Adv. Math. Fin. App., 2022, **7**(3), P. 527-533

Advances in Mathematical Finance & Applications

www.amfa.iau-arak.ac.ir
Print ISSN: 2538-5569
Online ISSN: 2645-4610
Doi:10.22034/AMFA.2021.1891263.1363

Research Paper

A Kurganov-Tadmor Numerical Method for Option Pricing under the Constant Elasticity of Variance Model

Sakineh Ghiasi[a], Nouredin Parandin[b, *]

[a]*Department of Mathematics, Payame Noor University, (PNU), P.O.BPX,1935-3697 Tehran, Iran.*
[b]*Department of Mathematics, Kermanshah Branch, Islamic Azad University, Kermanshah, Iran,*

ARTICLE INFO

Article history:
Received 2020-01-20
Accepted 2021-04-25

Keywords:
Kurganov-Tadmor method
Option price
Constant elasticity of variance

ABSTRACT

The primary goal of option pricing theory is to calculate the probability that an option will be exercised at expiration and assign a dollar value to it. Options pricing theory also derives various risk factors or sensitivities based on those inputs, since market conditions are constantly changing, these factors provide traders with a means of determining how sensitive a specific trade is to price fluctuations, volatility fluctuations, and the passage of time. In this study, we derive a new exact solution for pricing European options using Kurganov-Tadmor when the underlying process follows the constant elasticity of variance model. This method was successfully applied to nonlinear convection-diffusion equations by Kurganov and Tadmor. Also, we provide computational results showing the performance of the method for European option pricing problems. The results showed that the proposed method is convenient to calculate the option price for $K = 3, \beta = \frac{-3}{4}, and\ N = 200$.

1 Introduction

The problem of option pricing plays a very important role both in modern financial theory and in practice. In the year 1973, the Black-Scholes model theory was published and has gradually become the most widely used mathematical model for option pricing problems [2]. Forecasting prices and pricing on goods and services had been an important issue and much research has been done in this area [8-13]. Moreover, the capacity to price risks and devise ideal venture methodologies within the sight of an unsure "arbitrary" market is the foundation of the current money hypothesis. The authors [10] initially thought about the easiest such issue of an alleged "European call option " at first explained by Black and Scholes utilizing Ito stochastic math for a market model by a Log-Brownian stochastic cycle. A basic and incredible formalism was introduced which permitted them to sum up the investigation to an enormous class of stochastic cycles, for example, ARCH, jump, or Lévy measures. They additionally address the instance of related Gaussian cycles, which is demonstrated to be a decent portrayal of three distinctive market indices (MATIF, CAC40, FTSE100). Their fundamental outcome is the presentation of the idea of an optimal system in the feeling of (functional) minimization of the risk regarding the

Corresponding author Tel.: +989188875373
E-mail address: *n-parandin@yahoo.com*

portfolio. If the risk might be made to evaporate for specific constant uncorrelated quasi-Gaussian ' stochastic cycles (counting Black and Scholes model), this is not true anymore for more general stochastic cycles. The estimation of the leftover risk is acquired and proposes the idea of risk amended option costs. Within the sight of exceptionally huge deviations, for example, in Lévy measures, new rules for balanced fixing of the options costs are examined. The researchers likewise applied their technique to different kinds of options, 'Asian', 'American', and discussed new possibilities ('double-decker' ...). But, we know the constant volatility assumption does not seem right in real cases [3, 17]. One of the most popular stochastic volatility models, which is widely used in practice, is the constant elasticity of variance (CEV) model. It was first proposed in [7, 5] as an alternative to the Black-Scholes model of underlying asset price movements.
Many types of research have been done in this area such as [14]. The paper Concentrated on the constant elasticity of variance (CEV) model for examining the optimal investment technique before and after retirement in a characterized commitment annuity plan where advantages are paid under the type of annuities; annuities should be ensured during a specific fixed timeframe. Utilizing Legendre transform, double hypothesis, and variable change strategy, we infer the unequivocal answers for the power and exponential utility functions in two unique periods (before and after retirement). Every arrangement contains an altered factor that mirrors an investor's decision to fence the unpredictability risk. To research the impact of the adjusted factor on the optimal methodology, we break down the property of the altered factor. The outcomes show that the dynamic conduct of the altered factor for the power utility fundamentally relies upon the time and the investor's risk aversion coefficient, though it just relies upon the time in the outstanding case [5]. It is notable that the old-style Black–Scholes model with steady instability doesn't completely mirror the stochastic idea of financial markets. Thus, there is a requirement for more sensible models that better reflect random market developments like the European option price. In this paper, we investigate a Kurganov-Tadmor (KT) numerical method for valuing European options on assets evolving under the CEV process. This scheme, recently introduced in [16]. An outline of this paper is as follows. In section 2 present a Kurganov-Tadmor method for the options pricing followed by a detailed description of the formula for the value of a European put option through the CEV model. Section 4 applies the method to value the CEV options.

2 Structures

2.1 constant elasticity of variance (CEV)

We model a European put option in a regime-switching economy where the drift rate and volatility are subject to random shifts between two states. The asset-price dynamics in a regime-switching economy have been described previously in [4-23]. However, for completeness, we start this section by briefly describing the constant elasticity of variance (CEV) model as well. The CEV model describes the following relationship between the volatility and price,

$$dS_t = (r - d)S_t dt + \sigma S_t^{\beta+1} dW_t \tag{1}$$

Where r is the risk-free interest rate, d is the dividend yield, σ is a volatility parameter and W_t is the Wiener process. The parameter β is known as the elasticity of the local volatility function. If $\beta > 0$ ($\beta < 0$) volatility and price are inversely (positively) related. Since the financial market often exhibits volatility skews of negative slope, the situation $\beta > 0$ is rarely considered in research. When $\beta = 0$ prices are lognormally distributed and the variance of returns is constant, as is assumed in the

Black-Scholes model. In the case $\beta = -\frac{1}{2}$ we get the Cox-Ingersoll-Ross (CIR) model [18]. Under the CEV model (1) a European put with strike K has price V(S, t) which is the solution of the initial-boundary value problem

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^{2\beta+2}\frac{\partial^2 V}{\partial S^2} + (r-d)S\frac{\partial V}{\partial S} - rV, \qquad S \geq 0,\ \ 0 \leq t \leq T, \tag{2}$$

With initial condition $\mathrm{V(S,t)} = \max(\mathrm{K} - \mathrm{S}, 0)$and boundary conditions given by $\mathrm{V}(0,\mathrm{t}) = \mathrm{K}\, e^{-rt}$ and $\mathrm{V(S,t)} \to 0$ as $S \to \infty$.

Analytical solution of European options under the CEV model (1) derived in [9-19]. Numerically solution of European option pricing under CEV (2) studied in the many piece of literature option. For instance, Wong and Zhao [22] proposed a Crank-Nicolson method for pricing European and American options under the CEV model. Thakoor et al [20] presented a fourth-order numerical method for the CEV European option pricing problem. Zhang et al [24] investigated a multi quadric quasi-interpolations method for pricing European and American options on assets evolving by the CEV process. In order to achieve the option pricing, in this range, including that with The Kurganov-Tadmor method, we have proposed this scheme, which allows combining properties of CEV and Kurganov-Tadmor algorithms. Therefore, we decided to pay attention to Kurganov-Tadmor's scheme which on one hand is already explicitly implemented in option pricing and has been repeatedly tested, and on the other hand, it is simple enough to be used as part of the scheme.

2.2 The Kurganov-Tadmor method for option pricing under the CEV model

Kuganov and Tadmor [21] introduced a high -resolution method for solution of nonlinear convection-diffusion equations

$$\frac{\partial}{\partial t}\, u(t,x) + \frac{\partial}{\partial x}\, f\big(u(t,x)\big) = \frac{\partial}{\partial x} Q\big(u(t,x), u_x(t,x)\big). \tag{3}$$

Now, the Black-Scholes equation (2) is discretized according to the Kurganov-Tadmor method. We want to transform the Black-Scholes PDE to the general form

$$\frac{\partial}{\partial t}\, u(t,x) + \frac{\partial}{\partial x}\, \mathcal{F}(u) = \frac{\partial}{\partial x} \mathcal{Q}(u, u_x) + S(t,x,u), \tag{4}$$

Where $\mathcal{S}$is the source term and

$$\mathcal{F}(s,v) = (2(\beta+1)\beta\sigma^2 sv - r)sv, \tag{5}$$

$$\mathcal{Q}(s,v) = \Big((\beta+1)\sigma^2 s^{2\beta} v - r\Big)\, s, \tag{6}$$

$$S(v) = ((2\beta+1)(\beta+1)\sigma^2 - 2r)v. \tag{7}$$

For uniform spatial grid points $\{s_i\}_{i=0}^{N}$for $N \in \mathbb{N}$ width Δs and mid-cells $\left[s_{i-\frac{1}{2}}, s_{i+\frac{1}{2}}\right]$ where $s_{i\pm\frac{1}{2}} = s_i \pm \frac{\Delta s}{2}$ the semi-discrete method for the Black-Scholes equation (2) takes the form

$$\frac{dV_i}{dt} = -\frac{1}{\Delta s}\left(H_{i+\frac{1}{2}}(t) - H_{i-\frac{1}{2}}(t)\right) + \frac{1}{\Delta s}\left(P_{i+\frac{1}{2}}(t) - P_{i-\frac{1}{2}}(t)\right) + S(v), \tag{8}$$

With

$$H_{i+\frac{1}{2}}(t) = \frac{1}{2}\left(\mathcal{F}\left(s_{i+\frac{1}{2}}, V^{+}_{i+\frac{1}{2}}\right), \mathcal{F}\left(s_{i+\frac{1}{2}}, V^{-}_{i+\frac{1}{2}}\right)\right) + \frac{a_{i+\frac{1}{2}}(t)}{2}\left(V^{+}_{i+\frac{1}{2}} - V^{-}_{i+\frac{1}{2}}\right), \tag{9}$$

$$H_{i-\frac{1}{2}}(t) = \frac{1}{2}\left(\mathcal{F}\left(s_{i-\frac{1}{2}}, V^{+}_{i-\frac{1}{2}}\right), \mathcal{F}\left(s_{i-\frac{1}{2}}, V^{-}_{i-\frac{1}{2}}\right)\right) + \frac{a_{i+\frac{1}{2}}(t)}{2}\left(V^{+}_{i-\frac{1}{2}} - V^{-}_{i-\frac{1}{2}}\right), \tag{10}$$

Where $a_{i+\frac{1}{2}}(t) = \left|\mathcal{F}\left(s_{i+\frac{1}{2}}\right)\right|$ and

$$V^{+}_{i+\frac{1}{2}}(t) = V_{i+1}(t) - \frac{1}{2}\Delta s\ (V_s)_{i+1}(t), \tag{11}$$

$$V^{-}_{i+\frac{1}{2}}(t) = V_{i}(t) + \frac{1}{2}\Delta s\ (V_s)_{i}(t), \tag{12}$$

$$V^{+}_{i-\frac{1}{2}}(t) = V_{i}(t) - \frac{1}{2}\Delta s\ (V_s)_{i}(t), \tag{13}$$

$$V^{-}_{i-\frac{1}{2}}(t) = V_{i-1}(t) + \frac{1}{2}\Delta s\ (V_s)_{i-1}(t). \tag{14}$$

The derivative $(V_s)_i(t)$ is approximated with a min mod limiter such that the semi-discrete method fulfills the Total variation diminishing condition [21]. The generalized min mod limiter is defined as

$$(V_s)_i(t) = \text{min mod}\left(\theta\frac{V_i(t) - V_{i-1}(t)}{\Delta s}, \frac{V_{i+1}(t) - V_{i-1}(t)}{2\Delta s}, \theta\frac{V_{i+1}(t) - V_i(t)}{\Delta s}\right) \tag{15}$$

Where $0 \le \theta \le 1$ and the min mod function is defined as

$$\text{min mod}\ (x_1, x_2, \dots) = \begin{cases} min_i(x_i), & if\ x_i > 0, \forall i, \\ max_i(x_i), & if\ x_i < 0, \forall i \\ 0, & otherwise. \end{cases} \tag{16}$$

Also, P denotes an approximation of $Q$and obtained using basic forward and backward differencing,

$$P_{i+\frac{1}{2}}(t) = Q\ \left(\frac{V_{i+1} - V_{i-1}}{2\Delta s}\right), \tag{17}$$

Where the second-order approximation for the derivative is used. At the boundaries, we used the following second-order formulae to approximate the derivatives of Q

$$\frac{\partial}{\partial s}\ v(t, s_{min}) = \frac{-3V_0(t) + 4V_1(t) - V_2(t)}{2\Delta s} + \boldsymbol{\mathcal{O}}(\Delta s^2), \tag{18}$$

$$\frac{\partial}{\partial s}\ v(t, s_{max}) = \frac{V_{N-1}(t) - 4V_N(t) + 3V_{N+1}(t)}{2\Delta s} + \boldsymbol{\mathcal{O}}(\Delta s^2), \tag{19}$$

Where $V_0(t)$ represents the approximation $at\ s_{min}$ $and\ V_{N+1}(t)$ $at\ s_{max}$.

3 Conclusions

3.1 Results and Discussion

In all our examples, we price a European put option with a current stock price of s $\epsilon[0, 10]$, a maturity of $\mathrm{T} = 0.5$ year, an interest rate of $\mathrm{r} = 0.05$, $\mathrm{d} = 0$ dividend yield, and at-the-money volatility of $\sigma =$

25%. For numerical simulation, we use the min mod limiter (15) with $\theta = 1.5$. Table 1 shows numerical results for the different values of β and $N = 200$. Computed errors (the difference between exact and computed prices) are shown for Kurganov-Tadmor method.

Table 1: Evaluation of European puts option for the CEV process with $\beta = \frac{-1}{4}, \beta = \frac{-1}{2}$ and $\beta = \frac{-3}{4}$ at $s\epsilon[0,10]$

Evaluation of European puts option for the CEV process				
N	K	$\beta = \frac{-1}{4}$	$\beta = \frac{-1}{2}$	$\beta = \frac{-3}{4}$
200	3	0.0688	0.0029	0.0189
200	5	0.0224	0.0606	0.0705
200	10	0.1572	0.0154	01163

Fig. 1 shows that the proposed method is convenient to calculate the option price for $K = 3, \beta = \frac{-3}{4}, and\ N = 200$. In Figure 2 the solution of the 3D problem is plotted by $= 3, \beta = \frac{-3}{4}$ and $N = 200$
.

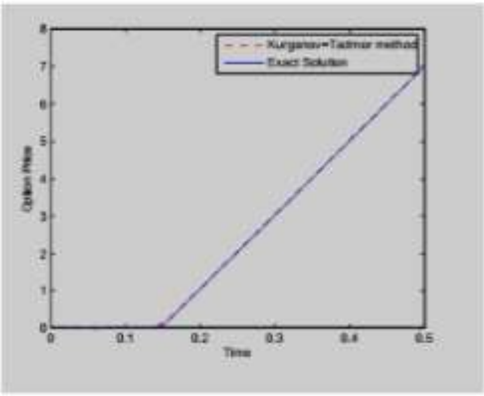

Fig. 1: Comparison between the exact solution and the numerical solution of a European call with the KT method

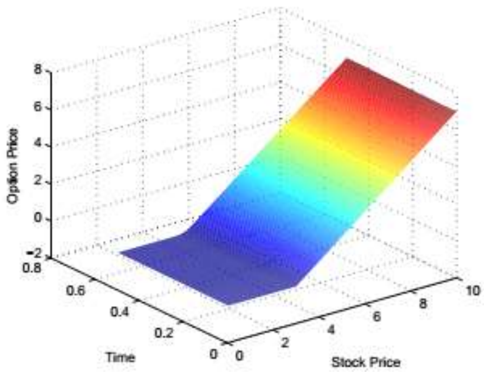

Fig. 2: Solution surface for European call option.

In this paper we decided to pay attention to Kurganov-Tadmor's scheme which on one hand is already explicitly implemented in option pricing and has been repeatedly tested, and on the other hand it is simple enough to be used as part of the scheme. Another important feature of this scheme is independence of approximating expressions. Thus, there is no need to use solution expansion on characteristics for calculation of option price for $K = 3, \beta \frac{-3}{4}, and\ N = 200$. In this study, a new exact closed-form solution for European options using Kurganov-Tadmor's scheme and CEV method is derived. The newly-obtained formula involves only the calculation of a single integral with a real integrand and can thus be very easily calculated, if numerical values are needed. Such a result is achieved through the analytic inversion of the Fourier transform. In this work, we showed the pricing of the European option when the underlying stock follows the constant elasticity of variance (CEV) process. For this aim, applying Its Lemma we obtain the PDE governing the CEV option value (2). For the valuation of the European option, we have proposed Kurganov-Tadmor numerical method. In table 1 and Figs.1, 2 to illustrate significant price changes with concerning parameters of option prices.

3.2 Suggestions for future work

An interesting extension of this paper would be to consider the case where the risk-free interest rate is state dependent. Extending the model such that r1≠r2 would yield similar partial differential equations (PDEs) under the restrictive assumption that the extra source of risk can be diversified (see [23] or [21]). However, if this assumption were not to be made, the pricing problem would lead to a non-trivial extension of the current model, which may further complicate the solution procedure. We are currently exploring this case.

References

[1] Bahirai. A., Alipour, M., Sadiq, R., *Option pricing in the presence of operational risk page*, Advances in Mathematical finance and applications, 2020, **5**(4), P.437-448. Doi:10.22034/amfa.2020.1897687.1409

[2] Black-Scholes F., Black- Scholes, M., *The pricing of options and corporate liabilities*, J. Polit. Econ, 1973, **81**, P. 637-659.

[3] Blattberg, R. C., Gonedes, N. J., *A comparison of the stable and student distributions as stochastic models for stock prices*, Journal of Business, 1974, **47**, P. 244-280. Doi: 10.1086/295634

[4] C.D. Fuh, R.H. Wang, J.C. Cheng, *Option pricing in a Black–Scholes model with Markov switching*, Working Paper, 2002.

[5] Cox, J.C., and Ross, S.A., *The valuation of options for alternative stochastic processes*, Journal of Financial Economics, 1976, **4**, P.145-166. Doi:10.1016/0304-405X(76)90023-4

[6] Cox, J.C., Ingersoll, J.E., and Ross, S.A., *A theory of the term structure of interest rates*, In Theory of valuation, 2005, P.129-164. Doi:10.2307/1911242.

[7] Cox, J., *Notes on option pricing I: Constant elasticity of variance diffusions*, Unpublished note Stanford University Graduate School of Business. 1975 Sep. Doi: 10.4236/jmf.2011.13009

[8] Davoodi Kasbi, A., Dadashi, I., Azinfar, K., *Stock price analysis using machine learning method* (Non-sensory-parametric backup regression algorithm in linear and nonlinear mode), 2021, **6**(2), P.285-301. Doi: 10.22034/amfa.2019.1869838.1232

[9] Emanuel, D., Mac, J., *Beth Further results on the constant elasticity of variance call option pricing model*, J. Finance. Quant. Anal, 1982, **17**(4), P.533-554.

[10] Esmaili, B., Souri, A., Mirlohi, S, M., Higher *moment portfolio Optimization with unequal weights based on generalized Capital Asset pricing model with independent and identically asymmetric power Distribution.* Advances in Mathematical Finance and Applications, 2021, **6**(2), P.263-283. Doi:10.22034/amfa.2020.1909590.1484

[11] Buffington, R. J., *Elliott American options with regime switching*, International Journal of Theoretical and Applied Finance, 2002, **5**, P. 497-514. Doi:10.1142/S0219024902001523.

[12] Buffington, R. J., *Elliott Regime switching and European Options Stochastic Theory and Control*, Lecture Notes in Control and Information Sciences,2002, **280**, P.73-82. Doi.10.1007/3-540-48022-6_5

[13] Bouchaud, J.P., Sornette, D., *The BlackScholes option pricing problem in mathematical finance: generalization and extensions for a large class of stochastic processes*, Journal of Physique I ,1994, **4**, P.863-881

[14] Jianwei Gao, *Optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model*, Mathematics and Economics,2009, **45**(1), P.9-18. Doi: 10.1016/j.insmatheco.2009.02.006.

[15] Khalegi, P., Aghai, M., Rezai, F., *Salience theory and pricing stock of corporates in Tehran stock exchange,* Advances in mathematical finance and applications, 2018, **3**(4), P.1-16. Doi:10.22034/AMFA.2019.577140.1120

[16] Kurganov, A., Tadmor, E., *New High-Resolution Central Schemes for Nonlinear Conservation Laws and Convection--Diffusion Equations*, Journal of Computational Physics, 2000, **160**, P.241-282. Doi:10.1006/jcph.2000.6459.

[17] Mac Beth, J. D., Merville, L. J., *An empirical examination of the Black-Scholes call option pricing model,* Journal of Finance, 1979, **34**, P. 1173-1186. Doi.1540-6261. 1979.tb00063.x

[18] Rezai, N., Elmi, Z., *Behaviour finance model and behaviour biases in stock price forecasting*, Advances in mathematical finance and applications, 2018, **3**(4), P.67-82. Doi:10.22034/AMFA.2019.576127.1118.

[19] Schroder, M., *Computing the constant elasticity of variance option pricing formula*, J. Finance, 1989, **44**(1), P.211-219. Doi:10.1111/j.1540-6261. 1989.tb02414.x

[20] Thakoor, N., Tangman, D., Bhuruth, M., *A new fourth-order numerical scheme for option pricing under the CEV model*, Appl. Math. Lett, 2013, **26**(1), P.160-164. Doi: 10.1016/j.aml.2012.08.002.

[21] Naik V., *Option valuation and hedging strategies with jumps in the volatility of asset returns*, Journal of Finance,1993, **48**(5), P. 1969-1984. Doi: 10.1111/j.1540-6261. 1993.tb05137.

[22] Wong, H., Zhao, J., *An artificial boundary method for American option pricing under the CEV model*, SIAM J. Numer. Anal, 2008, **46**(4), P. 2183-2209. Doi.101137/060671541.

[23] Guo, X., Information *and option pricings*, Journal of Quantitative Finance, 2001, **1**, P.38-44. Doi:10.1080/713665550.

[24] Zhang, S., Yang, H., and Yang, Y., *A multi quadric quasi-interpolations method for CEV option pricing model*, Journal of Computational and Applied Mathematics, 2019, **347**, P.1-11.

[25] Zomorodian, GH., Barzegar, L., Kazemi,S., Poortalebi, *Effect of oil price volatility and petroleum Bloomberg index on stock market returns of Tehran stock exchange using EGRACH model*, Advances in Mathematical Finance and Applications, 2016, **1**(2), P.69-84. Doi:10.22034/amfa.2016.527821