0/1316 |
| $x_1$ | 0/01 | 0/1 | 0/1 | 0/1 | 0/1 | 0/1 | 0/1 |
| $x_2$ | 0/23 | 0/24 | 0/24 | 0/2 | 0/23 | 0/2 | 0/25 |
| $x_3$ | 0/27 | 0/28 | 0/25 | 0/3 | 0/3 | 0/32 | 0/31 |
| $x_4$ | 0/25 | 0/24 | 0/25 | 0/23 | 0/2 | 0/21 | 0/24 |
| $x_5$ | 0/18 | 0/14 | 0/15 | 0/17 | 0/16 | 0/16 | 0/1 |
| *The second group* | | | | | | | |
| μ | 0/112 | 0/12 | 0/14 | 0/16 | 0/18 | 0/2 | 0/212 |
| *Minimize* $\sum_{i=1}^{5} \sqrt{F_{RR}} (\alpha_i + \beta_i) x_i$ | 0/0919 | 0/0942 | 0/1001 | 0/1059 | 0/1123 | 0/1206 | 0/1316 |
| $x_1$ | 0/2 | 0/23 | 0/22 | 0/21 | 0/21 | 0/12 | 0/1 |
| $x_2$ | 0/52 | 0/31 | 0/33 | 0/23 | 0/23 | 0/49 | 0/49 |
| $x_3$ | 0 | 0/05 | 0 | 0/15 | 0/15 | 0 | 0 |
| $x_4$ | 0/14 | 0/18 | 0/25 | 0/18 | 0/19 | 0/24 | 0/26 |
| $x_5$ | 0/14 | 0/23 | 0/2 | 0/23 | 0/22 | 0/14 | 0/15 |
| *Optimal stock portfolio* | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

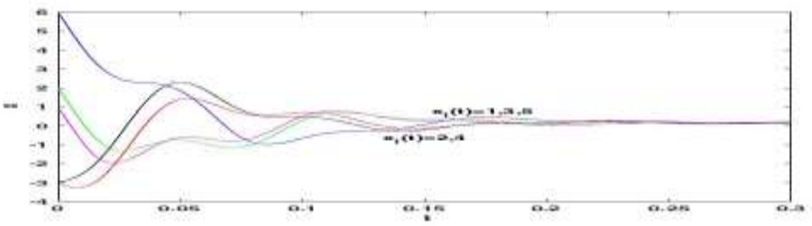


**Fig.5:** Figure 5. L-R type fuzzy number neural network model

The optimal solution of the first group (41) as follows $X^* = (0.1, 0.2, 0.3, 0.3, 0.1)^T$ and the optimal solution of the second group (42) as $X^* = (0.1, 0.2, 0.1, 0.4, 0.2)^T$ . According to Table 3, it can be seen that in the first group of example 5 shares, the higher the amount of investment in Siman Fars Noe stocks, the higher the return on investment. And the higher the investment in the shares of Piazar Agro, the lower the possible return. Also, for the second group, the higher the amount of investment in the shares of Azarab Ind-R, the higher the return on investment. And, the higher the investment in housing stock, the lower the return.

## 6.2 Triangular fuzzy Number

In addition to considering the previous assumptions of both stock groups, consider the lower and upper bounds of the total investment budget allocated to the i-th asset as follows.

$i = 1,2,3,4,5. \ \ l_i = 0.1, u_1 = 0.4 \ , u_2 = 0.4 \ , u_3 = 0.5 \ . u_4 = 0.6 \ . u_5 = 0.7 \ .$

Thus, the mean-variance possibility model (31) for the first group is formulated as follows.

Minimize $\quad 0/021x_1 + 0/15x_2 + 0/17x_3 + 0/16x_4 + 0/2x_5,$

Subject to

$$\begin{cases} 0/016x_1 + 0/04x_2 + 0/07x_3 + 0/06x_4 + 0/03x_5 \geq \mu \\ 0/1 \leq x_1 \leq 0/4. \\ 0/1 \leq x_2 \leq 0/4. \\ 0/1 \leq x_3 \leq 0/5. \\ 0/1 \leq x_4 \leq 0/6. \\ 0/1 \leq x_5 \leq 0/7. \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1. \end{cases} \tag{43}$$

For the other five shares with the same assumptions as before, Model (4) is also expressed as follows.

Minimize $\quad 0/017x_1 + 0/13x_2 + 0/17x_3 + 0/22x_4 + 0/22x_5$

Subject to

$$\begin{cases} 0/015x_1 + 0/021x_2 + 0/07x_3 + 0/03x_4 + 0/031x_5 \geq \mu \\ 0/1 \leq x_1 \leq 0/4. \\ 0/1 \leq x_2 \leq 0/4. \\ 0/1 \leq x_3 \leq 0/5. \\ 0/1 \leq x_4 \leq 0/6. \\ 0/1 \leq x_5 \leq 0/7. \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1. \end{cases} \tag{44}$$

The optimal stock portfolio is possible for $\mu = 0/10.0/15.0/17.0/19$ for the first and second groups of stocks in Table 4. The optimal solution of the first group (43) as follows $X^*$=(0.21, 0.18, 0.13, 0.18, 0.3)$^T$ And the optimal solution of the second group (44) as $X^*$=(0.2, 0.05, 0.15, 0.22, 0.38)$^T$ . According to Table 4, it can be seen that in the first group of Example 5 shares, the higher the amount of investment in S*Mobarakeh Steel stocks, the higher the return on investment. And the higher the investment in Shazand Petr, the lower the return will be. Also, for the second group, the higher the amount of investment in Tehran oil refining stocks, the higher the return on investment. And the higher the investment in Maskan Invest stocks, the lower the return on investment.

**Table 4:** Some optimal stock portfolios

| *The first group* | | | | |
|---|---|---|---|---|
| μ | 0/1 | 0/15 | 0/17 | 0/19 |
| $x_1$ | 0/19 | 0/2 | 0/28 | 0/34 |
| $x_2$ | 0/29 | 0/29 | 0/23 | 0/19 |
| $x_3$ | 0/19 | 0/19 | 0/17 | 0/17 |
| $x_4$ | 0/14 | 0/18 | 0/18 | 0/1 |
| $x_5$ | 0/19 | 0/14 | 0/14 | 0/2 |
| *The second group* | | | | |
| μ | 0/1 | 0/15 | 0/17 | 0/19 |
| $x_1$ | 0/28 | 0/25 | 0/24 | 0/23 |
| $x_2$ | 0/14 | 0/17 | 0/14 | 0/15 |
| $x_3$ | 0 | 0 | 0/02 | 0/03 |
| $x_4$ | 0/26 | 0/25 | 0/26 | 0/23 |
| $x_5$ | 0/32 | 0/33 | 0/34 | 0/36 |

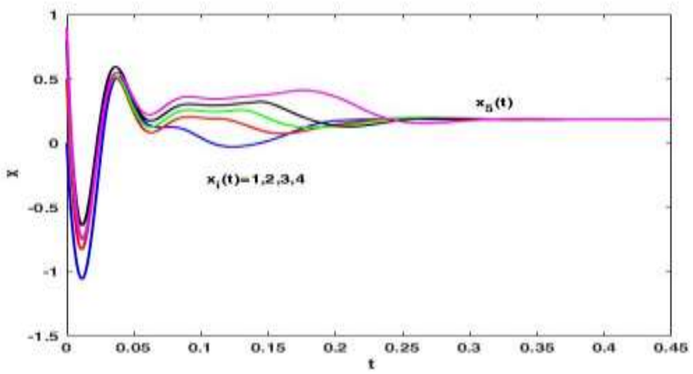



**Fig. 6:** Figure 6. Triangular fuzzy number neural network model

### 6.3 Trapezoidal fuzzy number

Considering the previous assumptions and the bottom and top edges of $x_i$ which is as follows. $(u_1.u_2.u_3.u_4.u_5)$ = (0/4, 0/4, 0/4, 0/4, 0/5, 0/6), $(l_1.\ l_2.\ l_3.\ l_4.\ l_5)$ = (0/1, 0/1, 0/1, 0/1, 0/1),

Using models (39) and (40), the optimal and low probability stock portfolios are shown in Tables 5 and 6, respectively. In this way, the optimal possibility portfolio model of the first group is formulated as follows.

**Table 5**. Some optimal portfolios are possible below the of the first and second groups

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *The first group* | | | | | | | | | | | | |
| Maximize M*(r) (%) | 5/19 | 6/02 | 6/95 | 7/73 | 7/89 | 8/82 | 9/75 | 10/6 | 11/18 | 11/73 | 12/26 | 12/79 |
| σ(%) | 1/23 | 1/4 | 1/6 | 1/62 | 1/8 | 2/0 | 2/2 | 2/4 | 2/6 | 2/8 | 3 | 3/2 |
| $x_1$ | 0/04 | 0/03 | 0/02 | 0/025 | 0/15 | 0/18 | 0/28 | 0/28 | 0/37 | 0/26 | 0/24 | 0/23 |
| $x_2$ | 0/2 | 0/23 | 0/17 | 0/17 | 0/14 | 0/14 | 0/14 | 0/16 | 0/18 | 0/15 | 0/16 | 0/18 |
| $x_3$ | 0/05 | 0/08 | 0/08 | 0/08 | 0/015 | 0/173 | 0 | 0/02 | 0/03 | 0/02 | 0/04 | 0/01 |
| $x_4$ | 0/01 | 0/032 | 0/032 | 0/01 | 0/18 | 0/17 | 0/22 | 0/22 | 0/23 | 0/22 | 0/23 | 0/24 |
| $x_5$ | 0/2 | 0/06 | 0/32 | 0/04 | 0/27 | 0/18 | 0/3 | 0/32 | 0/29 | 0/35 | 0/33 | 0/35 |
| $\Sigma\, x_i$ | 0/5 | 0/422 | 0334 | 0/325 | 0/755 | 0/843 | 0/94 | 1 | 1 | 1 | 1 | 1 |
| *The second group* | | | | | | | | | | | | |
| Maximize M*(r) (%) | 5/19 | 6/02 | 6/95 | 7/73 | 7/89 | 8/82 | 9/75 | 10/6 | 11/18 | 11/73 | 12/26 | 12/79 |
| σ(%) | 1/23 | 1/4 | 1/6 | 1/62 | 1/8 | 2 | 2/2 | 2/4 | 2/6 | 2/8 | 3 | 3/2 |
| $x_1$ | 0/07 | 0/021 | 0/02 | 0/04 | 0/1 | 0/163 | 0/11 | 0/28 | 0/21 | 0/09 | 0/08 | 0/09 |
| $x_2$ | 0/16 | 0/22 | 0/164 | 0/11 | 0/14 | 0/16 | 0/22 | 0/15 | 0/17 | 0/29 | 0/25 | 0/3 |
| $x_3$ | 0/01 | 0/081 | 0/08 | 0/09 | 0 | 0/04 | 0/1 | 0/01 | 0/06 | 0/18 | 0/13 | 0/16 |
| $x_4$ | 0/01 | 0 | 0/02 | 0/05 | 0/165 | 0/19 | 0/16 | 0/23 | 0/22 | 0/16 | 0/17 | 0/13 |
| $x_5$ | 0/25 | 0/1 | 0/05 | 0/035 | 0/35 | 0/29 | 0/35 | 0/33 | 0/34 | 0/28 | 0/37 | 0/32 |
| $\Sigma\, x_i$ | 0/5 | 0/422 | 0/334 | /0325 | 0/755 | 0/843 | 0/94 | 1 | 1 | 1 | 1 | 1 |

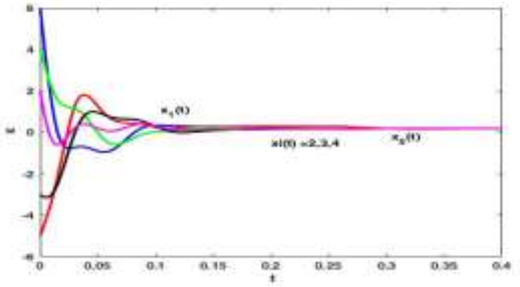


**Fig. 7:** Figure 7: Trapezoidal fuzzy number neural network model of the first group

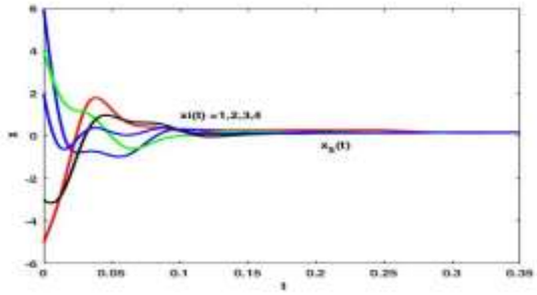


**Fig 8:** Figure 8: Trapezoidal fuzzy number neural network model of the second group

**Table 6**. Some high potential optimization portfolios of the first and second groups

| *The first group* | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Maximize M*(r) (%) | 9/21 | 10/32 | 11/56 | 12/79 | 14/03 | 15/3 | 16/49 | 17/65 | 18/77 | 19/72 | 20/66 | 21/6 |
| σ(%) | 1/62 | 1/8 | 2 | 2/2 | 2/4 | 2/6 | 2/8 | 3 | 3/2 | 3/4 | 3/6 | 3/8 |
| $x_1$ | 0/03 | 0/04 | 0/08 | 0/12 | 0/05 | 0/178 | 0/11 | 0/16 | 0/15 | 0/16 | 0/19 | 0/21 |
| $x_2$ | 0/21 | 0/24 | 0/18 | 0/16 | 0/24 | 0/16 | 0/215 | 0/198 | 0/21 | 0/28 | 0/18 | 0/17 |
| $x_3$ | 0/05 | 0/11 | 0/03 | 0/022 | 0/119 | 0/04 | 0/08 | 0/07 | 0/09 | 0/17 | 0/08 | 0/05 |
| $x_4$ | 0/01 | 0/042 | 0/07 | 0/12 | 0/08 | 0/18 | 0/18 | 0/22 | 0/23 | 0/15 | 0/23 | 0/23 |
| $x_5$ | 0/2 | 0/13 | 0/271 | 0/278 | 0/28 | 0/28 | 0/32 | 0/31 | 0/32 | 0/24 | 0/33 | 0/34 |
| $\Sigma x_i$ | 0/5 | 0/562 | 0/631 | 0/7 | 0/769 | 0/838 | 0/905 | 0/958 | 1 | 1 | 1 | 1 |
| *The second group* | | | | | | | | | | | | |
| Maximize M*(r) (%) | 9/21 | 10/32 | 11/56 | 12/79 | 14/03 | 15/3 | 16/49 | 17/65 | 18/77 | 19/72 | 20/66 | 21/6 |
| σ(%) | 1/62 | 1/8 | 2 | 2/2 | 2/4 | 2/6 | 2/8 | 3 | 3/2 | 3/4 | 3/6 | 3/8 |
| $x_1$ | 0/07 | 0/09 | 0/11 | 0/12 | 0/179 | 0/15 | 0/2 | 0/16 | 0/24 | 0/28 | 0/29 | 0/27 |
| $x_2$ | 0/18 | 0/172 | 0/16 | 0/17 | 0/14 | 0/2 | 0/16 | 0/19 | 0/18 | 0/16 | 0/17 | 0/17 |
| $x_3$ | 0/03 | 0/01 | 0/021 | 0/026 | 0 | 0/04 | 0/035 | 0/078 | 0/03 | 0/02 | 0 | 0/02 |
| $x_4$ | 0/01 | 0/04 | 0/07 | 0/098 | 0/18 | 0/158 | 0/19 | 0/19 | 0/26 | 0/23 | 0/24 | 0/24 |
| $x_5$ | 0/21 | 0/25 | 0/27 | 0/286 | 0/27 | 0/29 | 0/32 | 0/34 | 0/29 | 0/31 | 0/3 | 0/3 |
| $\Sigma x_i$ | 0/5 | 0/562 | 0/631 | 0/7 | 0/769 | 0/838 | 0/905 | 0/958 | 1 | 1 | 1 | 1 |

$$\text{Maximize} \quad 0/016x_1 + 0/04x_2 + 0/07x_3 + 0/06x_4 + 0/03x_5, \tag{45}$$

$$\text{Subject to} \quad \begin{cases} 0/1 \le x_1.\ x_2.\ x_3.\ x_4.\ x_5 \le 0/4. \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1, \quad i = 1, 2, \dots,\ n \end{cases}$$

And the optimal possibility portfolio model of the second group is obtained as follows.

$$\text{Maximize} \quad 0/015x_1 + 0/021x_2 + 0/07x_3 + 0/03x_4 + 0/031x_5, \tag{46}$$

$$\text{Subject to} \quad \begin{cases} 0/1 \le x_1,\ x_2,\ x_3,\ x_4,\ x_5 \le 0/4. \\ x_1 + x_2 + x_3 + x_4 + x_5 = 1, \quad i = 1, 2, \dots,\ n \end{cases}$$

The optimal possible low solution (45) is $X^{*}= (0.28, 0.16, 0.02, 0.22, 0.32)^{T}$ and the optimal high possible solution is $X^{*}= (0.1, 0.4, 0.02, 0.18, 0.3)^{T}$. According to Table 6, it can be seen that the low possibility portfolio for the first group of example 5 shares, the more investment in the shares of Piazar Agro and the less investment in the shares of Siman Fars Noe will have the highest possible return. Also, for a high probability portfolio, the first group will have the highest possible return on investment in Shazand Petr and less investment in Siman Fars Noe shares.

The optimal possible low solution (46) is $X^{*}= (0.04, 0.26, 0.12, 0.18, 0.4)^{T}$ and the optimal high possible solution is $X^{*}= (0.24, 0.17, 0.01, 0.2, 0.38)^{T}$.

According to Table 6, it can be seen that the low possibility portfolio for the second group of example 5 shares, the lower the amount of investment in Loghman Pharm shares and the higher the amount of investment in Maskan Invest shares, will have the highest possible return. Also, for the high possibility portfolio of the second group, the less the investment in the shares of E.Kh Shargh and Palayesh Tehran and the more the investment in the shares of Azarab Ind-R, the higher the possible return.

## 7 Conclusions

Examining the optimal share portfolio through fuzzy number certainty theory in models such as Markowitz, fractional model, qualitative model, Min-Max model, we concluded that none of them is as suitable as the mean-variance because the mean-variance model has several possible possibilities. Follows simultaneously and provides maximum return and minimum risk. The issue of the stock portfolio is always one of the most attractive issues in the field of finance which deals with the selection of stocks and the allocation of weight per share.

In this study, fuzzy numbers are used as a powerful tool to describe the ambiguity of indeterminate space. Also, the model of mean and variance patterns of possibility for stock portfolio based on the mean and low and high variance of the possibility of fuzzy numbers was presented. Simple linear programming models for optimal stock portfolio selection were also obtained, assuming that the assets of trapezoidal fuzzy numbers, triangular fuzzy numbers, and fuzzy numbers are of the L-R type. The mean, variance, and covariance, respectively, replace the mean, variance, and covariance of Markowitz, which were based on the probability distribution. This model's parameters can examine experts' opinions better than Markowitz's mean-variance method. In addition, it can better explain the rate of return and risk of stock portfolio selection.

For example, we used the Tehran Stock Exchange by examining the trapezoidal, triangular, and LR fuzzy number models. It is proved that the use of the model provides an optimal stock portfolio that has less risk and higher returns for the desired investment. Therefore, with the simple linear programming models obtained for the optimal stock portfolio for the domestic stock exchange, it is observed that the risk of the stock model is less than the probable model, which gives better results than the Markowitz mean-variance model.

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