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Perfect 3-colorings of the cubical graph and its applications in agriculture management

Hamed Karami[1], Morteza Karami[2], and Ali Hosseinzadeh[2]*

[1] *Department of Agriculture Extension and Education, College of Agricultural and Development, University of Tehran, Tehran, Iran*
[2] *Department of Agriculture, Islamshahr Branch, Islamic Azad University, Islamshahr, Iran*

Abstract

In this paper, we enumerate the parameter matrices of all perfect 3-colorings of the cubical graph Q_3. We also present some basic results for a cubic connected graph of order8. It has a wide range of applications in agriculture production and banking systems.

Key Words: Perfect Coloring; Equitable Partition; Cubical graph

Introduction

The concept of a perfect m-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). The concept has been referred to as "equitable partition" in the literature too (see [8]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Therefore, some effort has been made on enumerating the parameter matrices of some Johnson graphs, including $J(6,3)$, $J(7,3)$, $J(8,3)$, $J(8,4)$, and $J(v,3)$ (v odd) (see [2, 3, 7]).

Fon-Der-Flass enumerated the parameter matrices (perfect 2-colorings) of n-dimensional hypercube Q_n for $n < 24$. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix (see [4, 5, 6]).

Some results for the existence of perfect 2-colorings in generalized Petersen graphs are presented ([1]).

In this article, we enumerate the parameter matrices of all perfect 3-colorings of Q_3.

Definition and Concepts

In this section, we give some basic definitions and concepts.

Definition .1 The Hypercube graph Q_n has vertices, respectively, edges given by

$V(Q_n) = \{x = (x_1, \cdots, x_n): x_i \in \mathbb{Z}_2\}$,
$E(Q_n) = \{xy: \text{x and y differinpreciselyonecoordinate}\}$.

The graph Q_3 is called the cubical graph. It is clear that the cubical graph is a 3-regular connected graph.

Definition .2 For a graph G and an integer m, a mapping $T: V(G) \to \{1, \cdots, m\}$ is called a perfect m-coloring with matrix $A = (a_{ij})_{i,j\in\{1,\cdots,m\}}$, if it is subjective, and for all i, j, for every vertex of color i, the number of its neighbors of color j is equal to a_{ij}. The matrix A is called the parameter matrix of a perfect coloring. In the case $m = 3$, we call the first color white, the second color black, and the third color red. In this paper, we generally show a parameter matrix by

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Remark .3 in this paper, we consider all perfect 3-colorings, up to renaming the colors; i.e, we identify the perfect 3-coloring with the matrices

$$\begin{bmatrix} a & c & b \\ g & i & h \\ d & f & e \end{bmatrix}, \begin{bmatrix} e & d & f \\ b & a & c \\ h & g & i \end{bmatrix},$$
$$\begin{bmatrix} e & f & d \\ h & i & g \\ b & c & a \end{bmatrix}, \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}, \begin{bmatrix} i & g & h \\ c & a & b \\ f & d & e \end{bmatrix},$$

Obtained by switching the colors with the original coloring.

Perfect 3-colorings of a cubic connected graph of order 8

This section presents some results concerning the necessary conditions for the existence of perfect 3-colorings of a cubic connected graph of order 8 with a given parameter matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

The simplest necessary condition for the existence of perfect 3-colorings of a cubic connected graph with the matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is

$$a + b + c = d + e + f = g + h + i = 3.$$

Also, it is clear that we cannot have $b = c = 0$, $d = f = 0$, or $g = h = 0$, since the graph is connected. In addition, we have $b = 0$, $c = 0$, $f = 0$ if $d = 0$, $g = 0$, $h = 0$, respectively.

The next proposition gives a formula for calculating the number of white, black and red vertices in a perfect 3-coloring.

Proposition .1 Let T be a perfect 3-coloring of a graph G with the matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

1. If $b, c, f \neq 0$, then
$$|W| = \frac{|V(G)|}{\frac{b}{d}+1+\frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b}+1+\frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f}+1+\frac{g}{c}}.$$

2. If $b = 0$, then
$$|W| = \frac{|V(G)|}{\frac{c}{g}+1+\frac{ch}{fg}}, |B| = \frac{|V(G)|}{\frac{f}{h}+1+\frac{fg}{ch}}, |R| = \frac{|V(G)|}{\frac{h}{f}+1+\frac{g}{c}}.$$

3. If $c = 0$, then

$$|W| = \frac{|V(G)|}{\frac{b}{d}+1+\frac{bf}{dh}}, |B| = \frac{|V(G)|}{\frac{d}{b}+1+\frac{f}{h}}, |R| = \frac{|V(G)|}{\frac{h}{f}+1+\frac{dh}{bf}}.$$

4. If $f = 0$, then

$$|W| = \frac{|V(G)|}{\frac{b}{d}+1+\frac{c}{g}}, |B| = \frac{|V(G)|}{\frac{d}{b}+1+\frac{cd}{bg}}, |R| = \frac{|V(G)|}{\frac{g}{c}+1+\frac{bg}{cd}}.$$

Proof. (1): Consider the 3-partite graph obtained by removing the edges uv such that u and v are the same color. By counting the number of edges between parts, we can easily obtain $|W|b = |B|d$, $|W|c = |R|g$, and $|B|f = |R|h$. Now, we can conclude the desired result from $|W| + |B| + |R| = |V(G)|$.
The proof of (2), (3), (4) is similar to (1).
In this section, without any restrictions on generality, we assume $|W| \le |B| \le |R|$.
In the next Lemma, under the condition $|W| = 1$, we enumerate all matrices that can be a parameter matrix for a cubic connected graph.

Lemma .2 Let G be a cubic connected graph. If T is a perfect 3-coloring with the matrix A, and $|W| = 1$, then A should be one of the following matrices,

$$A_1 = \begin{bmatrix}0&3&0\\1&1&1\\0&1&2\end{bmatrix}, A_2 = \begin{bmatrix}0&3&0\\1&0&2\\0&1&2\end{bmatrix}, A_3 = \begin{bmatrix}0&3&0\\1&0&2\\0&2&1\end{bmatrix}, A_4 = \begin{bmatrix}0&0&3\\0&1&2\\1&1&1\end{bmatrix}, A_5 = \begin{bmatrix}0&0&3\\0&0&3\\1&1&1\end{bmatrix}, A_6 = \begin{bmatrix}0&0&3\\0&0&3\\1&2&0\end{bmatrix}, A_7 = \begin{bmatrix}0&1&2\\1&1&1\\1&1&1\end{bmatrix}, A_8 = \begin{bmatrix}0&2&1\\1&0&2\\1&1&1\end{bmatrix}, A_9 = \begin{bmatrix}0&1&2\\1&2&0\\1&0&2\end{bmatrix}, A_{10} = \begin{bmatrix}0&1&2\\1&0&2\\1&1&1\end{bmatrix}, A_{11} = \begin{bmatrix}0&1&2\\1&0&2\\1&2&0\end{bmatrix}.$$

Proof. Let $A = \begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$ be a parameter matrix with $|W| = 1$. Consider the white vertex. It is clear that none of its adjacent vertices are white; i.e, $a = 0$. Therefore, we have two cases below.

1. The adjacent vertices of the white vertex are the same color.

If they are black, then $b = 3$ and $c = 0$. From $c = 0$, we get $g = 0$. Also, since the graph is connected, $f, h \neq 0$. Hence, we obtain the following matrices:

$$\begin{bmatrix}0&3&0\\1&1&1\\0&1&2\end{bmatrix}, \begin{bmatrix}0&3&0\\1&0&2\\0&1&2\end{bmatrix}, \begin{bmatrix}0&3&0\\1&0&2\\0&2&1\end{bmatrix}, \begin{bmatrix}0&3&0\\1&1&1\\0&2&1\end{bmatrix}, \begin{bmatrix}0&3&0\\1&0&2\\0&3&0\end{bmatrix}, \begin{bmatrix}0&3&0\\1&1&1\\0&3&0\end{bmatrix}.$$

If the adjacent vertices of the white vertex are red, then $c = 3$, and $b = 0$. From $b = 0$, we get $d = 0$. Also, since the graph is connected, $f, h \neq 0$. Hence, we obtain the following matrices:

$$\begin{bmatrix}0&0&3\\0&1&2\\1&1&1\end{bmatrix}, \begin{bmatrix}0&0&3\\0&1&2\\1&2&0\end{bmatrix}, \begin{bmatrix}0&0&3\\0&2&1\\1&1&1\end{bmatrix}, \begin{bmatrix}0&0&3\\0&2&1\\1&2&0\end{bmatrix}, \begin{bmatrix}0&0&3\\0&0&3\\1&1&1\end{bmatrix}, \begin{bmatrix}0&0&3\\0&0&3\\1&2&0\end{bmatrix}.$$

Finally, by using Remark 2.3 and the fact that $|W| \le |B| \le |R|$, it is obvious that there are only six matrices in (1), as shown by A_1, A_2, A_3, A_4, A_5, and A_6.

2. The adjacent vertices of the white vertex are different colors.

It immediately gives that $b, c \neq 0$. Also, it can be seen that $d = g = 1$. An easy computation, as in (1), shows that there are only five matrices that can be a parameter matrix in this case, as shown byA_7, A_8, A_9, A_{10}, and A_{11}.

We now present two lemmas which can be used to reach our goal.

Lemma .3 Let G be a cubic connected graph of order8. If T is a perfect 3-coloring with the matrixA, and$|W| = |B| = 2, |R| = 4$, then A should be one of the following matrices

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

Proof. First, suppose that $b, c \neq 0$. As $|W| = 2$, by Proposition 3.1, it follows that $\frac{b}{d} + \frac{c}{g} = 3$. From $b + c \leq 3$, we have $b = 2, c = g = d = 1$, or $c = 2$, $b = g = d = 1$. If $b = 2$, $c = g = d = 1$, we get a contradiction of $|B| = 2$. If $c = 2$, $b = d = g = 1$, then we conclude from $|B| = 2$ and $|R| = 4$ that $h = 1$ and $f = 2$. Therefore, $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Second, suppose that $b = 0$ and, in consequence, $d = 0$. As $|R| = 4$, by Proposition 3.1, it follows that $\frac{g}{c} + \frac{h}{f} = 1$. Therefore, $c = f = 2$, $g = h = 1$, or $c = f = 3, h = 2$,
$g = 1$, or $c = f = 3, g = 2, h = 1$. If $c = f = 2$ and $g = h = 1$, then $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$. In the other two cases, we get a contradiction of $|B| = 2$.

Third, suppose that $c = 0$ and, in consequence, $g = 0$. As $|B| = 2$, by Proposition 3.1, it follows that $\frac{d}{b} + \frac{f}{h} = 3$. Therefore $d = 2$, $b = f = h = 1$, or $f = 2$, $b = h = d = 1$. If $d = 2$, $b = f = h = 1$, then we get a contradiction of $|R| = 4$. If $f = 2$, and $b = h = d = 1$, then $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

Finally, note that the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ is the same as the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ up to renaming the colors by Remark 2.3.

Lemma .4 Let G be a cubic connected graph of order 8. If T is a perfect 3-coloring with the matrix A, and $|W| = 2$, and $|B| = |R| = 3$, then A should be one of the following matrices

$$\begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Proof. First, suppose that $b, c \neq 0$. As $|W| = 2$, by Proposition 3.1, it follows that $\frac{b}{d} + \frac{c}{g} = 3$. From $b + c \leq 3$, we get $b = 2$, $c = g = d = 1$, or $c = 2$, $b = g = d = 1$. If $b = 2$, and $c = g = d = 1$, we get a contradiction of $|B| = 3$. If $c = 2$ and $b = d = g = 1$, then, from Proposition 3.1, we have $f = 2$, and $h = 3$, which is a contradiction of $g + h \leq 3$.

Second, suppose that $b = 0$ and, in consequence, $d = 0$. As $|R| = 3$, by Proposition 3.1, it follows that $\frac{g}{c} + \frac{h}{f} = \frac{5}{3}$. Therefore, $c = 3$, $g = 2$, $h = f = 1$, or $f = 3$, $h = 2$, $c = g = 1$. If $c = 3$, $g = 2$, and $h = f = 1$, then $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$. In the other case, we get a contradiction of $|W| = 2$.

Third, suppose that $c = 0$ and, in consequence, $g = 0$. As $|B| = 3$, by Proposition 3.1, it follows that $\frac{d}{b} + \frac{f}{h} = \frac{5}{3}$. Therefore, $h = 3$, $f = 2$, $b = d = 1$, or $b = 3$, $d = 2$, $f = h = 1$. If $h = 3$, $f = 2$, and $b = d = 1$, then we get a contradiction of $|W| = 2$. If $b = 3$, $d = 2$, and $f = h = 1$, then $A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Finally, note that the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ is the same as the matrix $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ up to renaming the colors, by Remark 2.3.

Perfect 3-colorings of the cubical graph

In this section, we enumerate the parameter matrices of all perfect 3-colorings of the cubical graph Q_3.

Theorem .1 There are no perfect 3-colorings with the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ for the cubical graph.

Proof. Contrary to our claim, suppose that T is a perfect 3-coloring with the matrix $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ for the cubical graph. without any restrictions on generality, suppose that $T((000)) = 2$. Therefore, again without any restrictions on generality, suppose that $T((100)) = T((001)) = 3$ and $T((010)) = 1$. From $T((100)) = T((001)) = 3$, we can easily see that $T((110)) = T((011)) = T((101)) = 3$ which is a contradiction of $T((010)) = 1$.

Theorem .2 There are no perfect 3-colorings with the matrices $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ for the cubical graph.

Proof. Contrary to our claim, suppose that T is a perfect 3-coloring with the matrix $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ for the cubical graph. Without any restrictions on generality, suppose that $T((000)) = 1$. It gives $T((100)) = T((001)) = T((010)) = 2$. Therefore, two of the vertices (110), (011) or (101) are white which is a contradiction.

Theorem .3 The cubical graph Q_3 has perfect 3-colorings only with the matrices $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$.

Proof. As it has been shown in section 3, only the matrices listed in Lemmas 3.2, 3.3, 3.4 can be parameter matrices. By using Proposition 3.1 and easy computation, it can be easily seen that there are no perfect 3-colorings with the matrices listed in Lemma 3.2 for the cubical graph. Hence, from Theorem 4.1 and Theorem 4.2, it suffices to show that there are perfect 3-colorings with the matrices $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$. Consider two mappings T_1 and T_2 as follows,

$T_1((001)) = T_1((011)) = 1, T_1((100)) = T_1((110)) = 2,$
$T_1((000)) = T_1((010)) = T_1((101)) = T_1((111)) = 3,$
and

$T_2((000)) = T_2((111)) = 1, T_2((010)) = T_2((101)) = 2,$

$T_2((100)) = T_2((001)) = T_2((110)) = T_2((011)) = 3.$

It is clear that T_1 and T_2 are perfect 3-colorings with the matrices $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$, respectively.

It is known that enumerating parameter matrices of the cubical graph has a wide range of applications in agriculture production and its development and banking systems.

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