

A Thermoelasticity Solution for Thick Cylinders Subjected to Thermo-Mechanical Loads under Various Boundary Conditions

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Abstract: In this paper, a thermoelasticity solution for steady state response of thick cylinders which are subjected to pressure and external heat flux in inner surface is presented. Displacement field obeys the kinematics of the first order shear deformation theory (FSDT). It is assumed that the temperature varies both along the length and the thickness. The variation of the temperature occurs linearly through the thickness. Using energy method, the equilibrium equations and general boundary conditions are derived for the cylinder. Based on the developed analytical solution, adequate numerical results are depicted to provide an insight into the influence of the thermal and mechanical loads and boundary conditions on thermo-mechanical behavior of cylinder. Results show that shear stresses are noticeable at boundaries; moreover, temperature, displacement fields and stresses are strongly depended on length. Furthermore, the capability of the proposed method to solve any axisymmetrically cylindrical shells with general boundary conditions and thermo-mechanical loading is proven.

Keywords: Axisymmetric, Thermoelasticity, Thick Cylinders, Thermo-Mechanics

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1 INTRODUCTION

Cylindrical shells are of the most common features in almost all aspects of nuclear reactor design. The popularity of using such structures as portions of containment vessels, support skirts, coolant and flow inlet and outlet nozzles, fuel sleeves, and even as moderators makes the cylindrical shell structures' phenomena a favourable study for recent investigations. In many of these applications such cylinders are subjected to displacement and thermal gradient in both axial and radial directions. Consequently, it is important to analyse them to promote their reliability and to justify the structural capability of the design.

Some investigations are conducted on the elastic analysis of cylinders [1-6]. For instance, an analytical solution for elastic analysis of functionally graded (FG) cylinder based on classic method (Lamé's solution) is presented by Hongjun et al., [1]. They assumed that material properties vary linearly and exponentially. For FG cylinder with power law distribution of material properties is conducted by Zhifei et al., [2]. In [1-2] effects of materials non-homogeneity on mechanical characteristics of cylinder are studied and their results show that multi-layer method discontinuity predicts the circumferential stress. Ghannad and Zamani Nejad [3] presented a closed-form solution for thick walled cylindrical shells under pressure. The analytical solution is obtained based on the classical shell theory (CST) for plane strain and plane stress states. Considering transverse shear deformation, a stress analysis of cylinders based on FSĐT formulation is presented by Ghannad and Zamani Nejad [4]. Elastic response of thick cylindrical shells with variable thickness are studied by Eipakchi et al., [5] and Ghannad et al., [6] for homogenous and FG material, consequently.

The thermo-mechanical analysis of cylinders is presented in [7-13] based on CST. Obata and Noda [7] studied one-dimensional steady thermal stresses in a circular hollow cylinder and a hollow sphere, utilizing the perturbation method. Their work indicates the influence of inside radius size on stresses and the available temperature regions. Zimmerman and Lutz [8] presented solutions for the problem of uniform heating of a circular cylinder based on Frobenius series method and determined the effective thermal expansion coefficient. Jabbari et al., [9] obtained axisymmetric mechanical and thermal stresses for a thick hollow cylinder subjected to the temperature and pressure at inner and outer surfaces. Eslami et al., [10] and Bayat et al., [11] presented the study for thick sphere. An analytical solution for the coupled thermoelasticity of thick cylinders under radial temperature or mechanical shock load is obtained based on the Fourier expansion and Eigen function methods by Jabbari et al., [12]. The

thermo-mechanical behavior of cylinder is carried out in [13-19], considering shear stress effect in cylinder.

Kim and Noda [13-14] studied the two-dimensional unsteady thermoelastic problem of an infinite hollow cylinder and plate using the Green's function approach. In a later study, a non-axisymmetric case of the previous problem was solved by Jabbari et al., [15] using non-axisymmetric temperature distribution by expanding displacements and temperature distribution in the Fourier series.

It is noteworthy to state that the aforementioned researches [13-15] studied the thermo-mechanical behavior of cylinder in radial and circumferential directions. Shao [16] studied the thermo-mechanical stresses of hollow cylinders with the finite length. Results of the study prove that the proposed method is only suitable for the simply supported boundary conditions and where the temperature is constant at both ends. Jabbari et al., [17-18] introduced analytical solutions for two dimensional and three-dimensional steady-state thermoelastic problems of the circular hollow cylinder, using the generalized Bessel function and Fourier series. Arefi and Rahimi [19] presented an analytical solution for thermo elastic behavior of clamped-clamped cylinder under thermal and mechanical loads based on FSĐT. In this case, 1D heat conduction in finite length of the cylinder through radial direction is considered. Results indicate that there are considerable differences between radial displacements from classic method (1D) and FSĐT (2D) analysis. Further investigation on the elasticity, stability, buckling, and vibration responses of cylindrical shell have also been reported in literatures [20-23].

Stress analysts frequently derive the solutions of specific thermoelasticity problems; however, most of the published solutions are for specific problems and require additional analytical work when extended to more general situations. Although the heat conduction in finite length of the cylinder is often investigated in radial direction, but in real situations the heat conduction can be two dimensional in a finite length of the cylinder. It is of great concern for this paper to provide a general analysis to be applicable to almost any axisymmetrically, thermo-mechanically loaded cylindrical shell. The temperature may vary in the axial direction and linearly through the thickness. The generalized thermo-mechanical boundary conditions, giving the conditions on the two ends of cylinder, are given. These formulae can be easily applied to analyse the cylinders or to analyse the cylinders joined to other shells or structures.

The results are obtained for two cases in which the first case is mechanically clamped-clamped having constant temperature at two ends and the second case is mechanically clamped-free and thermally insulated-

constant temperature at two ends. The effects of different boundary conditions on the temperature, displacements, and components of stresses as well as two-dimensional distributions of temperature and displacement fields through the cylinder are brought out through number of parametric studies. The analytical solution with the proposed arbitrary thermal and mechanical boundary conditions at the two ends of cylinder is used to analyse the thermal and mechanical behaviors. Using the energy method, the derivation and implementation of an analytical method with its boundary conditions are discussed to solve the problems. The advantage of this method is shown in evaluation of the temperature, displacement, and stress fields. The results are in good agreement with finite element method (FEM).

2 ANALYSIS

A thick-walled axisymmetric cylindrical shell with an inner radius r_i , outer radius r_o and length L subjected to internal pressure P and heat flux H is considered (Figure 1).

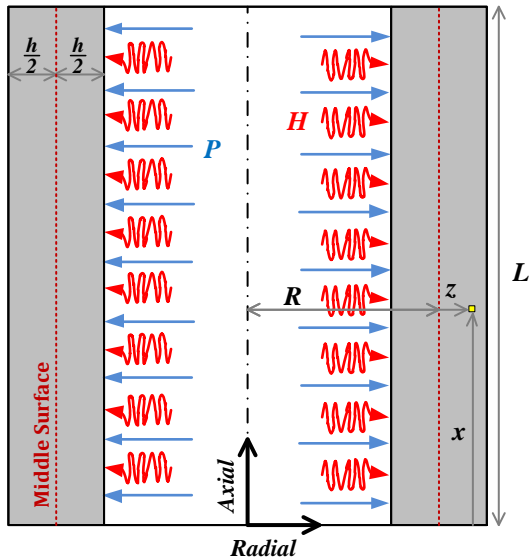


Fig. 1 Cross section of the thick-walled cylinder under thermo-mechanical loading

In figure 1, the location of typical point m, (r) within the thickness of cylindrical shell element can be determined by R and z , as $r=R+z$ in which R is the distance of middle surface from the axial direction, and z is the distance of typical point from the middle surface. Obviously, z and x must be within the ranges $-h/2 \leq z \leq h/2$ & $0 \leq x \leq L$ where h is the thickness of cylinder.

Displacement field (U_z, U_x) and the temperature change from reference temperature ($\Theta=T-T^*$) must be function of z or r and x , which are expressed as follow in the FSDT [4]-[24].

$$\begin{cases} U_z(x, z) = U_z^0(x) + zU_z^1(x) \\ U_\theta(x, z) = 0 \\ U_x(x, z) = U_x^0(x) + zU_x^1(x) \\ \Theta_z(x, z) = \Theta_z^0(x) + z\Theta_z^1(x) \end{cases} \quad (1)$$

Where $U_z^0(x)$ and $U_x^0(x)$ are the displacement components and $\Theta_z^0(x)$ is the component of temperature change at middle surface. The $U_z^1(x)$ and $U_x^1(x)$ functions are used to determine the displacement and temperature change field.

The strain-displacement relations (kinematic equations) are [4]:

$$\begin{cases} \varepsilon_z = \frac{\partial U_z}{\partial z} = U_z^1(x) \\ \varepsilon_\theta = \frac{U_z}{r} = \frac{U_z(x, z)}{R+z} = \frac{U_z^0(x)}{R+z} + \frac{U_z^1(x)}{R+z} z \\ \varepsilon_x = \frac{\partial U_x}{\partial x} = \frac{dU_x^0(x)}{dx} + \frac{dU_x^1(x)}{dx} z \\ \gamma_{zx} = \frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} = \left(U_x^1(x) + \frac{dU_z^0(x)}{dx} \right) + \frac{dU_z^1(x)}{dx} z \end{cases} \quad (2)$$

The thermal field-temperature change relations are [25]:

$$\begin{cases} e_z = -\frac{\partial \Theta}{\partial z} = -\Theta^1(x) \\ e_x = -\frac{\partial \Theta}{\partial x} = -\frac{d\Theta^0(x)}{dx} - \frac{d\Theta^1(x)}{dx} z \end{cases} \quad (3)$$

In addition, the stress tensor and heat flux vector components which are based on constitutive equations for isotropic materials are as follow [4-25]:

$$\begin{cases} \begin{bmatrix} \sigma_z \\ \sigma_\theta \\ \sigma_x \end{bmatrix} = \frac{E}{(1-2\nu)(1+\nu)} \begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_z \\ \varepsilon_\theta \\ \varepsilon_x \end{bmatrix} - \frac{E\alpha}{(1-2\nu)} \Theta \\ \tau_{zx} = \frac{E}{2(1+\nu)} \gamma_{zx} \\ \begin{bmatrix} h_z \\ h_x \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} e_z \\ e_x \end{bmatrix} \end{cases} \quad (4)$$

Where σ_i , ε_i , h_i , and e_i are the stresses, strains, heat fluxes, and thermal fields in the axial (x), circumferential (θ) and radial (z) direction. In Eqs. (3)

E , ν , α , and k are modulus of elasticity, Poisson's ratio, coefficient of thermal expansion, and thermal conduction coefficient. In addition, the value of effective stress based on von Mises failure theory is:

$$\sigma_{eff} = \frac{1}{\sqrt{2}} \left[(\sigma_z - \sigma_\theta)^2 + (\sigma_\theta - \sigma_x)^2 + (\sigma_z - \sigma_x)^2 + 6\tau_{zx}^2 \right]^{0.5} \quad (5)$$

The mechanical and thermal resultants defined:

$$\begin{Bmatrix} N_z^m \\ N_\theta^m \\ N_x^m \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_z (1 + \frac{z}{R}) \\ \sigma_\theta \\ \sigma_x (1 + \frac{z}{R}) \end{Bmatrix} dz \quad (6)$$

$$\begin{Bmatrix} M_\theta^m \\ M_x^m \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_\theta \\ \sigma_x (1 + \frac{z}{R}) \end{Bmatrix} z dz \quad (7)$$

$$\begin{Bmatrix} Q_x^m \\ M_{zx}^m \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} 1 \\ z \end{Bmatrix} \tau_{zx} (1 + \frac{z}{R}) dz \quad (8)$$

$$\begin{Bmatrix} N_z^t \\ N_x^t \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} h_z \\ h_x \end{Bmatrix} (1 + \frac{z}{R}) dz \quad (9)$$

$$M_x^t = \int_{-h/2}^{h/2} h_x (1 + \frac{z}{R}) z dz \quad (10)$$

Where equations (6)-(8) define mechanical resultants and equations (9) and (10) define thermal resultants. On the basis of the principle of virtual work, the variations of thermo-mechanical energy are equal to the variation of the external thermo-mechanical work $\delta U = \delta W$; where U is the total thermo-mechanical energy and W is the total external work due to internal pressure and heat flux. The thermo-mechanical energy is [26]:

$$\begin{cases} U = \iiint_{\Omega} U^* d\Omega; d\Omega = r dr d\theta dx = (R+z) dz d\theta dx \\ U^* = \frac{1}{2} (\sigma_z \varepsilon_z + \sigma_\theta \varepsilon_\theta + \sigma_x \varepsilon_x + \tau_{zx} \gamma_{zx} - h_z e_z - h_x e_x) \end{cases} \quad (11)$$

and the external work is [26]:

$$W = \iint_S (PU_z - H\Theta) dS, dS = r_i d\theta dx = (R - \frac{h}{2}) d\theta dx \quad (12)$$

The variation of the thermo-mechanical energy and external work are:

$$\begin{cases} \delta U = \int_0^L \int_0^{2\pi} \int_{-h/2}^{h/2} (\sigma_z \delta \varepsilon_z + \sigma_\theta \delta \varepsilon_\theta + \sigma_x \delta \varepsilon_x \\ + \tau_{zx} \delta \gamma_{zx} - h_z \delta e_z - h_x \delta e_x) (R+z) dz d\theta dx \\ \delta W = \int_0^L \int_0^{2\pi} (P \delta U_z - H \delta \Theta) (R - \frac{h}{2}) d\theta dx \end{cases} \quad (13)$$

Substituting Eqs. (2)-(4) into Eqs. (13), and drawing upon calculus of variation and the virtual work principle, by simplifications we will have:

$$\begin{cases} R \frac{dN_x^m}{dx} = 0 \\ R \left(Q_x^m - \frac{dM_x^m}{dx} \right) = 0 \\ R \left(\frac{N_\theta^m}{R} - \frac{dQ_x^m}{dx} \right) = P(R - \frac{h}{2}) \\ R \left(\frac{M_\theta^m}{R} + N_z^m - \frac{dM_{zx}^m}{dx} \right) = -P \frac{h}{2} (R - \frac{h}{2}) \\ R \frac{dN_x^t}{dx} = H(R - \frac{h}{2}) \\ R \left(N_z^t - \frac{dM_x^t}{dx} \right) = H \frac{h}{2} (R - \frac{h}{2}) \end{cases} \quad (14)$$

and the boundary conditions are:

$$\begin{aligned} & \left[N_x^m \delta U_x^0 + M_x^m \delta U_x^1 + Q_x^m \delta U_z^0 \right. \\ & \left. + M_{zx}^m \delta U_z^1 + N_x^t \delta \Theta^0 + M_x^t \delta \Theta^1 \right]_{0,L} = 0 \end{aligned} \quad (15)$$

Eq. (15) states the boundary conditions, which must exist at the two ends of the cylinder. The number of thermal and mechanical resultants aren't equal to number of equations in relation (14), thus for solving the set of differential Eq. (14) thermal and mechanical resultants need to be expressed in terms of the components of temperature and displacement field. Substituting Eqs. (2)-(10) into Eq. (14), set of differential Eq. (14) are rewritten as follow:

$$\begin{cases} [A] \frac{d^2}{dx^2} \{y\} + [B] \frac{d}{dx} \{y\} + [C] \{y\} = \{F\} \\ \{y\} = \{U_z^0 \quad U_z^1 \quad U_x^0 \quad U_x^1 \quad \Theta^0 \quad \Theta^1\}^T \end{cases} \quad (16)$$

Where $[A]_{6 \times 6}$, $[B]_{6 \times 6}$, $[C]_{6 \times 6}$ are the coefficients' matrices, and $\{F\}$ is force vector. The nonzero elements of $[A]_{6 \times 6}$, $[B]_{6 \times 6}$, $[C]_{6 \times 6}$ and $\{F\}$ are calculated by the relations given in Appendix A.

3 ANALYTICAL SOLUTION

To solve the set of differential equations (16), the $\{y\}$ is changed to $\{y^*\}$, and integrating the first and fifth equation in the set of Eq. (14).

$$\begin{cases} [A^*] \frac{d^2}{dx^2} \{y^*\} + [B^*] \frac{d}{dx} \{y^*\} + [C^*] \{y^*\} = \{F^*\} \\ \{y^*\} = \left\{ U_z^0 \quad U_z^1 \quad \frac{dU_x^0}{dx} \quad U_x^1 \quad \Theta^0 \quad \Theta^1 \right\}^T \end{cases} \quad (17)$$

where $[A^*]_{6 \times 6}$, $[B^*]_{6 \times 6}$, $[C^*]_{6 \times 6}$ are the coefficients, therefore the new arrangement of $[A]_{6 \times 6}$, $[B]_{6 \times 6}$, $[C]_{6 \times 6}$ are as below:

$$[A^*] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{24} & 0 & 0 \\ A_{31} & A_{32} & 0 & 0 & 0 & 0 \\ A_{41} & A_{42} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{65} & A_{66} \end{bmatrix} \quad (18)$$

$$[B^*] = \begin{bmatrix} 0 & 0 & 0 & A_{14} & 0 & 0 \\ B_{21} & B_{22} & A_{23} & 0 & B_{25} & B_{26} \\ 0 & 0 & 0 & B_{34} & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (19)$$

$$[C^*] = \begin{bmatrix} B_{11} & B_{12} & A_{13} & 0 & B_{15} & B_{16} \\ 0 & 0 & 0 & C_{24} & 0 & 0 \\ C_{31} & C_{32} & B_{33} & 0 & C_{35} & C_{36} \\ C_{41} & C_{42} & B_{43} & 0 & C_{45} & C_{46} \\ 0 & 0 & 0 & 0 & A_{55} & A_{56} \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \quad (20)$$

The force vector $\{F^*\}$ is as follow:

$$\{F^*\} = \begin{Bmatrix} K_1 \\ 0 \\ P(R - \frac{h}{2}) \\ -P(R - \frac{h}{2}) \frac{h}{2} \\ H(R - \frac{h}{2}) \frac{x^2}{2} + K_2 x + K_3 \\ H \frac{h}{2} (R - \frac{h}{2}) \end{Bmatrix} \quad (21)$$

Where K_1 , K_2 , and K_3 are constant, the results of integrating. The Eq. (21) is a set of linear non-homogenous differential equations with constant coefficients. Defining the differential operator $P(D)$, Eq. (21) is written as:

$$\begin{cases} P(D) = [A^*]D^2 + [B^*]D + [C^*] \\ D = \frac{d}{dx}, D = \frac{d^2}{dx^2} \end{cases} \quad (22)$$

Thus:

$$P(D)\{y^*\} = \{F^*\} \quad (23)$$

The differential Eq. (23) has the general solution including general solution for homogeneous case $\{y^*\}_g$ and particular solution $\{y^*\}_p$, as follow:

$$\{y^*\} = \{y^*\}_g + \{y^*\}_p \quad (24)$$

For the general solution for homogeneous case, $\{y^*\} = \{\xi\}e^{mx}$ is substituted in $P(D)\{y^*\} = \{0\}$,

$$([A^*]m^2 + [B^*]m + [C^*])\{\xi\}e^{mx} = \{0\} \quad (25)$$

Eq. (25) is eigenvalue problem for non-trivial solution $e^{mx} \neq 0$, the determinant of the coefficient must be considered zero.

$$\det([A^*]m^2 + [B^*]m + [C^*]) = 0 \quad (26)$$

The result of the Eq. (26) is an eight-order polynomial which is a function of m , the solution of which having an 8 eigenvalues m_i . The eigenvalues are 4 pairs of conjugated roots. By calculating eigenvectors $\{\xi\}_i$ corresponding to eigenvalues, the general solution for Eq. (23) is obtained:

$$\{y^*\}_g = \sum_{i=1}^8 C_i \{\xi\}_i e^{m_i x} \quad (27)$$

The constants C_i are obtained by applying boundary conditions. In Eq. (23), $\{F^*\}$ is quadratic polynomial, thus particular solution $\{y^*\}_p = \{y^*\}_{p2} x^2 + \{y^*\}_{p1} x + \{y^*\}_{p0}$ having $\{y^*\}_{p2}$, $\{y^*\}_{p1}$ & $\{y^*\}_{p0}$ unknown coefficients vectors are obtained by substituting particular solution in Eq. (23). Therefore, general solution is:

$$\{y^*\} = \sum_{i=1}^8 C_i \{\xi\}_i e^{m_i x} + \{y^*\}_{p2} x^2 + \{y^*\}_{p1} x + \{y^*\}_{p0} \quad (28)$$

There are 11 constants in $\{y^*\}$, having 8 constants for general solution and 3 constants for particular solution. Due to an integration in $\{y^*\}$ another constant K_4 appears in the general problem $\{y\}$. Now there are 12 constants in $\{y\}$ that are obtained by applying thermo-mechanical boundary conditions at two ends of the cylinder.

Boundary conditions can be express as displacement field and temperature or mechanical resultants and thermal resultants and or combination of them at both ends of the cylinder as follow:

$$x=0,L; \{y\} = \begin{Bmatrix} U_z^0 \\ U_z^1 \\ U_x^0 \\ U_x^1 \\ \Theta^0 \\ \Theta^1 \end{Bmatrix} \text{ or } \begin{Bmatrix} N_x^m \\ M_x^m \\ Q_x^m \\ M_{zx}^m \\ N_x^l \\ M_x^l \end{Bmatrix} \text{ or } \begin{Bmatrix} U_z^0 \text{ or } N_x^m \\ U_z^1 \text{ or } M_x^m \\ U_x^0 \text{ or } Q_x^m \\ U_x^1 \text{ or } M_{zx}^m \\ \Theta^0 \text{ or } N_x^l \\ \Theta^1 \text{ or } M_x^l \end{Bmatrix} \quad (29)$$

As previously mentioned, by applying boundary conditions at both ends of the cylinder, the analytical solution is obtained. Mathematical operations that are expressed for solving the set of differential equations are programmed in MAPLE 13.

4 RESULTS AND DISCUSSION

In this section, the aforementioned analytical solution is carried out for two cases with different boundary conditions. Cylinders have inner radius $r_i=40 \text{ mm}$, outer radius $r_o=60 \text{ mm}$ and length $L=800 \text{ mm}$. The Young's modulus of elasticity, coefficient of thermal expansion and thermal conduction coefficient of cylinders are $E=200 \text{ Gpa}$, $\alpha=12 \times 10^{-6}/^\circ\text{C}$ & $k=45 \text{ w}/(m^\circ\text{C})$, respectively. The Poisson's ratio is 0.3. Both cylinders are subjected to pressure $P=80 \text{ MPa}$ and external heat flux $H=500 \text{ (w}/m^2)$ in inner surface. The reference temperature is assumed to be $T^*=25^\circ\text{C}$ for two cases. The non-dimensional parameters are defined as follow:

$$\begin{cases} \bar{z} = \left(\frac{2z}{h}\right), \bar{x} = \left(\frac{x}{L}\right), \bar{r} = \left(\frac{r}{r_i}\right) \\ \bar{U}_z = \left(\frac{U_z}{r_i}\right) \times 10^4, \bar{U}_x = \left(\frac{U_x}{r_i}\right) \times 10^4 \\ \bar{T} = \left(\frac{T}{T^*}\right), \bar{\sigma}_\theta = \left(\frac{\sigma_\theta}{P}\right) \\ \bar{\tau}_{zx} = \left(\frac{\tau_{zx}}{P}\right), \bar{\sigma}_{eff} = \left(\frac{\sigma_{eff}}{P}\right) \end{cases} \quad (30)$$

4.1. Clamped-Clamped & Temperature-Temperature

At first case, two ends of the cylinder are considered to have $T=50^\circ\text{C}$ and they are mechanically clamped. Mechanical and thermal behaviors of the cylinder for this case are shown in Figure 2. Behaviors of the cylinder are discussed in radial and axial directions, Non-dimensional radial and axial displacements are shown in Figure 2a and 2b depicting the effects of the boundary conditions on displacement. Temperature distribution is shown in Figure 2c. The maximum temperature is occurred in the middle length of the cylinder, corresponds to the thermal loading.

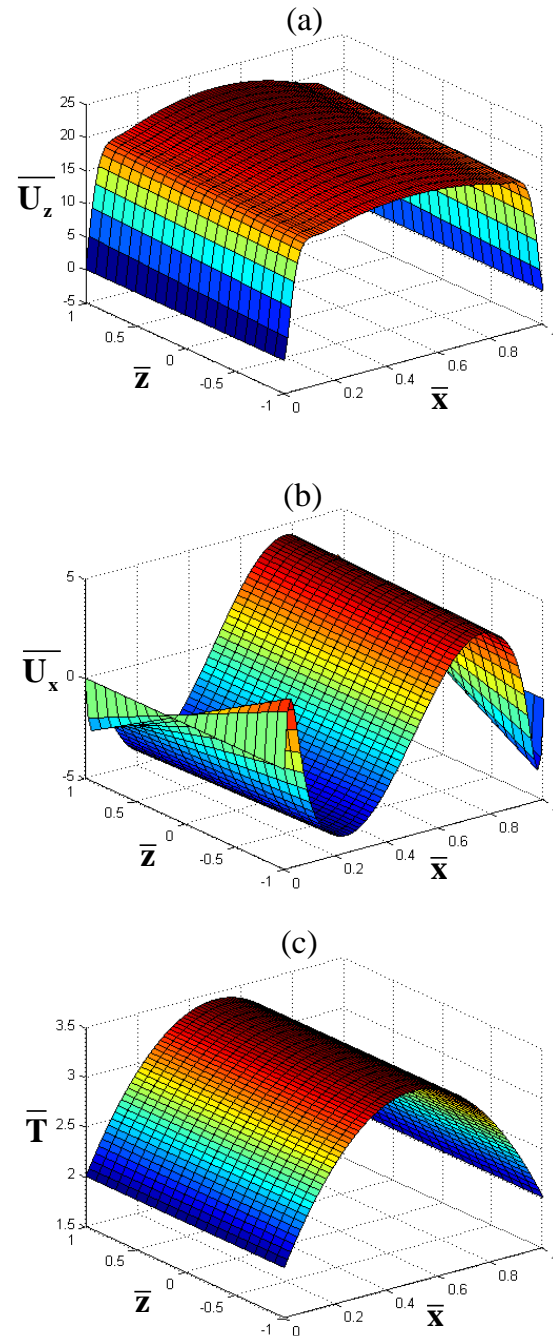


Fig. 2 Distribution of non-dimensional radial (a) and axial (b) displacement and temperature (c) in cylinder

The temperature variation versus cylinder layers and the radial and axial displacements variation versus cylinder heights are plotted in Figures 3-5. A good agreement between FEM and analytical results is observed. Figure 3 shows that the temperature increases from both ends to the middle length of the cylinder and the maximum temperature happens at the middle length of the cylinder. Also from Figure 3 it is concluded that

temperature profile changes along the axial axis and it is independent of radial axis and temperature change. Figure 4 shows that radial displacement has the same behavior as temperature having the maximum value in the middle length of cylinder. But radial displacement against the temperature profile varies in axial and radial axis. Figure 5 shows that the maximum of axial displacement occurs in regions near boundaries and is zero at the middle length. Figure 5 demonstrates the independency of axial displacement from radial axis in regions far away from boundaries and varies along axial direction.

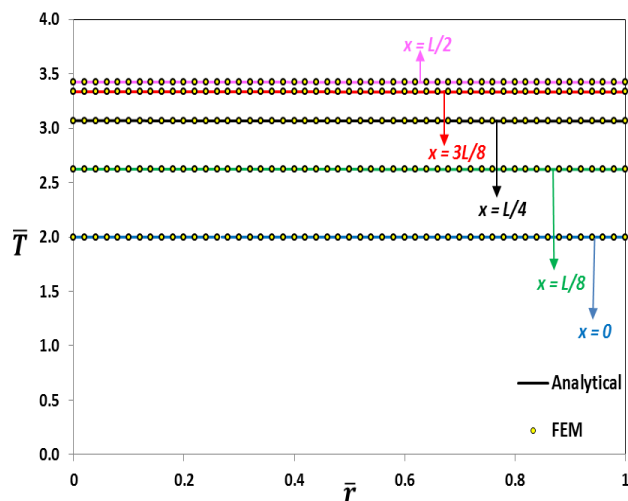


Fig. 3 Distribution of temperature through the thickness in the different heights of cylinder

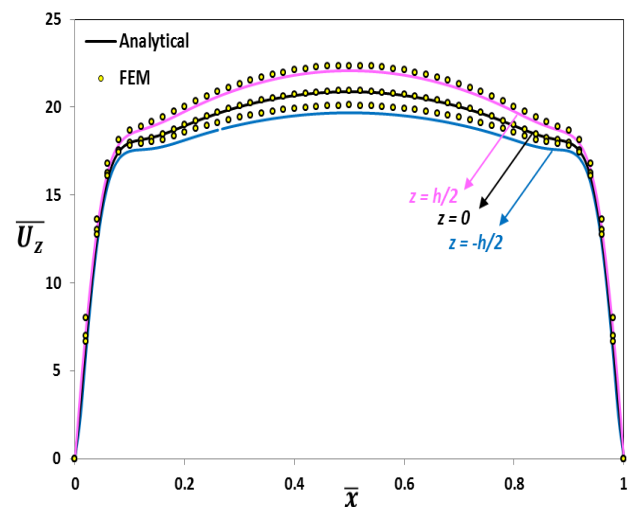


Fig. 4 Distribution of radial displacement along the cylinder surfaces

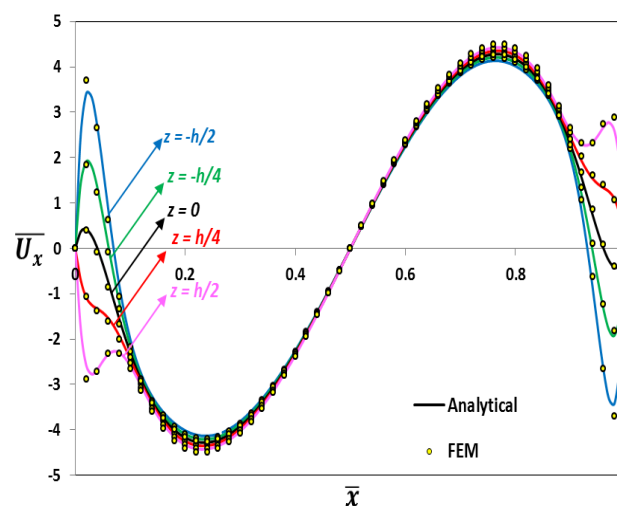


Fig. 5 Distribution of axial displacement along the cylinder surfaces

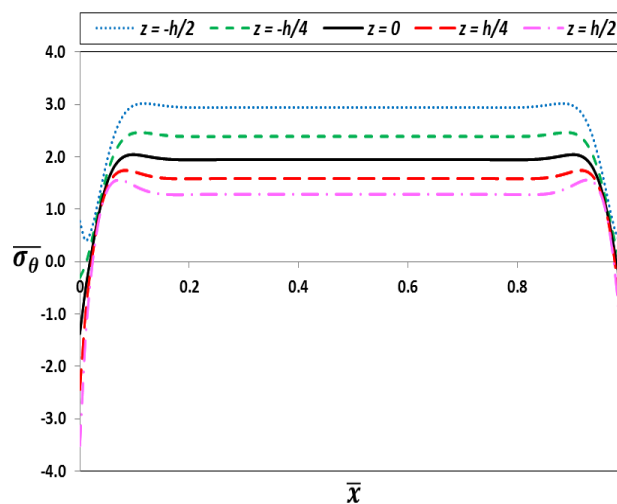


Fig. 6 Distribution of circumferential stress along the surfaces.

Circumferential, shear, and effective stresses for different layers of cylinder are shown in Figures 6-8. From Figures 6-8, it is concluded that at the regions far away from boundaries stresses leads to constant values. But the critical values of stresses occurred in boundaries in the outer surfaces; hence for precise engineering design, designers should use this presented methodology. Figures 6-8 show that the stresses in far regions of the boundaries are independent of axial axis and changes along radial direction. Figure 7 illustrates that shear stress is significant at boundaries leading to negligible values at far regions from boundaries.

A comparison between Figures 6-8 demonstrates that the circumferential stress has dominant values between stresses at far region from boundaries, so the behavior of von Misses stress is the same as circumferential stress at far regions from boundaries. But due to the shear stress effect at the boundaries the behavior of von Misses stress is different from circumferential stress.

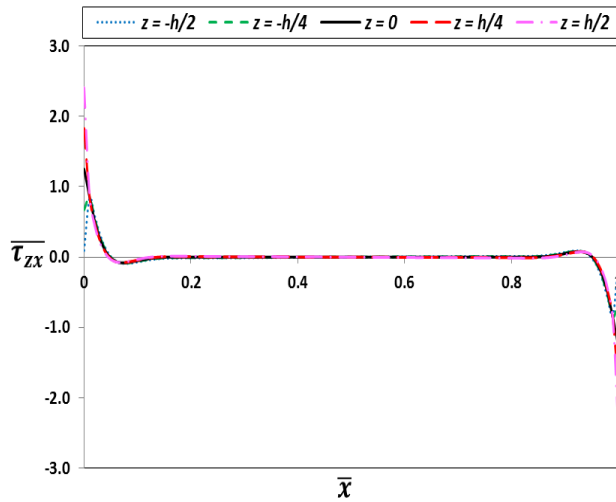


Fig. 7 Distribution of shear stress along the surfaces

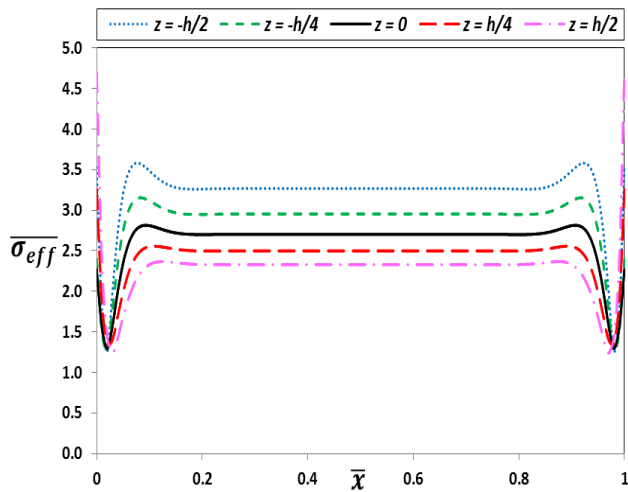


Fig. 8 Distribution of von misses stress along the surfaces

4.2. Clamped-Free & Temperature-Insulated

At second case, one end of the cylinder is considered mechanically clamped at $T=50^{\circ}C$, and another end is considered mechanically free and is insulated thermally. Effects of boundary conditions on mechanical and thermal behaviors are depicted in

figure 9. Maximum values of displacements and temperature occur in the free and insulated end of the cylinder, respectively.

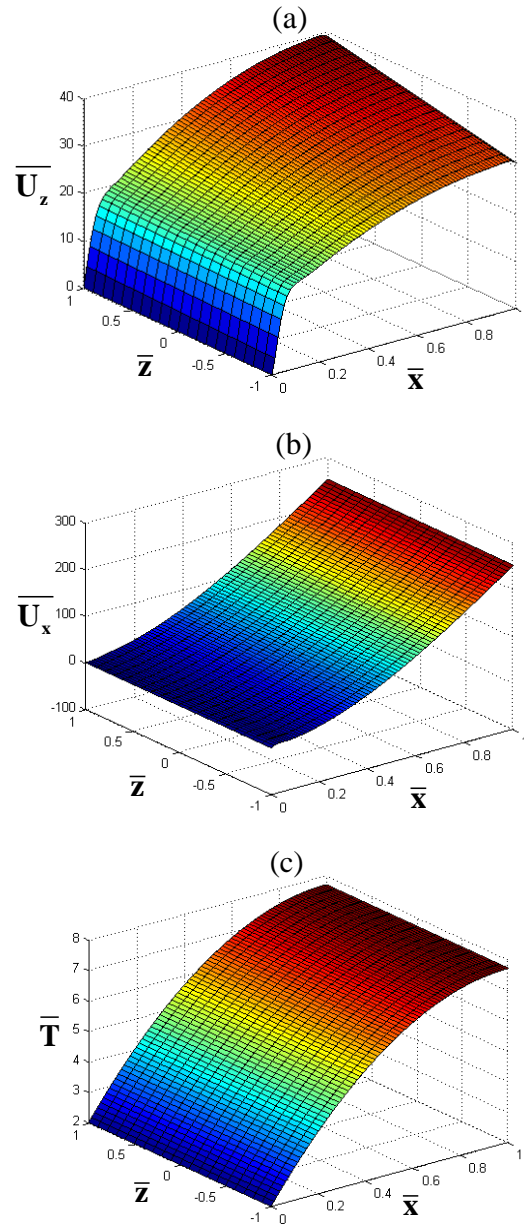


Fig. 9 Distribution of non-dimensional radial (a) and axial (b) displacement and temperature (c) in cylinder

Temperature distribution, radial and axial displacements for different heights and surfaces of cylinder are shown in Figures 10-12. Changes in the boundary conditions have made great effects on mechanical and thermal behaviors of the cylinder. In general, the maximum values of temperature and displacements are much greater than the first case and occur at the insulated and free ends of the cylinder, respectively. In the second case study, the maximum

values of temperature and radial displacement are approximately two times greater than the values for the first case and the maximum of axial displacement is 60 times greater than the value for the first case. Also Figure 12 shows that changing the boundary conditions have made the axial displacement profile to be independent of the radial direction and changes along the axial direction.

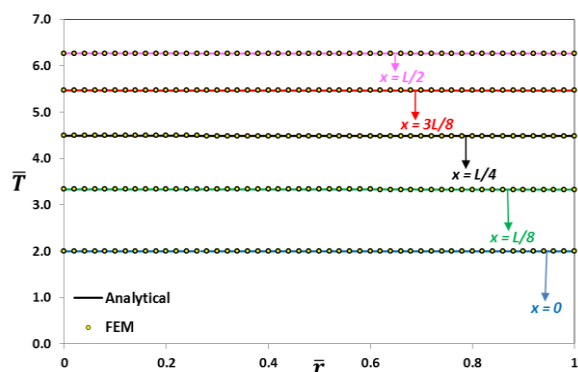


Fig. 10 Distribution of temperature through the thickness in the different heights of cylinder

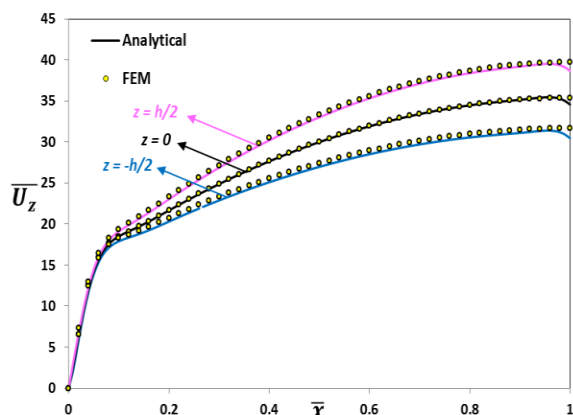


Fig. 11 Distribution of radial displacement along the cylinder surfaces

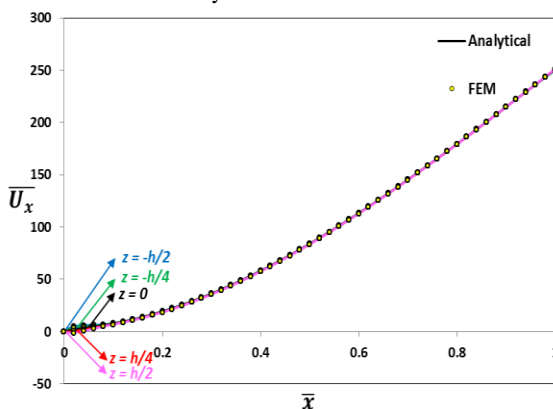


Fig. 12 Distribution of axial displacement along the cylinder surfaces

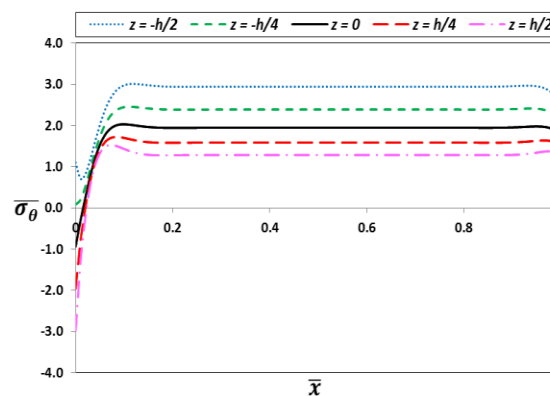


Fig. 13 Distribution of circumferential stress along the surfaces

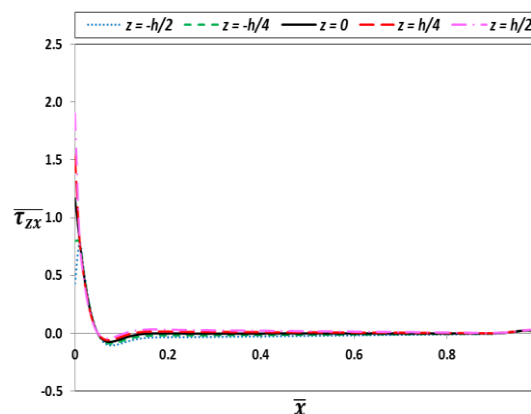


Fig. 14 Distribution of shear stress along the surfaces

Circumferential, shear and von Mises stresses are shown in Figure 13-15, respectively. The effects of boundary conditions on stresses are obvious, spatially at free end of the cylinder.

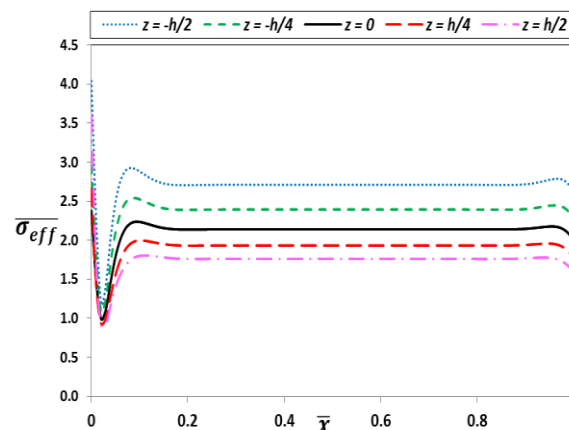


Fig. 15 Distribution of von mises stress along the surfaces

Table 1 shows the results of T , U_r , U_x , σ_{eff} and σ_θ in middle and inner surface. The obtained results from two methods were compared and good agreements are found.

Table 1 Numerical data from Analytical and FEM calculations for in $x=L/2$.

clamped-clamped & temperature-temperature ends						
Surface	Method	T (°C)	U _r (mm)	U _x (mm)	σ _{eff} (Mpa)	σ _θ (Mpa)
Middle	Analytical	85.56	0.08353	0	216.22	155.62
	FEM	85.55	0.08369	0	216.73	156.18
Inner	Analytical	85.61	0.07874	0	261.39	235.23
	FEM	85.63	0.08042	0	289.20	207.53
clamped-free & temperature-insulated ends						
Surface	Method	T (°C)	U _r (mm)	U _x (mm)	σ _{eff} (Mpa)	σ _θ (Mpa)
Middle	Analytical	156.67	0.11999	0.33539	171.19	155.62
	FEM	156.66	0.12016	0.33505	172.02	156.18
Inner	Analytical	156.72	0.10791	0.33647	216.85	235.23
	FEM	156.74	0.10960	0.33601	257.22	207.53

5 CONCLUSION

This paper presents analytical solution for obtained axisymmetric cylinder which is subjected to an internal pressure and external heat flux with arbitrary boundary conditions. Using energy method, first-order shear deformation and first-order temperature theory, the equilibrium equations are derived. The ordinary system of differential equations having constant coefficients is solved analytically. Results show that boundary conditions can change the mechanical and thermal behavior significantly. At the free-end case, the displacement and stress increase and when it is insulated, the temperature increase. For two cases the shear stress is important at boundary points and it is negligible at far regions from clamped boundaries. In general, the extracted conclusions are classified as follow:

(1) Comparison between the present results (two-dimensional thermal analysis) with the literature (one dimensional thermal analysis) indicates that the displacement and temperature field are strongly depended on axial direction of the cylinder, even in far regions from both ends. This difference implies that a two dimensional analysis is inevitable for analysis of the cylinders with finite length.

(2) The advantage of this method is its mathematical power to which it can handle arbitrary boundary conditions at the two ends of the cylinder for the axisymmetric thermo-mechanical analysis of thick cylinder with finite length. The proposed method can be extended to other kinds of mechanical and thermo-mechanical problems. It is necessary to check the solution convergence in analytical series solutions reported in literature, but in this method no check is needed.

(3) This method is suitable for modelling and studying the effect of various end condition of the cylinder; especially the effect of shear deformations at two ends

of the cylinder. These shear deformations tend to significant displacement gradient and consequently significant shear stresses. Both deformations and stresses can be useful in analysis and design of the cylindrical pressure vessels.

(4) Results show that with the same loading conditions, changes in boundary conditions can cause superior effects on thermal and mechanical behavior like temperature distribution, displacements profile and stress distribution in the cylinder. So the designer should take severe attention to the different boundary conditions and apply appropriate safety factor in their design.

6 APPENDIX

The nonzero elements of $[A]$ are calculated as follows:

$$\begin{cases}
 A_{13} = \int_{-h/2}^{h/2} \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} (R+z) dz \\
 A_{14} = \int_{-h/2}^{h/2} \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} (R+z)z dz \\
 A_{23} = -\int_{-h/2}^{h/2} \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} (R+z)z dz \\
 A_{24} = -\int_{-h/2}^{h/2} \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} (R+z)z^2 dz \\
 A_{31} = -\int_{-h/2}^{h/2} \frac{KE}{2(1+\nu)} (R+z) dz \\
 A_{32} = -\int_{-h/2}^{h/2} \frac{KE}{2(1+\nu)} (R+z)z dz \\
 A_{41} = -\int_{-h/2}^{h/2} \frac{KE}{2(1+\nu)} (R+z)z dz \\
 A_{42} = -\int_{-h/2}^{h/2} \frac{KE}{2(1+\nu)} (R+z)z^2 dz \\
 A_{55} = -\int_{-h/2}^{h/2} k(R+z) dz; A_{56} = -\int_{-h/2}^{h/2} k(R+z)z dz \\
 A_{65} = \int_{-h/2}^{h/2} k(R+z)z dz; A_{66} = \int_{-h/2}^{h/2} k(R+z)z^2 dz
 \end{cases} \quad (A.1)$$

where K is the shear correction factor that is embedded in the shear stress term. It is assumed that in the static state, for cylindrical shells $K=5/6$ [4]. Also the nonzero elements of $[B]$ are calculated as follow:

$$\left. \begin{aligned}
 B_{11} &= \int_{-h/2}^{h/2} \frac{\nu E}{(1-2\nu)(1+\nu)} dz \\
 B_{12} &= \int_{-h/2}^{h/2} \frac{\nu E}{(1-2\nu)(1+\nu)} (R+2z) dz \\
 B_{15} &= -\int_{-h/2}^{h/2} \frac{E\alpha}{(1-2\nu)} (R+z) dz \\
 B_{16} &= -\int_{-h/2}^{h/2} \frac{E\alpha}{(1-2\nu)} (R+z)z dz \\
 B_{21} &= \int_{-h/2}^{h/2} \frac{E}{(1+\nu)} \left(\frac{K(R+z)}{2} - \frac{\nu z}{(1-2\nu)} \right) dz \\
 B_{22} &= \int_{-h/2}^{h/2} \frac{E}{(1+\nu)} \left(\frac{K(R+z)}{2} - \frac{\nu(R+2z)}{(1-2\nu)} \right) z dz \\
 B_{25} &= \int_{-h/2}^{h/2} \frac{E\alpha}{(1-2\nu)} (R+z)z dz \\
 B_{26} &= \int_{-h/2}^{h/2} \frac{E\alpha}{(1-2\nu)} (R+z)z^2 dz \\
 B_{33} &= \int_{-h/2}^{h/2} \frac{\nu E}{(1-2\nu)(1+\nu)} dz \\
 B_{34} &= \int_{-h/2}^{h/2} \frac{E}{(1+\nu)} \left(\frac{\nu z}{(1-2\nu)} - \frac{K(R+z)}{2} \right) dz \\
 B_{43} &= \int_{-h/2}^{h/2} \frac{\nu E}{(1-2\nu)(1+\nu)} (R+2z) dz \\
 B_{44} &= \int_{-h/2}^{h/2} \frac{E}{(1+\nu)} \left(\frac{\nu(R+2z)}{(1-2\nu)} - \frac{K(R+z)}{2} \right) z dz
 \end{aligned} \right\} \quad (A.2)$$

and the nonzero elements of $[C]$ are calculated as follows:

$$\left. \begin{aligned}
 C_{24} &= \int_{-h/2}^{h/2} \frac{KE}{2(1+\nu)} (R+z) dz \\
 C_{31} &= \int_{-h/2}^{h/2} \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} (R+z)^{-1} dz \\
 C_{32} &= \int_{-h/2}^{h/2} \frac{E}{(1+\nu)(1-2\nu)} \left(\nu + \frac{z(1-\nu)}{(R+z)} \right) dz \\
 C_{35} &= -\int_{-h/2}^{h/2} \frac{E\alpha}{(1-2\nu)} dz \\
 C_{36} &= -\int_{-h/2}^{h/2} \frac{E\alpha}{(1-2\nu)} z dz \\
 C_{41} &= \int_{-h/2}^{h/2} \frac{E}{(1+\nu)(1-2\nu)} \left(\frac{z(1-\nu)}{(R+z)} + \nu \right) dz \\
 C_{42} &= \int_{-h/2}^{h/2} \frac{E}{(1-2\nu)} \left(\frac{z^2}{(R+z)} + (R+z) \right) dz \\
 C_{45} &= -\int_{-h/2}^{h/2} \frac{E\alpha}{(1-2\nu)} (R+2z) dz \\
 C_{46} &= -\int_{-h/2}^{h/2} \frac{E\alpha}{(1-2\nu)} (R+2z)z dz \\
 C_{66} &= -\int_{-h/2}^{h/2} k(R+z) dz
 \end{aligned} \right\} \quad (A.3)$$

The force vector $\{F\}$ is calculated, using heat flux and pressure at inner surface:

$$\{F\} = (R - \frac{h}{2}) \left\{ 0 \quad 0 \quad P \quad -P \frac{h}{2} \quad H \quad H \frac{h}{2} \right\}^T \quad (A.4)$$

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