Analytical Solutions for Buckling of Functionally Graded Circular Plates under Uniform Radial Compression using Bessel Function

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Abstract: The current study presents a new analytical method for buckling analysis of circular plates with constant thickness and Poisson's ratio, made of functionally graded material subjected to radial loading. Governing equations are based on energy method for thin plates. The boundary conditions of the plate are assumed to be simply supported and clamped. The stability equations were obtained by using conservation of energy. The critical buckling load and first mode shape in terms of Bessel function of the first kind were obtained using Variational Calculus method. Increase in buckling capacity and improvement in the behavior of functionally graded plates in comparison to homogenous plates have been investigated.

Keywords: Analytical Solution, Buckling, Functionally Graded Circular Plate, Variational Calculus

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1 INTRODUCTION

New design techniques are needed for applying new enhanced materials able to bearing high pressure in developing industrial structures. One of these materials is functionally graded material (FGM) that has been used in frequent applications in industry. The idea of FGMs was primarily formed for aeronautics field. The concept of functionally graded materials (FGMs) was first emerged in 1984 in Japan as a thermal resistant material for aircrafts, and space shuttles [1].

Materials whose compounds and functions vary evenly or step-wisely from one side to the other side are called functionally graded materials. Prevalently, FG material frequently shows smooth variations of function and property. Moreover, in functionally graded materials, the property of one side differs from that of the other side. Therefore, there are several behaviors within a material. For instance, one side may have the high mechanical strength and the other side may have the high thermal resistant property; hence, there are "two qualities" that even can be contrary properties, in one material. FGM provides two discordant properties such as thermal conductivity and thermal barrier property in one material contemporaneously.

The profit of FGMs is the capability of maintaining their embedded materials under high temperature gradients without dispossessing their structure entirety [1]. These materials are usually made of a commixture of ceramic and metal. The ceramic constituent has low thermal conductivity and the metal constituent disallows fracture of material due to high temperature gradients [2-3]. FGM has an exquisite capability to diminish thermal stresses, especially in high-temperature applications. At present, it enables manufacturing industries to fabricate light-weight, firm and enduring materials which are utilizable in many fields such as energy conversion material, structural material and others. Structure grading technology is also employed for cancer research.

There are few works done about the buckling of functionally graded structures in comparison with comprehensive analyses on isotropic and composite plates and shells. By assuming that the material properties across the structure are formed by a spatial distribution of the local reinforcement volume fraction $V_f = v_f(x,y,z)$, Feldman and Aboudi [4] studied the elastic bifurcation buckling of functionally graded plates under uniaxial loading. Thermal buckling of functionally graded rectangular thin plates was investigated by Javaheri and Eslami [5]. By classical and higher order of shear deformation theories of plates, the basic equations of the plate were derived; also under several types of thermal loads the closed form solutions were obtained.

Deflection and stress analyses for sandwich plates were made by Zenkour [6]. Nonlinear post-buckling of functionally graded circular plates subjected to thermal and mechanical loadings was studied by Ma and Wang [7]. Buckling of FG plates subjected to transverse loads was studied for circular and rectangular shapes [8–10]. Analytical and numerical elaborating of the mechanical behavior of FG plates subjected to transverse load was studied by Chi and Chung [11-12]. Vibration and thermal buckling of FG plates have been investigated by several researchers for rectangular and circular plates [13–19]. An exact solution for buckling of functionally graded circular plates based on higher order of shear deformation plate theory under uniform radial compression has been investigated by M. M. Najafizadeh and H.R. Heydari [20].

In this work it is aimed to derive analytical solution for FG circular plates having constant thickness while subjected to compressive loads. Buckling under critical loading and first mode shapes have been obtained by mathematical approaching. Increase of critical buckling load for the case of FG plate with comparison of homogenous plate has been investigated.

2 BASIC EQUATIONS

The Poisson's ratio, ν, across the plate thickness is assumed to be constant. The relationship between stress and deflection of the plate in axisymmetric bending is as follows [21].

$$\sigma_r = -\frac{E(z)z}{1-\nu^2}\left(\omega'' + \frac{\nu}{r}\omega'\right) \tag{1}$$

$$\sigma_\theta = -\frac{E(z)z}{1-\nu^2}\left(\frac{1}{r}\omega'' + \nu\omega'\right) \tag{2}$$

Where σ_r and σ_θ are polar components of stress, and ω is the deflection of plate in z-coordinate (Fig. 1). The terms ω' and ω'' are first and second derivatives of ω with respect to 'r'.

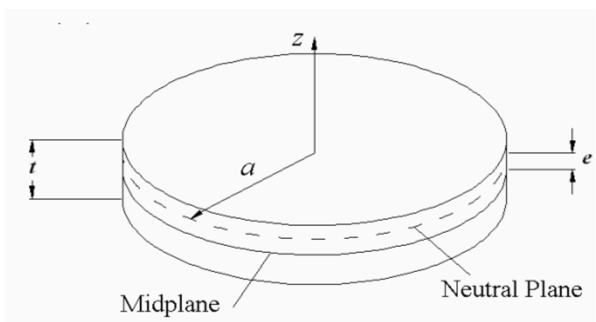

Fig. 1 Origin of z-coordinate

Fig. 1 shows the neutral plane which is located in position ‘e’ from the mid-plane, where the radius of the plate is ‘a’.The plate is subjected to pure bending [21], and thus the summation of all infinitesimal forces in any arbitrary radius must be set equal to zero. After using Eq. (1), the bellow simplified equation will be derived.

$$\int_{-(t/2)-e}^{(t/2)-e} E(z)zdz = 0 \tag{3}$$

Young’s modulus i.e. E(z), varies along the thickness of plate and follows Eq. (4) [22].

$$E(z) = (E_m - E_c)\left(\frac{t-2z-2e}{2t}\right)^n + E_c \tag{4}$$

Where the subscripts ‘m’ and ‘c’ denote the metallic and ceramic constituents, respectively, and ‘n’ is a material constant ($n \geq 0$). After substituting Eq. (4) into Eq. (3), amount of ‘e’is obtained as Eq. (5).

$$e = \frac{nt(E_c - E_m)}{2(n+2)(nE_c + E_m)} \tag{5}$$

Strain energy of circular plates with axisymmetric pure bending is shown in Eq. (6) [21].

$$\pi D \int_0^a \left(\left(\omega'' + \frac{\omega'}{r}\right)^2 - \frac{2(1-\nu)\omega'\omega''}{r}\right) rdr \tag{6}$$

In above equation, ‘D’ is flexural rigidity and can be obtained from the bellow equation.

$$D = \int_{-(t/2)-e}^{(t/2)-e} \frac{E(z)z^2}{1-\nu^2} dz \tag{7}$$

After substituting Eq. (4) into Eq. (7), flexural rigidity of FG plate can be obtained as bellow.

$$\begin{aligned} D &= \frac{(A^* E_c^2 + B^* E_c E_m + 12E_m^2)t^3}{C^*(1-\nu^2)\big((n^2+2n)E_c + (n+2)E_m\big)} \\ A^* &= n^4 + 4n^3 + 7n^2 \\ B^* &= 4n^3 + 16n^2 + 28n \\ C^* &= 12n^2 + 60n + 72 \end{aligned} \tag{8}$$

The work done by compressive radial load has been defined by the following equation [23].

$$\pi P \int_0^a (\omega')^2 rdr \tag{9}$$

The work done by compressive radial load is set equal to strain energy, and the critical amount of uniform compressive load can be obtained as bellow [24].

$$P = \frac{D\int_0^a \left(\left(\omega'' + \frac{\omega'}{r}\right)^2 - \frac{2(1-\nu)\omega'\omega''}{r}\right) rdr}{\int_0^a (\omega')^2 rdr} \tag{10}$$

3 ANALYTICAL BUCKLING ANALYSIS

For obtaining buckling load, we can use $\overline{\omega}(r)$ instead of $\omega(r)$ in Eq. (10), where $\eta(x)$ is an arbitrary continuous function that has at least one derivative where $\eta(0) = \eta(L) = 0$.

$$\overline{\omega}(r) = \omega(r) + \xi\eta(x) \tag{11}$$

For minimizing Eq. (10), the bellow equation must be satisfied.

$$\lim_{\xi \to 0} \frac{\partial P}{\partial \xi} = 0 \tag{12}$$

After replacing $\overline{\omega}(r)$ into Eq. (13), we have integrations in the general form of $\int_0^L \mathrm{f}\big(\mathrm{r}, \overline{\omega}(\mathrm{r}), \overline{\omega}'(\mathrm{r}), \overline{\omega}''(\mathrm{r})\big)\mathrm{dx}$ in numerator and denominator. After manipulations, we can obtain Eq. (13), in which f_ω, $f_{\omega'}$ and $f_{\omega''}$ are derivatives of ‘f’ with respect to ω, ω' and ω'' respectively.

$$f_\omega - \frac{\partial f_{\omega'}}{\partial r} + \frac{\partial^2 f_{\omega''}}{\partial r^2} = 0 \tag{13}$$

The terms of Eq. (13) for the numerator of Eq. (10) are as bellow.

$$\begin{aligned} f_\omega &= 0 \\ \frac{\partial f_{\omega'}}{\partial r} &= 0.6r\omega^{(3)} + 2\omega'' - 2\,\omega'/r \\ \frac{\partial^2 f_{\omega''}}{\partial r^2} &= 2r\omega^{(4)} + 4.6\omega^{(3)} \end{aligned} \tag{14}$$

After substituting Eqs. (14) and similarly for denominator of Eq. (13), and by minimizing Eq. (10), the requesting ordinary differential equation will be obtained as bellow.

$$\begin{aligned} &r^3\omega^{(4)} + 2r^2\omega^{(3)} + D^*\omega'' + E^*\omega' = 0 \\ &D^* = \big((P/D)r^3 - r\big) \\ &E^* = \big((P/D)r^2 + 1\big) \end{aligned} \tag{15}$$

Solution of above ODE is shown as bellow, where $J(\nu, x)$ and $Y(\nu, x)$ are the Bessel functions of the first and second kinds, respectively. They satisfy Bessel’s equation: $x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0$.

$$\omega = C_1 + C_2 J(0, G^*) + C_3 Y(0, G^*) + C_4 ln(r)$$
$$G^* = r\sqrt{P/D} \quad (16)$$

For pinned and clamped plates, the first derivative of ω in Eqs. (16) with respect to 'r' at origin of the polar coordinate (i.e. $r = 0$) must be vanished. Therefore Eq. (16) changes as bellow. The dimensionless ratio Pa^2/D for pinned and clamped plates has constant amount and is assumed equal to C_p and C_c, respectively. In general, Pa^2/D, is assumed equal to C.

$$\omega = C_1 + C_2 J\left(0, r\sqrt{C}/a\right) + C_3 H^*$$
$$H^* = Y\left(0, r\sqrt{C}/a\right) - 2ln(r)/\pi \quad (17)$$

The expression $\psi(r) = \omega(r)/\omega_{max}$ is dimensionless deflection. $\psi(0)$ and $\psi(a)$ are equal to 1 and 0 respectively. The second and third terms in $\psi(0)$ are equal to zero and a real number which is a function of radius of plate (i.e.$f(a)$), respectively. For pinned and clamped plates, the constants C_2 and C_3 in $\psi(r)$ can be expressed in terms of C_1.

$$C_2 = \left(J^* - \left(J^* + \pi f(a)\right)C_1\right)/I^*$$
$$C_3 = \left(K^* C_1 + \pi J\left(0, \sqrt{C}\right)\right)/I^*$$
$$I^* = J^* + \pi J\left(0, \sqrt{C}\right)f(a) \quad (18)$$
$$J^* = 2ln(a) - \pi Y\left(0, \sqrt{C}\right)$$
$$K^* = \pi\left(1 - J\left(0, \sqrt{C}\right)\right)$$

In simply-supported plate at $r = a$, the moment M_r must be vanished. Therefore, Eq. (19) for pinned plates must be satisfied.

$$a\omega''(a) + \nu\omega'(a) = 0 \quad (19)$$

After satisfying above equation, for pinned FG, intact laminated through the thickness or homogeneous plates, the constant C_1 can be expressed as follows.

$$\left(\pi J\left(0, \sqrt{C}\right)L^* + M^* K^*\right)/(L^* K^* - N^*) \quad (20)$$

The parameters L^*, $M^*$and $N^*$are as bellow.

$$L^* = \left((1 - \nu)O^* - CY\left(0, \sqrt{C}\right)\right)/a$$
$$M^* = \left((1 - \nu)\left(\sqrt{C}J\left(1, \sqrt{C}\right)\right) - CJ\left(0, \sqrt{C}\right)\right)/a \quad (21)$$
$$N^* = M^*\left(K^* + \pi f(a)\right)$$
$$O^* = 2/\pi + \sqrt{C}Y\left(1, \sqrt{C}\right)$$

For clamped plate the first derivative of $\psi(r)$ with respect to r at $r = a$ must be vanished. Eq. (22) show the constant C_1 for clamped FG, laminated or homogeneous plates.

$$C_1 = \pi\left(Y\left(1, \sqrt{C}\right)J^* + J\left(0, \sqrt{C}\right)P^*\right)/Q^*$$
$$P^* = 2/\sqrt{C} + \pi J\left(1, \sqrt{C}\right) \quad (22)$$
$$Q^* = \pi\left(J^* + \pi f(a)\right)Y\left(1, \sqrt{C}\right) - K^* P^*$$

After substituting Eqs. (18) into ω in Eqs. (17), in order to have a smooth and compatible dimensionless deflection for any arbitrary amounts of radius (i.e. a), it is obvious that the coefficient C_3 must be vanished. Eq. (23) for pinned and clamped FG, intact laminated through the thickness or homogeneous plates shows the dimensionless deflection of first mode shape.

$$\psi(r) = \pi\left(J\left(0, r\sqrt{C}/a\right) - J\left(0, \sqrt{C}\right)\right)/K^* \quad (23)$$

After substituting above equation in Eq. (10), the minimum amount of P has been obtained for $C_p = 4.1978$. The dimensionless deflection for first mode of pinned plates has been shown as bellow.

$$\psi(r) = 1.2435\, J(0{,}2.049\, r/a) - 0.2435 \quad (24)$$

Because of symmetry, deflection function of first mode in clamped plate is an even function; therefore, the Fourier series contain only cosine terms. For this case, the terms of the Fourier series are defined as follows.

$$\omega(r) = A_0/2 + \sum_{n=1}^{\infty} A_n cos(n\pi r/a) \quad (25)$$

The integral of the multiplication of two orthogonal functions (i.e. $\omega'(r)$ and $\omega''(r)$) is vanished, as shown bellow.

$$\int_0^a \omega'(r)\omega''(r)dr = 0 \quad (26)$$

After substituting Eq. (23) into Eq. (26), the minimum amount of C for satisfying above equation occurs for $C_c = 14.6820$. The dimensionless deflection for first mode of clamped plates has been shown as bellow.

$$\psi(r) = 0.7129\, J(0{,}3.8317\, r/a) + 0.2871 \quad (27)$$

For elaborating the behavior of FGP in comparison to homogeneous plates, we may concern dimensionless buckling critical load for circular FGP divided by the dimensionless buckling critical load for the circular homogeneous plate with equal radii. This amount is equal to the ratio of the flexural rigidity of FGP divided by the flexural rigidity of homogeneous plate (i.e. R). By substituting $n = 0$ into Eq. (8) the flexural rigidity of FGP turns into the homogeneous plate's flexural rigidity as Eq. (28).

$$D = E_m t^3 / \left(12(1-\nu^2)\right) \tag{28}$$

The dimensionless parameter 'R' has been shown as bellow.

$$R = \frac{A^*(E_c/E_m)^2 + B^*(E_c/E_m) + 12}{C^*\left((n^2+2n)(E_c/E_m) + n + 2\right)} \tag{29}$$

The parameter 'n' takes values greater than or equal to zero; therefore the parameter 'R' is between 1 and E_c/E_m.

4 RESULTS AND DISCUSSIONS

For numerical solution, homogeneous and functionally graded circular plates with arbitrary amounts of r/a, and n are assumed. Fig. 2 shows the buckling dimensionless first mode shape of pinned homogeneous, intact laminated through the thickness and functionally graded circular plates in general for arbitrary amounts of the ratio r/a.

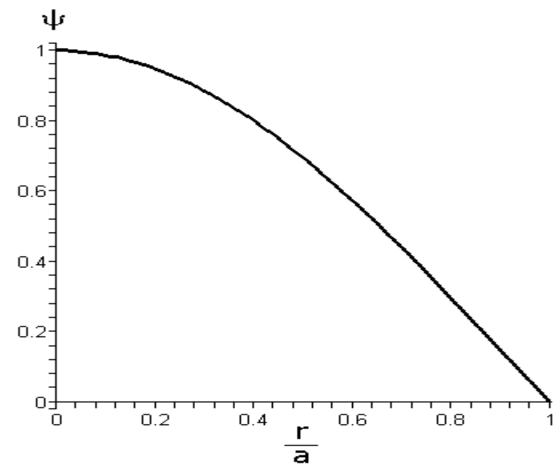

Fig. 2 Dimensionless deflection for pinned plate

Fig. 3 shows the buckling dimensionless first mode shape of clamped homogeneous, intact laminated through the thickness and functionally graded circular plates in general for arbitrary amounts of the ratio r/a. It has been seen that by increasing the power of the function $\mathrm{E(z)}$, the buckling loads of the pinned and clamped plate are increased; and at great powers, the ratio of the FG plate's buckling load to the homogeneous plate's buckling load approaches to $\mathrm{E_c/E_m}$. The result of this analysis is shown in Fig. 4.

The above figure shows that the improvement of buckling capacity in clamped where pinned circular plates are the same and there is no difference between them.

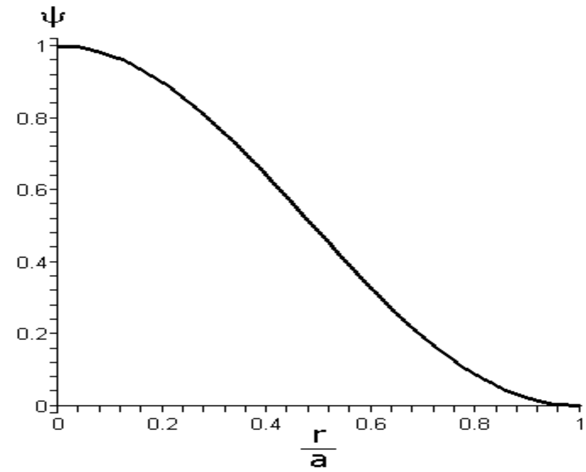

Fig. 3 Dimensionless deflection for clamped plate

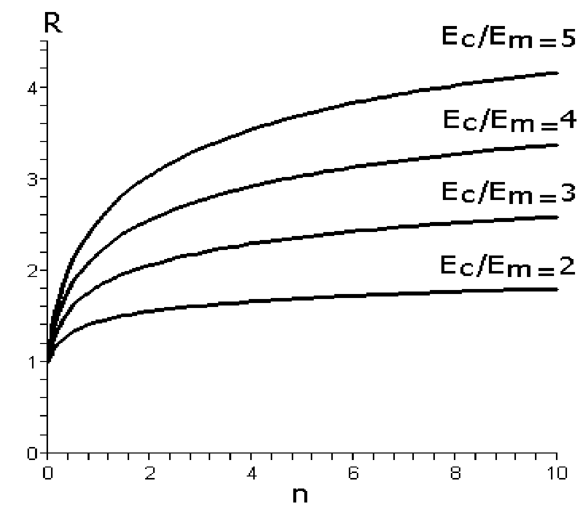

Fig. 4 Improvement of the buckling capacity

Table 1 shows the result validation for FG circular pinned and clamped plates [25].

Table 1 Result validation

	$\mathrm{Pa^2/D}$	
	Pinned	Clamped
Current work	4.2	14.68
Samsam and Eslami	4.2	14.68

7 CONCLUSION

This paper has presented a novel method for analyzing the buckling behavior of circular functionally graded plates. Exact analytical solutions for buckling of functionally graded circular simply supported or clamped plates with constant thickness and Poisson's ratio were presented. The critical buckling load and dimensionless first mode shape were obtained using variation calculus for two mentioned boundary conditions. Validity of solutions was proved by surveying the literature. This work proves that dimensionless first mode shape of buckling for prismatic functionally graded plates is similar to prismatic homogeneous plates. Because of symmetrical conditions for prismatic functionally graded plates, first mode shape in this case is similar to first mode shape of homogeneous plates and also is dependant upon boundary conditions of the plate. The dimensionless first mode shape has been obtained in terms of Bessel function of first kind for clamped and simply supported boundary conditions. The increase of capacity in using FG plates instead of homogenous plates has been investigated analytically by calculating buckling loads for both kinds of plates.

REFERENCES

[1] Woo, J. and Meguid, S. A. "Nonlinear analysis of functionally graded plates and shallow shells", International Journal of Solids and Structures, Vol. 38, 2001, pp. 7409-7421.

[2] Ge, C.-C., Li, J.-T., Zhou, Z.-J., Cao, W.-B., Shen, W.-P., Wang, M.-X., Zhang, N.-M., Liu, X., and Xu, Z.-Y., "Development of functionally graded plasma-facing materials", Journal of Nuclear Materials, Vol. 283-287, 2000, pp. 1116–1120.

[3] Put, S., Vleugels, J., Anné, G. and Van der Biest, O. "Functionally graded ceramic and ceramic-metal composites shaped by electrophoretic deposition", Colloids Surf A, Physicochem. Eng. Aspects, Vol. 222, No. 1-3, 2003, pp. 223-232.

[4] Feldman, E., and Aboudi, J., "Buckling analysis of functionally graded plates subjected to uniaxial loading", Composite Structure", Vol. 38, 1997, pp. 29-36.

[5] Javaheri, R., and Eslami, M. R., "Thermal buckling of functionally graded plates based on higher order theory", Journal of Thermal Stresses, Vol. 25, 2002, pp. 603-625.

[6] Zenkour, A. M., "A comprehensive analysis of functionally graded sandwich plates", part I-deflection and stresses, International Journal of Solids and Structures, Vol. 42, 2005, pp. 5224-5242.

[7] Ma, L. S. and Wang, T. J., "Nonlinear bending and post-buckling of a functionally graded circular plate under mechanical and thermal loadings", International Journal of Solids and Structures, Vol. 40, 2003, pp. 3311-3330.

[8] Zenkour, A. M., "A comprehensive analysis of functionally graded sandwich plates", part 2-buckling and free vibration, International Journal of Solids and Structures, Vol. 42, 2005, pp. 5243-5258.

[9] Ma, L. S. and Wang, T. J., "Relationships between axisymmetric bending and buckling of FGM circular plates based on third-order plate theory and classical plate theory", International Journal of Solids and Structures, Vol. 41, 2003, pp. 85-101.

[10] Yang, J. and Shen, H. S., "Non-linear analysis of functionally graded plates under transverse and in-plane loads", International Journal of Non-Linear Mechanics, Vol. 38, 2003, pp. 467-482.

[11] Chi, S. H. and Chung, Y. L., "Mechanical behavior of functionally graded material plates under transverse load", Part I: analysis, Vol. 43, 2006, pp. 3657-3674.

[12] Chi, S. H. and Chung, Y. L., "Mechanical behavior of functionally graded material plates under transverse load", part II: numerical results, International Journal of Solids and Structures, Vol. 43, 2006, pp. 3675-3691.

[13] Shen H. S., "Thermal post buckling behavior of shear deformable plates with temperature-dependent properties", International Journal of Mechanical Sciences, Vol. 49, 2007, pp. 466-478.

[14] Morimoto, T., Tanigawa, Y. and Kawamura R., "Thermal buckling of functionally graded rectangular plates subjected to partial heating", International Journal of Mechanical Sciences, Vol. 48, 2006.

[15] Ganapathi, M., and Prakash, T., "Thermal buckling of simply supported functionally graded skew plates", Composite Structures, Vol. 74, 2006, pp. 247-250.

[16] Prakash, T., and Ganapathi, M., "Asymmetric, flexural vibration and thermoelastic stability of FGM circular plates using finite-element method, Composites, Part B, Engineering, Vol. 37, 2006, pp. 642-649.

[17] Park, J. S. and Kim, J. H., "Thermal post-buckling and vibration analyses of functionally graded materials", Journal of Sound and Vibration, Vol. 289, 2006, pp. 77-93.

[18] Najafizadeh, M. M., and Heydari, H. R., "Thermal buckling of functionally graded circular plates based on higher order shear deformation plate theory", European Journal of Mechanics, A/Solids, Vol. 23, 2004, pp. 1085-1100.

[19] Wu, L., "Thermal buckling of a simply supported moderately thick rectangular FGM plate", Composite Structures, Vol. 64, 2003, pp. 211-218.

[20] Najafizadeh, M. M., and Heydari, H. R., "An exact solution for buckling of functionally graded circular plates based on higher order shear deformation plate theory under uniform radial compression", International Journal of Mechanical Sciences, Vol. 50, 2008, pp. 603-612.

[21] Ugural, A. C., "Stresses in plates and shells", 2nd edition, 1999, McGraw-Hill, Boston.

[22] Praveen, G. N., and Reddy J. N., "Nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates", International Journal of Solids and Structures, Vol. 35, 1998, pp. 4457-4476.

[23] Timoshenko S. P. and Gere, J., "Theory of elastic stability", 2nd edition, 1983, McGraw-Hill, New York.

[24] Naei, M. H., Masoumi, A., and Shamekhi, A., "Buckling analysis of circular functionally graded material plate having variable thickness under uniform compression by finite-element method", Proceedings of the Institution of Mechanical Engineers, Part C, Journal of Mechanical Engineering Science, Vol. 221, No. 11, 2007, pp. 1241-1247.

[25] Samsam, B. A., and Eslami, M. R., "Buckling of thick functionally graded plates under mechanical and thermal loads", Composite Structure, Vol. 78, 2007, pp. 433-439.