Increasing the Fundamental Frequency of the Cantilever Rotating Beam by Placing the Intermediate Elastic Support with Minimum Stiffness at the Optimum Point Based on the Courant's Maximum–Minimum Theorem using Finite-Element Analysis Software

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Received: 10 December 2020, Revised: 26 April 2021, Accepted: 1 May 2021

Abstract: In this paper, the effect of the optimal position and minimum stiffness of the elastic middle support on increasing the fundamental frequency of a rotating cantilever beam is investigated based on the Courant's maximum–minimum theorem using ABAQUS finite element software. First, the software analysis results are compared with the numerical analysis results for a non-rotating cantilever beam to confirm the accuracy of the software model. Next, by placing the middle elastic support at the optimal point selected based on the Courant theorem, the minimum stiffness of the elastic intermediate support for the maximum fundamental frequency of the rotating console beam was obtained. The results of this study prove that the Courant's maximum–minimum theorem is completely valid for rotating cantilever beams and can be used to improve the vibrational behavior of rotating engineering components. Finally, the minimum diameter of damping wire for the turbomachine blade is calculated as a practical application of the minimum stiffness of the intermediate elastic support for the rotating beam.

Keywords: Blade, Damping Wire, Fundamental Frequency, Intermediate Elastic Support, Rotating Beam, Stiffness

How to cite this paper: Mehdi Asgarikia, Farshad Kakavand and Hasan Seidi "Increasing the Fundamental Frequency of the Cantilever Rotating Beam by Placing the Intermediate Elastic Support with Minimum Stiffness at the Optimum Point Based on the Courant's Maximum–Minimum Theorem using Finite-Element Analysis Software" , Int J of Advanced Design and Manufacturing Technology, Vol. 14/No. 3, 2021, pp. 65–73.
DOI: 10.30495/admt.2021.1917477.1232

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1 INTRODUCTION

The most important factor in the failure of the rotating industrial structures is fatigue due to structure vibrations. Different factors, such as forced harmonics, stimulating the dynamics of the rotating structure, cause vibrations with different oscillation ranges, which can cause a resonance phenomenon and, consequently, a failure and blunt damage. Therefore, it is necessary to improve and increase the fundamental frequency of the rotating structure to ensure that it is as far away from the resonance amplitude as possible. Since the most important factor in the studying vibration of a structure is its destruction due to the resonance phenomenon, maximizing the fundamental frequencies of the structure is a vital part of mechanical design. Since many industrial rotating elements can be considered as a rotating cantilever beam, increasing the fundamental frequency of a rotating cantilever beam through the elastic intermediate support can be an engineering solution to avoid the resonance phenomenon. However, optimal position and minimum stiffness of the intermediate support is very important for designers. Figure 1 shows a cantilever beam with intermediate elastic support.

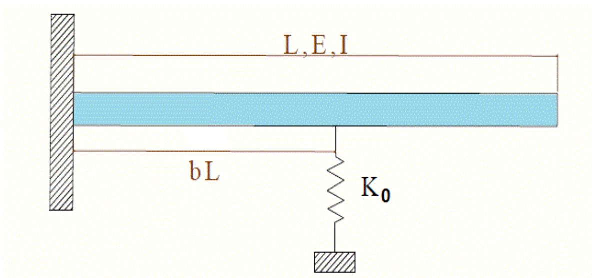


Fig. 1 Cantilever beam with intermediate elastic support.

Since many engineering structures are simulated by a beam with specific support conditions, useful studies have been performed to improve the dynamic behaviour of a beam with different boundary conditions and support conditions. Rayleigh [1] showed that adding elastic stiffness and/or kinetic constraints to the system (without changing the mass) does not reduce any of its fundamental frequencies. By extending Rayleigh's work, Courant [2] showed that the addition of n kinematical constraints to the system affects the Eigen frequencies of the system, as shown in the following inequality:

$$\omega_{i+n} \leq \mu_i \leq \omega_{i+n+1} \quad (1)$$

Where, μ_i is the eigenfrequency for ith mode of the restrained system and ω_i is the eigenfrequencies of mode i of the unrestrained system. Akesson and Olhoff [3] showed that to maximize the fundamental frequency of a cantilever beam, there is a certain minimum stiffness for the support added to the beam. In fact, increasing the stiffness of the intermediate support has no effect on the maximum fundamental frequency. Rao [4] developed the frequency equation of a beam with an intermediate support by continuous conditions at the support point. Albarracı´n et al. [5] calculated the influence of intermediate support when the end of the beam has elastic constraints. Wang et al. [6] obtained a new approach to the frequency analysis of a beam according to the position of simple support (or point support) using the discretization method. Additionally, they proposed a method to determine the optimal position of elastic supports based on frequency.

Rotary beams have a wide range of practical applications in industry, especially in the turbomachine and aerospace industries. Generally, a rotating beam is a beam rotating with a certain angular velocity about an axis perpendicular to its longitudinal axis. Modelling and analysis of dynamic and vibration behaviour of rotating elastic beams play an important role in the design and engineering applications such as turbine blades, compressor blades, helicopter blades, etc. Due to wide applications of rotating beams, vibration analysis of these beams is of great importance. Figure 2 Shows the schematic form of a rotating beam.

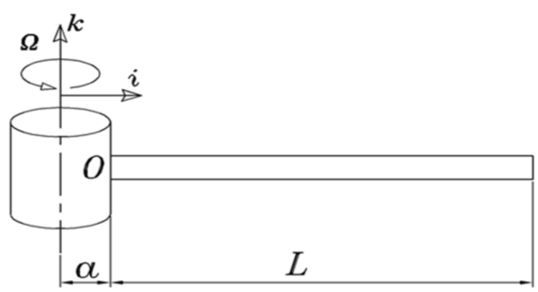


Fig. 2 Schematic form of a rotating beam.

In 2001, Lin et al. [7] examined the transverse and axial vibration of Timoshenko's beam. They considered the effect of the Carioles force to obtain the natural frequency of the system. They also derived the governing equations of motion using the D'alambert principle and virtual work and used power series to solve the equations. In 2013, Stoykov et al. [8] studied the nonlinear vibrations of a rectangular beam in three dimensions. They modelled the beam in the transverse vibration using Timoshenko's theory and in the rotational direction using the theory of St. and Nantes and extracted the equations of motion using the principle of virtual work. They also used the finite element method to solve the equations by considering impact and external harmonic forces. In 2006, Cheng et al. [9]

studied the transverse and longitudinal vibrations of a rotating beam. They extracted the governing equations using the Hamilton principle and then solved those using hypothetical modes. They showed that with an increase in rotational speed of the rotating beam, the natural bending frequency increases and its natural longitudinal frequency decreases. In 2006, Salarieh and Ghorashi [10] examined the transverse and torsional vibration of a rotating beam with a concentrated mass at its free end. In their study, they considered beam as Timoshenko and used Hamilton principle to derive equations, solved equations numerically, and finally compared the resulting answers with Euler-Bernoulli beam. In 2011, Ansari et al. [11] studied the transverse and torsional vibration of a rotating beam with a concentrated mass at its free end. They assumed the rotating beam as a Bernoulli-Euler beam and derived the governing equations from the Hamilton principle by taking into account the coupling effect and solved the governing equations using the hypothetical modes. In 2013, Bambill et al. [12] investigated the transverse vibration of non-uniform rotating beams. They derived the equations of motion of the rotating beam using Hamilton principle and hypothetical assumption of Timoshenko beam and then solved them using the finite-element method. In 2015, Tang et al. [13] studied the transverse vibration of a rotating beam. They considered an Euler-Bernoulli beam with a conical cross-section and used integral equations to calculate natural frequencies. In 2017, Chen et al. [14] examined the free transverse vibration of a Timoshenko rotating beam with a conical cross-section. In 2020, Ajinkya Baxy and Abhjit Sarkar [15] introduced the natural frequencies of a curved rotating cantilever beam based on a perturbation-based method. In fact, using both the Riley-Ritz method and the perturbation method, they formulated a new formulation for determining the natural frequencies of a curved rotating cantilever beam.

In the current paper, the minimum stiffness and the optimal position of the intermediate support were investigated to improve the vibration response of a rotating beam. The vibration analysis of rotating cantilever beams has been carried out by a large number of researchers. However, this is the first time that the maximum frequency increase of a rotating beam located on the intermediate elastic support at the optimal point has been investigated using the Courant's maximum–minimum theorem. The fundamental frequency of the rotating cantilever beam on intermediate elastic support is studied using the finite-element method by ABAQUS software. For this purpose, the first three fundamental frequencies of a non-rotating cantilever beam are computed using finite-element analysis software, and their results are compared with the finite-element analytical solution. Then, based on Courant's maximum–minimum theorem, the optimal position of intermediate elastic support is selected, the amount of increase in fundamental frequency is computed, the minimum stiffness of intermediate elastic support is determined. In order to verify the results obtained from software analysis, they are compared with the analytical results of studies conducted by Akesson and Olhoff [3] and Wang et al. [6]. Afterwards, the optimal position is chosen based on Courant's maximum–minimum theorem and minimum stiffness, and the increase in the fundamental frequency of a rotating beam is calculated by the finite-element analysis software. At the end, the minimum diameter of damping wire of the turbine blade is given as a practical example of a minimum stiffness of intermediate elastic support.

2 SOFTWARE SIMULATION AND COMPARISON OF RESULTS WITH FINITE-ELEMENT ANALYSIS SOLUTION

In this research, the geometric modelling is performed in ABAQUS software, which is one of the well-known software in the finite-element analysis. Since rotating beams are mostly applied in modelling rotating blades and slenderness ratio of these blades is not significantly high, Timoshenko theory is expected to be the proper theory to be applied in analysing these beams. In software analysis, for calculation of the fundamental frequency, a 1-meter length cantilever beam with a circular section and a slenderness ratio of L/R=20 is considered. Due to the uniformity of cross-section and constant physical properties along the beam, the beam element is used for problem analysis using the finite-element method. The type of loading used in the mechanical category is of rotational body force type so that it exerts the effects of centrifugal force on the geometric model. This is a great possibility in ABAQUS software that you can create the load of the rotating body force to define the loads caused by the rotation of the model. You can set the angular velocity or rotational acceleration. In any case, the load definition must include an axis of rotation, which is defined as follows:

-If you work in 3D space, you specify the location and direction of the axis by entering the global coordinates of the two points.

-If you work in 2D space, you specify the location of the axis by entering the coordinates of a point on the screen. The axis direction is always off-page.

-If you work in symmetrical axis space, the axis is always in the position and direction of the global positive Z axis.

-The beam materials are selected as structural steel and assumed completely uniform. In this analysis, temperature effects are assumed negligible. Physical properties of the beam under investigation are listed in "Table 1".

Table 1 Physical properties of the beam under investigation

E	207 Gpa
G	70 Gpa
A	0.0314 m^2
I_z	0.000078 m^4
ρ	7800 Kg/m^3
Ω	3000 rpm
δ	1

2.1. Optimum Position of Rigid Intermediate Support in Non-Rotating Cantilever Beam

According to Courant's maximum–minimum theorem [2], adding one extra rigid support at a distance:

x=b.L

From fixed support of cantilever beam (0<b<1) can increase structural frequency ω_i by an amount between ith and (i+1)th natural frequency. Schematic form of this statement can be observed in "Fig. 1" . Akesson and Olhoff [3] showed that the fundamental frequency of the cantilever beam increases to the second natural frequency of the no-support beam by placing additional support at the second modal node of the beam. The results of their studies for the optimal position of intermediate support are shown in "Fig. 3" .

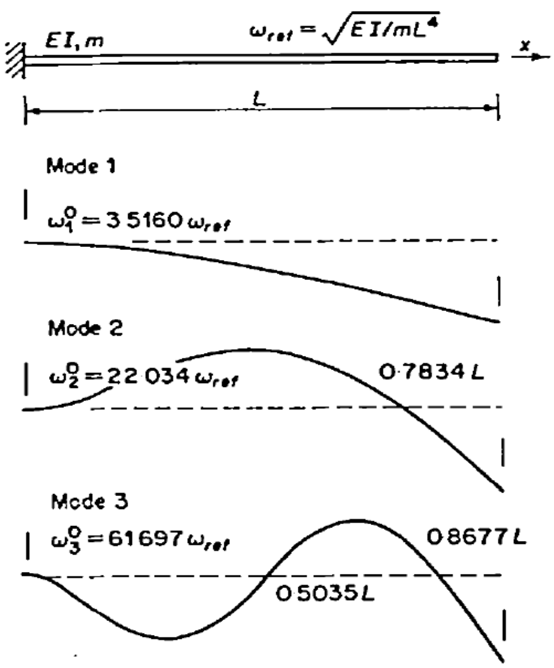
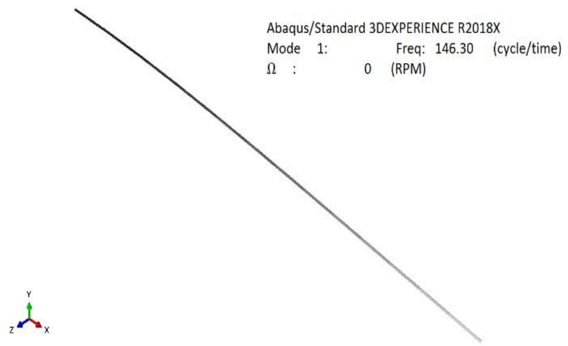


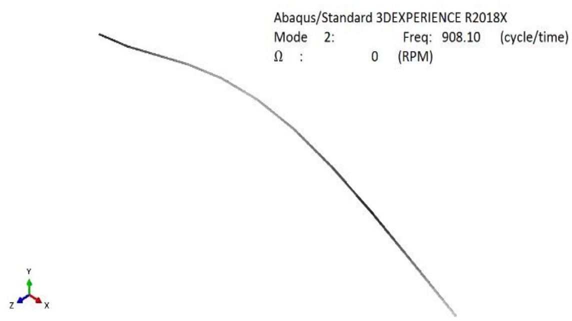
Fig. 3 The first three natural frequencies of the non-rotating cantilever beam [3].

According to the studies carried out by Akesson and Olhoff [3], if additional rigid support is placed at x=0.7834L which is the second modal node of the no-support beam, its natural frequency will be equal to the second natural frequency of the no-support beam ($\omega_1 = \omega_2^0$). Also, if two additional rigid supports are placed at x=0.5035L and x=0.8677L (locations of the third modal node of the no-support beam), first natural frequency of the beam will be equal to the third natural frequency of the no-support beam($\omega_1 = \omega_3^0$).

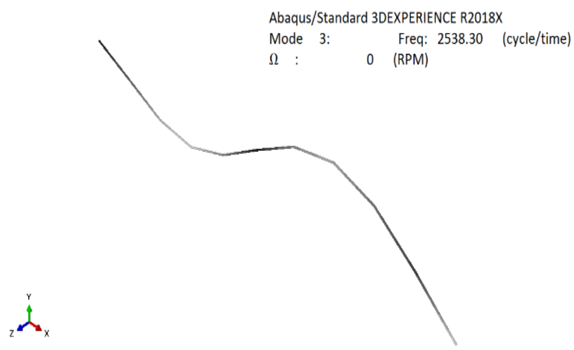
The mode shapes obtained from finite-element analysis software for first, second, and third modes of the non-rotating cantilever beam are shown in "Figs. 4a, 4b, and 4c" .



(a): Mode 1



(b): Mode 2



(c): Mode 3

Fig. 4 Software analysis of the first three natural frequencies of the non-rotating cantilever beam [3].

Moreover, the results of this analysis are given in "Table 2" . Comparison of results of software analysis with results of the analysis conducted by Akesson and Olhoff [3] confirms the correctness of software analysis.

Table 2 Comparison of results of software analysis with results of the analysis for natural frequency

Fundamental frequency	This study result (Hz)	Ref.[3] result (Hz)	Error %
ω_1^0	146.3	143.5	1.95
ω_2^0	908.1	900.3	0.86
ω_3^0	2538.3	2521.2	0.67

2.2. Minimum Stiffness of Intermediate Elastic Support in The Non-Rotating Cantilever Beam

Akesson and Olhoff [3] showed that the intermediate support added to the system should not necessarily be rigid. For this additional support, there is a non-dimensional minimum stiffness of K_0, and for maximization of the fundamental frequency, the support stiffness should not be less than K_0. They represented the result of their research in the following equation:

$$K_0 = K_s L^3/EI \quad (2)$$

Where, K_s denotes the minimum stiffness of the support. They argued that a non-dimensional beam stiffness of K_0=266.9 leads to minimum stiffness of support which can increase the first natural frequency of the restrained system to second natural frequency of the no-support system. Eq. (2) is entirely the same as the relation proposed by Wang et al. [6] in which minimum non-dimensional stiffness of the cantilever beam is given as K_0=266.87. Fig. 5 shows the software model of added support to the beam at the optimum point. By increasing the stiffness of elastic support from 0.001, a minimum stiffness of K_s=4.305e9 (N/m) is obtained at which fundamental frequency becomes maximum. By substituting into Eq. (2), a minimum non-dimensional stiffness of K_0=266.76 is achieved, which is close to the results of references [3] and [6].

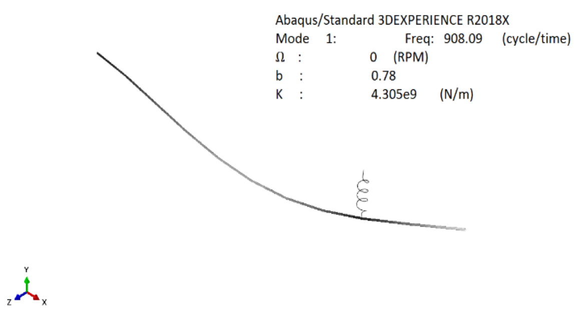
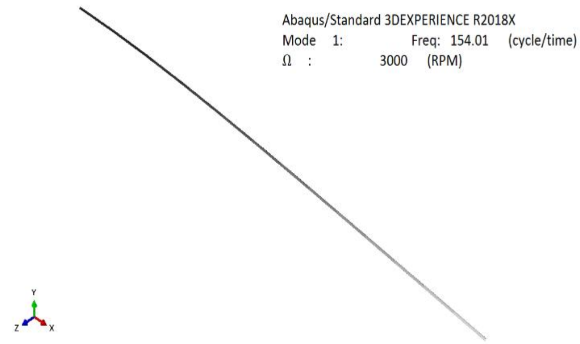


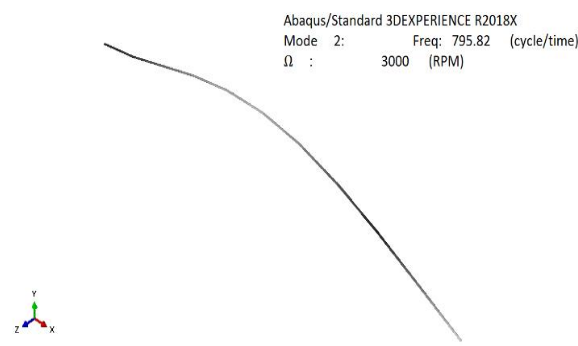
Fig. 5 The software model of added support to the beam.

3 MINIMUM STIFFNESS OF INTERMEDIATE ELASTIC SUPPORT FOR ROTATING CANTILEVER BEAM

As stated in the geometric simulation segment in the introduction and section 2, rotating beams are mostly applied in modelling the rotating blades. These blades are usually used in turbomachines, and their operating rotational speed is approximately Ω=3000 (rpm). We consider a rotating beam, as indicated in "Fig. 2", which rotates with a rotational speed of Ω=3000 (rpm) about the axis passing through support perpendicular to the longitudinal axis of the beam. According to beam properties shown in "Table 1" and applying an angular speed of Ω=314.15 (rad/s) =3000(rpm), first, finite-element modal analysis software is performed, and ω_1^0=154.01 (Hz) and ω_2^0 =795.82(Hz) are obtained for the first and second mode forms and natural frequencies of the beam with no intermediate support. Figs. 6a and 6b show the first and second mode shapes of rotating beam with no intermediate elastic support.



(a): Mode 1



(b): Mode 2

Fig. 6 The first and second modal shapes of rotating beam.

As mentioned before, in this study, Courant's maximum–minimum theorem is used and based on the results obtained from section 2-2, we place the elastic support at point x=0.7834L on the rotating beam. Then, elastic support stiffness is gradually increased from zero, and natural frequency corresponding to each dimensionless stiffness K_0 is derived. This is done until an increase in stiffness does not raise the fundamental frequency of the rotating beam. As a result, a non-dimensional stiffness of K_0=240.31 is achieved for rotating beam. The results of the influence of changing the stiffness of intermediate elastic support on an increase in the fundamental frequency of the rotating beam are shown in "Fig. 7”.

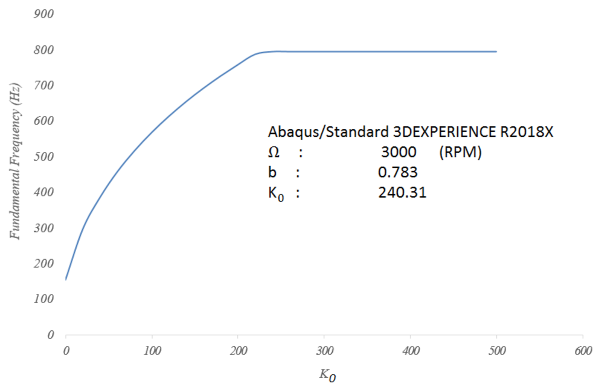
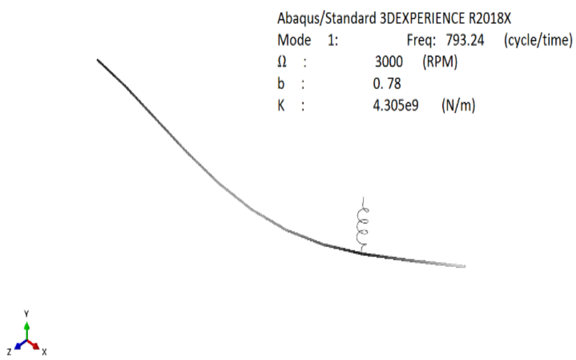
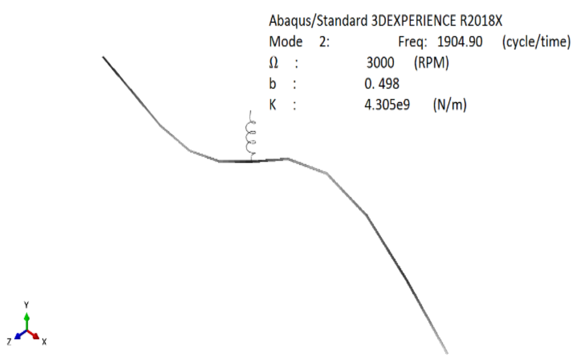


Fig. 7 Variation curve of the fundamental frequency with respect to the increase in stiffness of middle support.

As is evident in “Fig. 7”, by increasing the non-dimensional stiffness of intermediate support from K_0=240.31, the natural frequency of the beam does not exceed ω_1=793.24 (Hz), and K_0=240.31 is introduced as minimum support stiffness for obtaining the maximum fundamental frequency of the rotating beam.



(a): Mode 1



(b): Mode 2

Fig. 8 The first and second mode shapes of rotating beam with intermediate elastic support.

By applying the geometric properties of the beam as in Table 1 and angular velocity of Ω=314.15 (rad/s) =3000 (rpm) and K_0=240.31 for minimum non-dimensional stiffness, frequency analysis is carried out by software and ω_1=793.24 (Hz) and ω_2=1904.9 (Hz) are achieved for the first and second natural frequencies of the rotating beam with elastic support. “Figs. 8a and 8b” represent first and second mode shapes of rotating beam with intermediate elastic support.

The results in “Table 3” show that $\omega_1 = \omega_2^0$ that proves the Courant’s maximum–minimum theorem using finite-element software analysis. In this table, parameter b determines the optimum point for placing intermediate elastic support which by placing the support at that location, the unrestrained beam frequency ω_0 increases by ω. In fact, by adding elastic support with minimum stiffness at the optimum position based on Courant’s maximum–minimum theorem, the fundamental frequency of rotating beam can be increased significantly.

Table 3 Frequency improvement by optimum location of intermediate support

Vibration mode	ω_0	b	ω	Frequency increasing
1st mode	154.01	0.78	793.24	415.0%
2nd mode	795.82	0.49	1904.9	139.4%
	795.82	0.86	1904.9	139.4%

4 A PRACTICAL EXAMPLE, THE MINIMUM DIAMETER OF DAMPING WIRE IN TURBINE BLADES

One of the main reasons of failure in turbomachine blades, especially in low-pressure regions, dynamic fatigue is due to applied vibrational stresses in a way that values of these stresses, particularly in resonance condition which occurs during operation and turning off the turbine, are significantly higher than static stresses. Thus, it is attempted to take fundamental frequencies of blades far away from frequencies of different excitation sources in turbomachines. Since using blades with large geometric properties can increase productions costs as well as raising the inertia, one of the ways for improving fundamental frequencies is using a damping wire passing through a certain point in all blades. Figure 9 shows the damping wire in blades of the low-pressure zone of a steam power plant.

The use of a damping wire, as an engineering method, is recommended to improve the vibrational behaviour of the rotating blades in turbines. Damping wire is a mostly high-strength metal wire that passes through the blades of a turbine and connects the two adjacent blades at a certain point.

Fig. 9 Damping wire in blades.

Researchers have made a significant effort in modal analysis of turbine blade to estimate the critical operation conditions of turbine blades and prevent them from failure. In most attempts made in this field, blades are modelled as a beam, and only blade's vibrational frequencies are obtained in free mode.

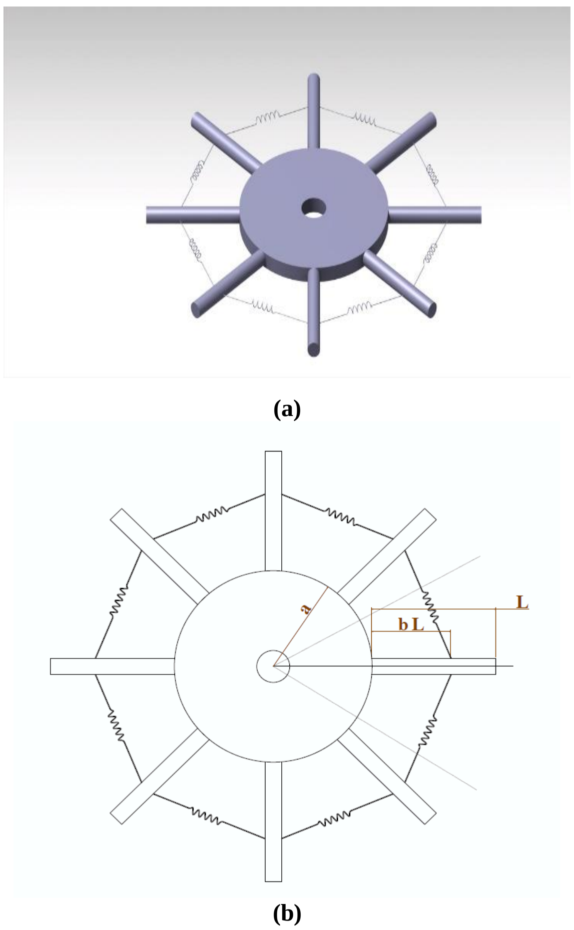


Fig. 10 Simplified form of the blades and damping wires passing through them.

If we consider damping wire between each pair of blades as a spring, we can assume that a row of the blades and damping wires passing through them have a simplified form shown in "Fig 10a" . If we consider one of the blades individually, as depicted in "Fig. 10b" , in the vibrational analysis of this blade, we are dealing with a rotating beam with intermediate elastic support.

In this blade, the optimum point for passing damping wire is the optimal position for placing intermediate support which is obtained by Courant's maximum–minimum theorem, and the minimum damping wire diameter is derived from minimum stiffness of intermediate elastic support using following relations:

$$k_{eq} = 2k = K_w \tag{3}$$

$$K_w = K_0 \frac{EI}{L^3} \tag{4}$$

$$k = \frac{E_w A}{L_w} = \frac{E_w \left(\frac{\pi d_s^2}{4}\right)}{L_w} \tag{5}$$

$$d_s = \sqrt{\frac{2K_w L_w}{\pi E_w}} \tag{6}$$

For instance, if the studied beam with properties given in "Table 1" is considered as a rotating blade of a turbomachine, by assuming damping wire length of L_w=5 (cm), the minimum damping wire diameter is extracted from Equation (5) which is equal to d_s=24.4 (mm).

5 RESULTS AND DISCUSSION

As stated in Section 2.1 and the results presented in "Table 2" , the determination of the optimal position of the intermediate support of the non-rotating beam, by finite element software analysis, confirms the Courant's maximum–minimum theorem and is fully consistent with the results of the analysis presented by Akesson and Olhoff [3].

This study shows that the results obtained from the finite element software analysis, regarding the minimum dimensionless stiffness K_0 of the intermediate elastic support for the non-rotating beam, are very close to the results obtained from the analytical solution performed by Akesson and Olhoff [3] and Wang et al. [6]. The results of this comparison are given in "Table 4" . The correctness of software analysis results is verified by comparing the software analysis results with results proposed by Akesson and Olhoff [3] and Wang et al. [6].

Table 4 Comparison of results of studies on Dimensionless minimum stiffness

Dimensionless minimum stiffness	This study result	Ref.[3] result	Ref.[6] result
K_0	266.76	266.9	266.87

The results show that the differences between non-dimensionless minimum stiffness K_0 obtained from software analysis and one suggested by Akesson and Olhoff [3] and Wang et al. [6] are 0.05% and 0.04%, respectively.

The results obtained from finite element software analysis, regarding the optimal position of the intermediate support show that the Courant theory is quite valid for a rotating beam. The results of this study also show that, as Akesson and Olhoff [3] have previously stated, there is a certain value for minimum dimensionless stiffness K_0 of the intermediate elastic support of the rotating beam that can lead to the maximum fundamental frequency.

Determining the minimum diameter of the damping wire of rotating blade can be a very good practical example of using an elastic intermediate support mounted on a rotating beam. The optimal position of the intermediate elastic support actually determines the optimal position of the damping wire on the blade, which increases the fundamental frequency of the blade and reduces the risk of resonance. Also, the minimum stiffness of the intermediate elastic support can lead to determining the minimum diameter of the damping wire, which is a good criterion for machine design and pave the way for further studies in this field.

6 CONCLUSION

In the present study, for the first time, Courant's maximum–minimum theorem has been used as a proven theory to increase the fundamental frequency of a non-rotating beam to improve the fundamental frequency of a rotating console beam. In this paper, the fundamental frequency increase of a rotating beam is investigated by placing an intermediate elastic abutment with the minimum stiffness in the desired position based on the Courant's maximum–minimum theorem using finite element analysis. The main conclusions of this study are presented as follows:

- The results of finite element software analysis, regarding the optimal position of the intermediate support are fully consistent with the results of the analytical solution for non-rotating beams.
- The minimum dimensionless stiffness obtained through finite element software analysis for non-rotating beam is very close to the results of the analytical solution.
- This research concludes that the Courant's maximum–minimum theorem for rotating beams is also quite valid.
- According to the results of this study, the use of Courant's maximum–minimum theorem to improve the fundamental frequency of a rotating beam with the approach of Timoshenko beam (slenderness ratio L/R=20) is valid.
- The method followed in this research can be a suitable model for designing and calculating the minimum diameter and optimal position of the damping wire of rotating blades in turbomachines.

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