Injection into Orbit Optimization using Orthogonal Polynomials

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Abstract: In this study, the problem of determining an optimal trajectory of a nonlinear injection into orbit problem with minimum time was investigated. The method was based on orthogonal polynomial approximation. This method consisted of reducing the optimal control problem to a system of algebraic equations by expanding the state and control vector as Chebyshev or Legendre polynomials with undetermined coefficients. The main characteristic of this technique was that it converted the differential expressions arising from the system dynamics and the performance index into some nonlinear algebraic equations, thereby greatly simplifying the problem solution. Our research effort focused on applying a Chebyshev series expansion to optimize the trajectory profile of a point-mass Satellite Launch Vehicle (SLV). This paper is divided as follows: first, the Chebyshev and Legender series expansion to optimization are introduced. Then, the flight mechanics model of the point-mass SLV is given. Next, our optimization problem is described and optimization results are presented and discussed.

Keywords: Chebyshev and legendre polynomials, Orthogonal functions, Point-mass SLV, Trajectory

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1 INTRODUCTION

Trajectory Optimization is a complex and important problem in all space missions and it has received a considerable amount of attention in the past few decades. Because of the complexity of most applications, trajectory optimization is most often solved numerically. Numerical methods for solving this problem and optimal control problems date back to nearly five decades ago during the 1950s, starting with the work of Bellman [1]. Methods for the trajectories have been broadly of two categories: indirect and direct techniques or their combinations. In an indirect method, first-order optimality conditions of the original optimal control problem are derived. The indirect approach leads to a Hamiltonian Boundary Value Problem (HBVP). Moreover, the most common indirect methods are the shooting method, the multiple-shooting method, and Collocation methods. In a direct method, the states and/or controls of the optimal control problem are discredited in some manner and the continuous-time optimal control problem is converted to a nonlinear optimization problem or a nonlinear programming problem (NLP). The most common direct methods are the direct shooting method, the direct multiple-shooting method, and the direct Collocation methods. In recent years, orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamic systems such as identification, analysis and optimal control.

The main characteristic of this technique is that it reduces the differential equation involved in the problem to an algebraic equation, thereby greatly simplifying the problem solution. Moreover, pseudo spectral methods have enjoyed much popularity. In a pseudospectral method, the states are approximated using a global polynomial and collocation is performed at the chosen points. Beforehand, a Chebyshev pseudospectral method for optimal control problems was presented and successfully applied on several problems [2]. This method employs Nth-degree Lagrange polynomial approximations for the state and control variables with the values of these variables at the Chebyshev–Gauss–Lobatto (CGL) points as the expansion coefficients [2].

Brusch [3] addressed the trajectory optimization for Atlas/Centaur launch vehicle by employing a classical augmented Lagrangian method, in which the solution was sought by alternately minimizing the augmented function with respect to independent variables and adjusting the Lagrange multipliers to satisfy constraints. Brauer et al. [4] employed an accelerated projected gradient method. These approaches involved considerable computer time for convergence. Well and Tandon [5] effectively used the recursive quadratic programming approach to three dimensional trajectory optimization.

Adimurthy [6] solved the 3D trajectory optimization problem through a diagonalized multiplier method in which multiplier update formula of Tapiaand Hanis was employed over a BFGS Hessian update.

Vathsal and Swaminathan [7] utilized a minimax technique by which an optimum pitch steering program could be designed for a SLV in the presence of large uncertainties in thrust, weight, aerodynamic coefficients, atmospheric density, and wind velocity experienced by the vehicle. Beltracchi [8] described a new approach to solve the all-up (ground to mission) trajectory optimization problem. The algorithm proposed in this paper does not require any all-up trajectories to be explicitly optimized, but separately simulates the booster and upper stage. The algorithm is based on solving the maximum transferable throw weight to a park orbit (for the booster), maximum transferable payload from the park orbit to the mission orbit (for the upper stage), and a coordination problem that adjusts the park orbit parameters to find the all-up optimum (maximum) payload to the mission orbit.

Weigel and Well [9] investigated an optimal ascent trajectory for a multistage SLV for dual payload problems. The optimization problems were formulated as multiphase problems with boundary and path constraints. Solutions were obtained using a direct multiple shooting method. Ping [10] proposed an inverse dynamics approach for trajectory optimization which could be useful in many difficult trajectory optimization and control problems. About using orthogonal functions and polynomial series, typical examples are the Walsh functions [11,12], the block-pulse functions [13,14], the Laguerre polynomials [15,16], the Legendre polynomials [17,18], the Chebyshev series [19 ,20], the Chebyshev wavelets [21], the Taylor series [22 ,23], and the Fourier series [24,25,26].

Our research effort focused on applying a numerical Chebyshev approach to optimize the trajectory profile of a point-mass SLV. A numerical algorithm based on a Chebyshev series expansion of control and state converts differential and integral expressions from the system dynamics and the performance index, the boundary conditions and other general constraints into systems of (non-linear) algebraic equations with unknown coefficients. The main characteristic of this technique was that it converted the differential expressions arising from the system dynamics and the performance index into some nonlinear algebraic equations, thereby greatly simplifying the problem solution. This paper is divided as follows: first, the Chebyshev and Legender series expansion to optimization are introduced. Then, the flight mechanics model of the point-mass SLV is given. Next, our optimization problem is described; optimization results are presented and discussed too. Finally, conclusions are drawn.

2 CHEBYSHEV AND LEGENDRE SERIES EXPANSION TO OPTIMIZATION

Recently, orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamic systems. In order to provide proper background, Chebyshev and Legendre polynomials are explained in this section.

Chebyshev and Legendre polynomials

To facilitate the presentation of the Chebyshev and Legendre algorithms, here we present some background on the Chebyshev and Legendre polynomials [19]. The Chebyshev polynomials of the first type are defined on the time interval $\tau \in [-1,1]$ as follows [27]:

$$T_r(\tau) = \mathrm{Cos}(r\theta),\ \theta = \mathrm{Arc}\cos\tau,\ -1 \le \tau \le 1 \quad (1)$$

Which are orthogonal with respect to the weight function $w(\tau) = 1/\sqrt{1-\tau^2}$ on the interval $[-1,1]$ and [19] [28]:

$$\int_{-1}^{1} \frac{T_m(\tau)T_n(\tau)}{\sqrt{1-\tau^2}} d\tau = \begin{cases} 0, & \text{for } n \ne m \\ \pi/2 & \text{for } n = m \ne 0 \\ \pi, & \text{for } n = m = 0 \end{cases} \quad (2)$$

Therefore, the first few Chebyshev polynomials are [27]:

$$\begin{aligned} &T_0(\tau) = 1, \\ &T_1(\tau) = \tau, \\ &T_2(\tau) = 2\tau^2 - 1, \\ &\vdots \\ &T_{r+1}(\tau) = 2rT_r(\tau) - T_{r-1}(\tau) \end{aligned} \quad (3)$$

Hence, the Legendre polynomials are defined on the time interval $\tau \in [-1,1]$ as follows [29]:

$$\int_{-1}^{1} L_n(X)L_m(X)dx = \frac{2}{2n+1}\delta_{mn} \quad (4)$$

δ_{mn} is Kronecker delta [29]:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \ne n \end{cases} \quad (5)$$

Which are orthogonal with respect to the weight function $w(\tau) = 1$ on the interval $[-1,1]$. Therefore, the Legendre polynomials are [29]:

$$\begin{aligned} &L_0(\tau) = 1 \\ &L_1(\tau) = \tau \\ &L_2(\tau) = \frac{3}{2}\tau^2 - \frac{1}{2}, \\ &L_3(\tau) = \frac{5}{2}\tau^3 - \frac{3}{2}\tau, \\ &\vdots \end{aligned} \quad (6)$$

With this recursive formula [29]:

$$L_{r+1}(\tau) = \frac{2r+1}{r+1}.\tau.L_r(\tau) - \frac{r}{r+1}L_{r-1}(\tau) \quad (7)$$

A function $f(\tau)$ can be approximated by Chebyshev or Legendre polynomials of length m as follows [27]:

$$f(\tau) = \frac{1}{2}a_0T_0(\tau) + \sum_{n=1}^{m} a_nT_n(\tau) \quad (8)$$

Where the coefficients $a_n, n = 0,1,2,\ldots,m$ are unknown. Moreover, the following property of polynomials will also be used [29]:

If

$$f(\tau) = \frac{1}{2}a_0T_0(\tau) + \sum_{n=1}^{\infty} a_nT_n(\tau) \quad (9)$$

Then

$$\int_{-1}^{1} f(\tau)d\tau = f_0 - \sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1} f_n \quad (10)$$

Mathematical formulation

The behavior of a dynamic system can be represented by the following set of ordinary differential equations [9]:

$$\begin{cases} \dfrac{dX}{d\tau} = F(X(\tau), U(\tau), \tau) \quad 0 \le \tau \le T_f \\ X(0) = X_0 \\ X(T_f) = X_f \end{cases} \quad (11)$$

And the state variable inequality constraints (SVIC) [1]:

$$S(X(\tau), \tau) \le 0,\ 0 \le \tau \le T_f \quad (12)$$

Where X and U are the vector functions of τ, $(X_1, X_2, \ldots, X_n)$ are the state variables, and $(U_1, U_2, \ldots, U_r)$ are the control variables.

The problem of optimal control is then to find the control $U_i, i = 1,2,..,r$ transferring the system Eq. (11) from the position $X_i = X_i(\tau_0)$ to the position $X_i = X_i(\tau_f)$ within the time $(\tau_f - \tau_0)$ while satisfying Eq. (12) and yielding the optimum of the performance index I, as given by [30]:

$$I = H[X(T_f), T_f] + \int_0^{T_f} G[X(\tau), U(\tau), \tau]\, d\tau \quad (13)$$

The vector function F, S and the scalar functions H and G are generally nonlinear, and are assumed to be continuously differentiable with respect to their arguments. In order to use the Chebyshev polynomials of the first order (or Legender), which are defined on $t \in [-1,1]$, $\tau \in [0, T_f]$ is transformed into $t \in [-1,1]$ by using the below transformation [10] [21]:

$$\tau = \frac{T_f}{2}(1+t) \quad (14)$$

It follows that those Eq. (11) and Eq. (13) are replaced by [10]:

$$\begin{cases} \dfrac{dx}{dt} = f(x(t),u(t),t), \quad -1 \le t \le 1 \\ x(-1) = x_{-1} = X_0 \\ x(+1) = x_{+1} = X_f \end{cases} \tag{15}$$

$$I = h[x(1),1] + \int_{-1}^{1} g[x(t),u(t),t]dt \tag{16}$$

$$s[x(t),t] \le 0, \quad -1 \le t \le 1 \tag{17}$$

The approximate solution of this optimal control problem is represented by Chebyshev (or Legender, Hereinafter, for simplification, "or Legender" can be ignored) series of order m for both the state and control [28]:

$$\begin{cases} x_m(t) = \dfrac{1}{2}a_0T_0(t) + \sum_{n=1}^{m} a_nT_n(t) \\ u_m(t) = \dfrac{1}{2}b_0T_0(t) + \sum_{n=1}^{m} b_nT_n(t) \end{cases} \tag{18}$$

Where $\alpha \equiv (a_0, a_1, \ldots, a_n)$, $\beta \equiv (b_0, b_1, \ldots, b_m)$ are the unknown coefficients. By substituting the Chebyshev approximations for the state and control from the Eq. (18) into Eq. (15) to Eq. (17), considering $B_n(\alpha,\beta,T_f)$ and $C_n(\alpha,T_f)$ to be the Chebyshev coefficients of $g[x_m(t),u_m(t),t]$ and $h[x_m(1),1]$ respectively, and using the property of Eq. (10), the following approximation for I is obtained [28]:

$$\begin{aligned} J(\alpha,\beta,T_f) \\ &= \frac{1}{2}C_0(\alpha,T_f) + \sum_{n=1}^{\infty} C_n(\alpha,T_f) + B_0(\alpha,\beta,T_f) \\ &\quad - \sum_{n=2}^{\infty} \frac{1+(-1)^n}{n^2-1} B_n(\alpha,\beta,T_f) \end{aligned} \tag{19}$$

For computational reasons, the infinite series is truncated at order m and we can calculate $\{B_n\}$ and $\{C_n\}$ by means of the following equations [28]:

$$\begin{cases} C_n(\alpha,T_f) = \dfrac{2}{k}\sum_{i=1}^{k} h(x_m(\cos\theta_i),T_f)\ \cos n\theta_i \\ B_n(\alpha,\beta,T_f) = \\ \quad \dfrac{2}{k}\sum_{i=1}^{k} g(x_m(\cos\theta_i),u_m(\cos\theta_i),T_f)\ \cos n\theta_i \end{cases} \tag{20}$$

$$n = 0,1,\ldots,N \ ; k > N \ ; \theta_i = \frac{2i-1}{k}.\frac{\pi}{2}$$

Where $t_i = \cos\theta_i$ refers to the roots of the Chebyshev polynomial of order K.

Approximation of the system dynamics

If the Eq. (18) is substituted into the Eq. (15) and if the Chebyshev series expansion of the right-hand side of Eq. (15) is truncated after the term of order M, the following is obtained [28]:

$$\frac{dx_m}{dt} = f_M(x_m(t).u_m(t),t,T_f) \tag{21}$$

Where

$$\begin{aligned} f_M(x_m(t).u_m(t),t,T_f) &= \frac{1}{2}A_0(\alpha,\beta,T_f)T_0(t) \\ &+ \sum_{n=1}^{m} A_n(\alpha,\beta,T_f)T_n(t) \end{aligned} \tag{22}$$

with

$$\begin{aligned} &A_n(\alpha,\beta,T_f) = \\ &\frac{2}{k}\sum_{i=1}^{k} f_M(x_m(\cos\theta_i),u_m(\cos\theta_i),T_f)\ \cos n\theta_i; \\ &\qquad n = 0,1,\ldots,M \ ; k > N \ ; \theta_i = \frac{2i-1}{k}.\frac{\pi}{2} \end{aligned} \tag{23}$$

The left-hand side of the Eq. (21) is of degree, $m-1$ while on the right-hand side, the polynomial is of degree M. If f is non-linear, M is set to be equal to $m-1$, but if f is linear, M is set to be equal to [28]. If $\{a'\}_n$ represents the Chebyshev coefficients of dx_m/dt, then equating the coefficients of same-order Chebyshev polynomials yields [28]:

$$\begin{aligned} &a'_0 = A_0, a'_1 = A_1, ..a'_{m-1} = A_{m-1} \\ &\quad A_m = 0,\ldots,A_M = 0 \end{aligned} \tag{24}$$

The relationship between $\{a'\}_n$ and $\{a_n\}$ can be expressed by [28]:

$$\begin{aligned} &a'_0 = A_0, a'_1 = A_1, ..a'_{m-1} = A_{m-1} \\ &\quad A_m = 0,\ldots,A_M = 0 \end{aligned} \tag{25}$$

In which, by Eq. (25), the following relationships in terms of α and β can be written [28]:

$$\begin{cases} F_{n-1} = A_{n-1}(\alpha,\beta,T_f) + \\ \quad A_{n+1}(\alpha,\beta,T_f) - 2na_n = 0 \ \ ; n = 1,2,\ldots,m \\ F_{n-1} = A_{n-1}(\alpha,\beta,T_f) + \\ \quad A_{n+1}(\alpha,\beta,T_f) = 0 \ \ ; n = m+1,\ldots,M+1 \\ A_{M+1}(\alpha,\beta,T_f) = A_{M+2}(\alpha,\beta,T_f) = 0 \end{cases} \tag{26}$$

These equations are the approximations for the system dynamics.

Approximation of the initial and final conditions

Another property of Chebyshev polynomials is:
$T_n(-1) = (-1)^n$ and $T_n(1) = 1, n = 1,2,\ldots$

Therefore, the initial and final conditions of the Eq. (15) are replaced by [28]:

$$\begin{cases} F_{M+1}(\alpha) = \frac{1}{2}a_0 + \sum_{n=1}^{m}(-1)^n a_n - x_{-1} = 0 \\ F_{M+2}(\alpha) = \frac{1}{2}a_0 + \sum_{n=1}^{m} a_n - x_1 = 0 \end{cases} \tag{27}$$

Equality and inequality constraints

Equality and inequality constraints of the form [21]:

$$\begin{cases} C(U,\tau) = 0 \\ C(X,\tau) = 0 \qquad 0 \le \tau \le T_f \\ C(X,U,\tau) = 0 \end{cases}$$
$$\text{Ineq}: \tag{28}$$
$$\begin{cases} S(U,\tau) \le 0 \\ S(X,\tau) \le 0 \qquad 0 \le \tau \le T_f \\ S(X,U,\tau) \le 0 \end{cases}$$

can be handled like the system dynamics: $x_m(t)$ and/or $u_m(t)$ are substituted, $C(.)$ and/or $S(.)$ are expanded in the Chebyshev series, and the Chebyshev balance principle is applied. In order to use Chebyshev polynomials, by using Eq. (14) [21]:

$$\begin{cases} c(u(t),t) = 0 \\ c(x(t),t) = 0 \qquad -1 \le t \le 1 \\ c(x(t),u(t),t) = 0 \end{cases}$$
$$\text{Ineq}: \tag{29}$$
$$\begin{cases} s(u(t),t) \le 0 \\ s(x(t),t) \le 0 \qquad -1 \le t \le 1 \\ s(x(t),u(t),t) \le 0 \end{cases}$$

We define an auxiliary function $y(t)$ by [21]:

$$s(x(t),u(t),t) = -\frac{1}{2}y_m^2(t) \quad -1 \le t \le 1 \tag{30}$$

$y(t)$ is approximated by a Chebyshev series of order m with the unknown coefficients $\{r_n\}$, i.e., [21]:

$$y_m(t) = \frac{1}{2}r_0 T_0(t) + \sum_{n=1}^{m} r_n T_n(t) \tag{31}$$

Where $\rho = (r_0, r_1, .., r_m)$ refers to the unknown coefficients. By substituting Eq. (31) and Eq. (18) into Eq. (30) and applying the Chebyshev balance, the principle is applied.

$$D_n(\alpha,\beta,\rho) = 0 \quad (n = 0,1,2,\ldots,m_2) \tag{32}$$

Where $\{D_n\}$ are the coefficients of the Chebyshev series expansion of $s(x(t),u(t),t) + \frac{1}{2}y_m^2(t)$,as truncated after the term of order m_2. They are calculated very accurately by means of the following formula [21] [27]:

$$D_n(\alpha,\beta,T_f) = \frac{2}{k} \sum_{i=1}^{k} \begin{bmatrix} s(x_m(\cos\theta_i), u_m(\cos\theta_i), \cos\theta_i) + \\ \frac{1}{2}y_m^2(\cos\theta_i) \end{bmatrix} \cos n\theta_i; \tag{33}$$
$$n = 0,1,\ldots,m_2 \;; k > m_2 \;; \theta_i = \frac{2i-1}{k}.\frac{\pi}{2}$$

Approximation of the performance index

The optimal control problem has been reduced to a parameter optimization problem which can be stated as follows. Find α , β and T_f (if free) so that $J(\alpha,\beta,T_f)$ is minimal, or maximal, according to the constraints Eq. (26) and Eq. (27). Many mathematical programming techniques can be used to solve this constrained problem. The solution proposed by Lagrange is to form an unconstrained problem by appending the constraints to the performance index, by means of Lagrange multipliers. If we define [27]:

$$L(\alpha,\beta,\rho,T_f) = J(\alpha,\beta,T_f) + \sum_{v=0}^{M+1}\lambda_v F_v(\alpha,\beta,T_f) + \sum_{v=0}^{m_2}\lambda_{v+M+2} D_v(\alpha,\beta,\rho) \tag{34}$$

Where $\{\lambda_n\}$ represents the Lagrange multipliers and the necessary conditions for the stationary are given by:

$$\begin{cases} \frac{\partial L}{\partial a_\eta} = 0 \quad \eta = 0,1,\ldots,m \\ \frac{\partial L}{\partial b_\eta} = 0 \quad \eta = 0,1,\ldots,m \\ \frac{\partial L}{\partial r_\eta} = 0 \quad \eta = 0,1,\ldots,m \\ F_{v,j} = 0 \quad v = 0,1,\ldots,M+1 \quad j = \text{number of states} \\ D_{k,v} = 0 \quad v = 0,1,\ldots,m_2 \quad k = \text{number of Constraints} \end{cases} \tag{35}$$

Hence, the determining equations for the unknowns are [21]:

$$\begin{cases} \frac{\partial J}{\partial a_\eta} + \sum_v \lambda_v \frac{\partial F_v}{\partial a_\eta} = 0 \quad \eta = 0,1,\ldots,m \\ \frac{\partial J}{\partial b_\eta} + \sum_v \lambda_v \frac{\partial F_v}{\partial b_\eta} = 0 \quad \eta = 0,1,\ldots,m \\ \frac{\partial J}{\partial r_\eta} + \sum_v \lambda_v \frac{\partial F_v}{\partial r_\eta} = 0 \quad \eta = 0,1,\ldots,m \\ F_{v,j} = 0 , v = 0,1,\ldots,M+1 \quad j = \text{number of state} \\ D_{k,v} = 0, v = 0,1,\ldots,m_2 \quad k = \text{number of Constraints} \end{cases} \tag{36}$$

If the final time T_f is free, then an additional equation is given by:

$$\frac{\partial L}{\partial T_f} = 0 \tag{37}$$

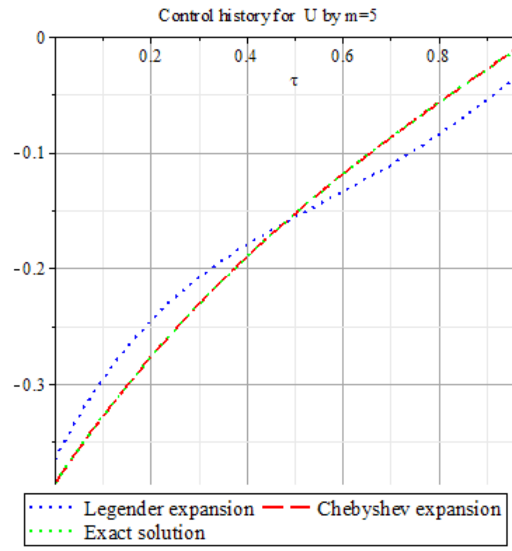


Fig. 1 The state variable for the Example 1 is illustrated.

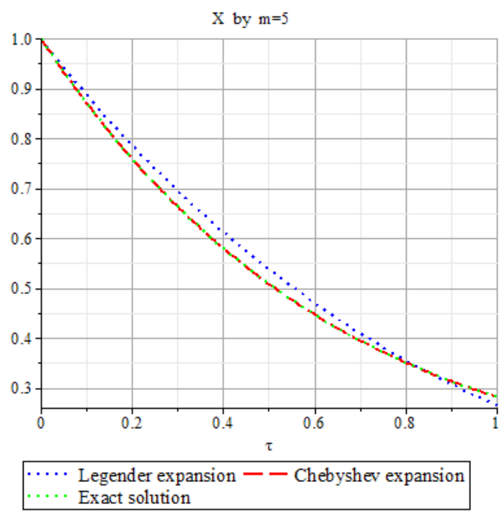


Fig. 2 The control variable for the Example 1 is illustrated.

Example 1

Find $U(\tau)$ that minimizes:

$$\begin{aligned} & I=\frac{1}{2}\int_0^1 (X^2+U^2)d\tau \\ & \frac{dX}{d\tau}=-X+U \quad 0\le\tau\le 1 \\ & X(0)=1 \end{aligned} \tag{38}$$

The exact solution of this example can be found by Pontryagin's maximum principle method as follows:

$$\begin{cases} U(\tau)=\dfrac{\sinh(\sqrt{2}(\tau-1))}{\sqrt{2}\cosh(\sqrt{2})+\sinh(\sqrt{2})} \\ X(\tau)=\dfrac{\sqrt{2}\cosh(\sqrt{2}(\tau-1))-\sinh(\sqrt{2}(\tau-1))}{\sqrt{2}\cosh(\sqrt{2})+\sinh(\sqrt{2})} \end{cases} \tag{39}$$

It can be seen that the Chebyshev and Legendre expansion with m = 5 can already offer a very precise result. A comparison between the results for the fifth-order Chebyshev, Legender approximation and the exact solution showed (Fig. 1 and Fig. 2) that the error in the Chebyshev series expansion was smaller than the Legender series expansion, because the Chebyshev series expansion was coincided to the exact solution. Therefore, the Chebyshev series expansion was used to solve the main Injection into Orbit Problem, as presented in the next section.

3 INJECTION IN TO ORBIT PROBLEM MODEL

Point-mass SLV definition

Consider an idealized point-mass SLV at the origin of inertial frame (x,y) at $t=0$, moving under the action of a constant propulsive force making a control angle $\beta(t)$ with the horizon. Obviously, the position and velocity vector of the vehicle will change due to the action of forces acting on it. The problem is to determine the optimal thrust-direction control to place the SLV at a given altitude with zero vertical velocity and the maximum horizontal velocity [31] [32].

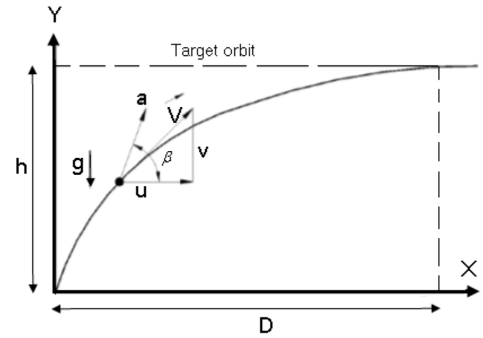


Fig. 3 Nomenclature for Injection into Orbit problem [5]

Dynamic model

The governing state-space equations for injection into the orbit problem are (see Figure 3):

$$\begin{cases} \dfrac{dx}{dt}=u \\ \dfrac{dy}{dt}=v \\ \dfrac{du}{dt}=a\,\mathrm{Cos}\beta \\ \dfrac{dv}{dt}=a\,\mathrm{Sin}\beta-g \end{cases} \tag{40}$$

Where g is the gravitational constant, h is the height of the target orbit, u is the horizontal component of velocity, v is the vertical component of velocity, x is satellite downrange, and y is satellite altitude.

3.3. Initial and terminal conditions

With its appropriate boundary conditions [31]:

$$u(t=0)=0\ ,\ v(t=0)=0$$
$$y(t=0)=0\ ,\ x(t=0)=0 \tag{41}$$

$$u(t=t_f)=U_f\ ,\ v(t=t_f)=0$$
$$y(t=t_f)=h\ ,\ x(t=t_f)=D \tag{42}$$

Non-dimensionalzing

Now, for a better physical understanding, the governing equations and the associated boundary conditions are non-dimensionalzed using a set of assumed reference parameters (u^*, y^*, t^*) [31]:

$$\bar{U}=\frac{u}{u^*}\ ,\bar{V}=\frac{v}{u^*}\ ,\ \bar{Y}=\frac{y}{y^*}\ ,\ \bar{X}=\frac{x}{y^*}$$
$$\tau=\frac{t}{t^*}\ ,\frac{d}{dt}=\frac{1}{t^*}\frac{d}{d\tau} \tag{43}$$

For optimal control problems where T_f is free, one usually utilizes the final time as a referencing condition for determining non-dimensional parameters. In addition, the reference parameters are [31]:

$$u^*=U_f\ ,y^*=h\ ,\ t^*=\frac{h}{U_f} \tag{44}$$

Now, by eliminating the first equation provided in the Eq. (40) and using the above non-dimensional state variable equations, the transformed equations become [31]:

$$\begin{cases}\dfrac{d\bar{U}}{d\tau}=\omega_1\, \mathrm{Cos}\beta\\ \dfrac{d\bar{V}}{d\tau}=\omega_1\, \mathrm{Sin}\beta-\omega_2\\ \dfrac{d\bar{Y}}{d\tau}=\omega_3\bar{V}\end{cases} \tag{45}$$

Where:

$$\omega_1=\frac{at^*}{u^*}\ ,\omega_2=\frac{g}{a}\ ,\ \omega_3=\frac{u^*t^*}{y^*} \tag{46}$$

With non-dimensional boundary conditions [31]:

$$\bar{u}(\tau=0)=0\ ,\ \bar{v}(\tau=0)=0\ ,\ \bar{y}(\tau=0)=0$$
$$\bar{u}(\tau=\tau_f)=1\ ,\ \bar{v}(\tau=\tau_f)=0\ ,\ \bar{y}(\tau=\tau_f)=1 \tag{47}$$

For a set of the assumed values of the parameters, (h, D, a, g) the geometrical parameters $(\omega_1, \omega_2, \omega_3)$ can be determined. For example, if $h=D=a=1$, and $g=0.33$, the geometrical parameters are $\omega_1=\omega_3=1\ \ \omega_2=0.33$.

4 TRAJECTORY OPTIMIZATION PROBLEM DESCRIPTION

Cost function

Different cost functions can be formulated for SLV trajectory profile optimization problem. One of the most common is the maximization of the final payload for a fixed SLV gross launch weight, minimization of fuel to be expended in flight and maximization of injection velocity for a given altitude or any of the significant orbital parameters. The payload optimization problem of directly maximizing the final mass has been dealt in the literature extensively [7]. However, for the present research effort, the problem was determining the control action of $\beta=\beta(t)$ required for transferring the point-mass SLV to the specified orbit with minimum time, so that the performance measure could be defined as:

$$I=\int_0^1 d\tau \tag{48}$$

Computational procedures

The task is now to use the equations presented in order to find the control history $\beta(t)$ which will propagate the initial state values, using the state differential equations and meeting specified point conditions while reaching the minimum amount of time. This formulation results in a problem of open loop optimal control. Transforming τ to the interval $[-1,1]$:

$$I=\frac{T_f}{2}\int_{-1}^{1} dt \tag{49}$$

$$\begin{cases}\dfrac{du}{dt}=\dfrac{T_f}{2}\mathrm{Cos}\beta\\ \dfrac{dv}{dt}=\dfrac{T_f}{2}(\mathrm{Sin}\,\beta-0.33)\\ \dfrac{dy}{dt}=\dfrac{T_f}{2}v\end{cases} \tag{50}$$

$$u(t=-1)=0\ ,v(t=-1)=0\ ,\ y(t=-1)=0$$
$$u(t=1)=1\ ,\ v(t=1)=0\ ,\ y(t=1)=1 \tag{51}$$

We first write analytically the Chebyshev series expansion of the third order $m=3$. Choosing $M=m-1=2$ for the first and second equations of Eq. (50) and $M=m=3$ for the third equation of Eq. (50), we come to

$$\begin{cases}\beta(t)=\dfrac{1}{2}a_0T_0(t)+\displaystyle\sum_{n=1}^{3}a_nT_n(t)\\ v(t)=\dfrac{1}{2}b_0T_0(t)+\displaystyle\sum_{n=1}^{3}b_nT_n(t)\\ y(t)=\dfrac{1}{2}c_0T_0(t)+\displaystyle\sum_{n=1}^{3}c_nT_n(t)\\ u(t)=\dfrac{1}{2}e_0T_0(t)+\displaystyle\sum_{n=1}^{3}e_nT_n(t)\end{cases} \tag{52}$$

The following expressions present the Taylor series for $\mathrm{Sin}\beta$ and $\mathrm{Cos}\beta$ about the base point t=0:

$$\begin{aligned} &\mathrm{Sin}\beta(t) \approx \beta(t) - \frac{(\beta(t))^3}{3!} \\ &\mathrm{Cos}\beta(t) \approx 1 - \frac{(\beta(t))^2}{2!} \end{aligned} \tag{53}$$

The unknown's $\alpha \equiv (a_0, a_1, a_2, \ldots, a_m)$, $\beta \equiv (b_0, b_1, b_2, \ldots, b_m)$, $\gamma \equiv (c_0, c_1, c_2, \ldots, c_m)$ and $\eta \equiv (e_0, e_1, e_2, \ldots, e_m)$ must satisfy the constraints:

$$\begin{cases} F_{0,1} \equiv A_{0,1} + A_{2,1} - 2b_1 = 0 \\ F_{1,1} \equiv A_{1,1} + A_{3,1} - 4b_2 = 0 \\ F_{2,1} \equiv A_{2,1} + A_{4,1} - 6b_3 = 0 \\ A_{4,1} = 0 \end{cases} \tag{54}$$

$$\begin{cases} F_{0,2} \equiv A_{0,2} + A_{2,2} - 2c_1 = 0 \\ F_{1,2} \equiv A_{1,2} + A_{3,2} - 4c_2 = 0 \\ F_{2,2} \equiv A_{2,2} + A_{4,2} - 6c_3 = 0 \\ A_{4,2} = 0 \end{cases} \tag{55}$$

$$\begin{cases} F_{0,3} \equiv A_{0,3} + A_{2,3} - 2e_1 = 0 \\ F_{1,3} \equiv A_{1,3} + A_{3,3} - 4e_2 = 0 \\ F_{2,3} \equiv A_{2,3} + A_{4,3} - 6e_3 = 0 \\ F_{3,3} \equiv A_{3,3} + A_{5,3} = 0 \\ A_{4,3} = A_{5,3} = 0 \end{cases} \tag{56}$$

With its appropriate boundary conditions:

$$\begin{cases} F_{3,1} \equiv \frac{1}{2} b_0 + \sum_{n=1}^{3} (-1)^n b_n - 0 = 0 \\ F_{4,1} \equiv \frac{1}{2} b_0 + \sum_{n=1}^{3} b_n - 0 = 0 \end{cases} \tag{57}$$

$$\begin{cases} F_{3,2} \equiv \frac{1}{2} c_0 + \sum_{n=1}^{3} (-1)^n c_n - 0 = 0 \\ F_{4,2} \equiv \frac{1}{2} c_0 + \sum_{n=1}^{3} c_n - 1 = 0 \end{cases} \tag{58}$$

$$\begin{cases} F_{4,3} \equiv \frac{1}{2} e_0 + \sum_{n=1}^{3} (-1)^n e_n - 0 = 0 \\ F_{5,3} \equiv \frac{1}{2} e_0 + \sum_{n=1}^{3} e_n - 1 = 0 \end{cases} \tag{59}$$

With $h = 0$ and $g = 0.5T_f$ **,** the approximate performance index to be minimized is:

$$\begin{aligned} L(\alpha, \beta, \gamma, \eta, T_f) &= T_f \\ &+ \sum_{v=0}^{3m+6} \lambda_v F_v(\alpha, \beta, \gamma, \eta T_f) \end{aligned} \tag{60}$$

Computational results

Eq. (35) or Eq. (36) and Eq. (54) to Eq. (59) give (7m+12) the determining equations for (7m+12) the unknowns $(T_f, \alpha_i, \beta_i, \gamma_i, \eta_i, \lambda_j)\ i = 1,2,..,m\ ,\ j = 0,1,2,..,3m+6$.

These nonlinear equations are solved with the iterative Newton method. It should be emphasized that the following solution (Table 1) is readily obtained for m=9 and m=11. For m = 9 and K = 20, the final results were obtained after 10.4 s computer time, whereas for m = 11 and K = 25, running time was 14.5 s. Hence, for m = 9, the parameter K was increased from 20 to 25, and it had no effect on the first five decimals of the results while the computer time was increased from 10.4 s to 12.6 s. It can be seen that with these coefficients, boundary conditions have been satisfied (Table 2).

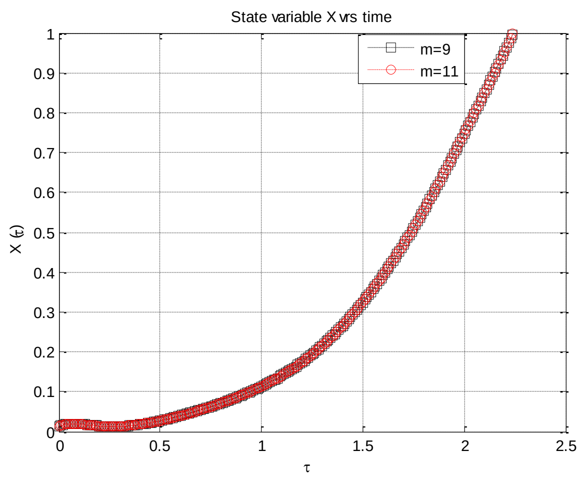
Table 1 Chebyshev coefficients for m=9 and m=11

Ninth-order Chebyshev approximation (m=9, K=20)	
T_f	2.21145 sec
$\beta(t)$	$a_0 = 0.66752,\ a_1 = -1.47354,\ a_2 = -0.38009,$ $a_3 = 0.17378,\ a_4 = 0.18288,\ a_5 = 0.02218,$ $a_6 = -0.07004,\ a_7 = -0.02725,$ $a_8 = 0.01023,\ a_9 = 0.02838$
$v(t)$	$b_0 = 0.65853,\ b_1 = 0.06694,\ b_2 = -0.35734,$ $b_3 = -0.09710,\ b_4 = 0.01772,\ b_5 = 0.02952,$ $b_6 = 0.00364,\ b_7 = -0.00922,$ $b_8 = -0.00745,\ b_9 = 0.00091$
$y(t)$	$c_0 = 0.93114,\ c_1 = 0.56160,\ c_2 = 0.04516,$ $c_3 = -0.07672,\ c_4 = -0.01970,\ c_5 = -0.00140,$ $c_6 = 0.00239,\ c_7 = 0.00111,$ $c_8 = -0.00069,\ c_9 = -0.00067$
$u(t)$	$e_0 = 0.85021,\ e_1 = 0.52997,\ e_2 = 0.11816,$ $e_3 = -0.01296,\ e_4 = -0.02822,\ e_5 = -0.01189,$ $e_6 = 0.00314,\ e_7 = 0.00420,$ $e_8 = 0.00237, e_9 = -0.00174$
Eleventh-order Chebyshev approximation (m=11, K=25)	
T_f	2.21131 sec
$\beta(t)$	$a_0 = 0.67372,\ a_1 = -1.48189,\ a_2 = -0.37478,$ $a_3 = 0.16386,\ a_4 = 0.18885,\ a_5 = 0.01151,$ $a_6 = -0.06193,\ a_7 = -0.04071,\ a_8 = 0.01969,$ $a_9 = 0.01068,\ a_{10} = 0.01371,\ a_{11} = -0.02895$
$v(t)$	$b_0 = 0.66447,\ b_1 = 0.07914,\ b_2 = -0.35337,$ $b_3 = -0.08785,\ b_4 = 0.01959,\ b_5 = 0.037469,$ $b_6 = 0.00537,\ b_7 = -0.00145, b_8 = -0.00633,$ $b_9 = 0.00813,\ b_{10} = 0.00170,\ b_{11} = 0.01058$
$y(t)$	$c_0 = 0.93104,\ c_1 = 0.56422,\ c_2 = 0.04475,$ $c_3 = -0.07433,\ c_4 = -0.02057,\ c_5 = 0.00083,$ $c_6 = 0.00154,\ c_7 = 0.00321, c_8 = -0.00154,$ $c_9 = 0.00135,\ c_{10} = -0.00125,\ c_{11} = 0.00248$
$u(t)$	$e_0 = 0.84843,\ e_1 = 0.53193,\ e_2 = 0.11633,$ $e_3 = -0.01151,\ e_4 = -0.02997,\ e_5 = -0.01031,$ $e_6 = 0.00131,\ e_7 = 0.00610, e_8 = 0.00072,$ $e_9 = 0.00074,\ e_{10} = -0.00195,\ e_{11} = 0.00045$

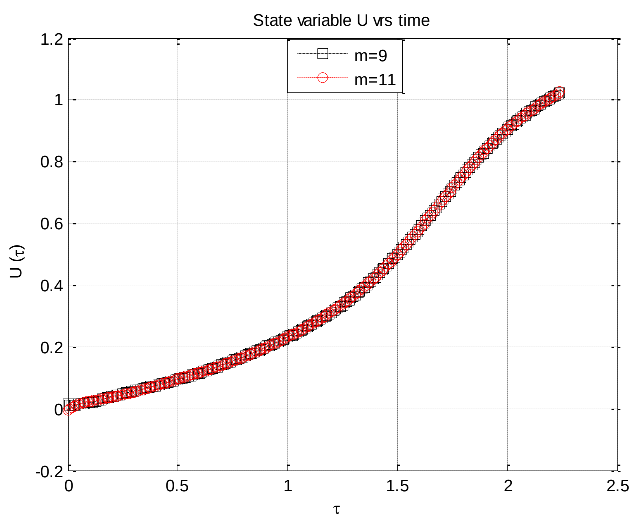
Table 2 shows that the errors on the boundary conditions are negligible and our solutions for point-mass SLV problem are sufficiently accurate. Hence, it can be concluded that Eleventh-order Chebyshev series expansion gives better results. Therefore, we find that we can get more accurate results by increasing m.

Table 2 The errors in the boundary conditions for m=9 and m=11

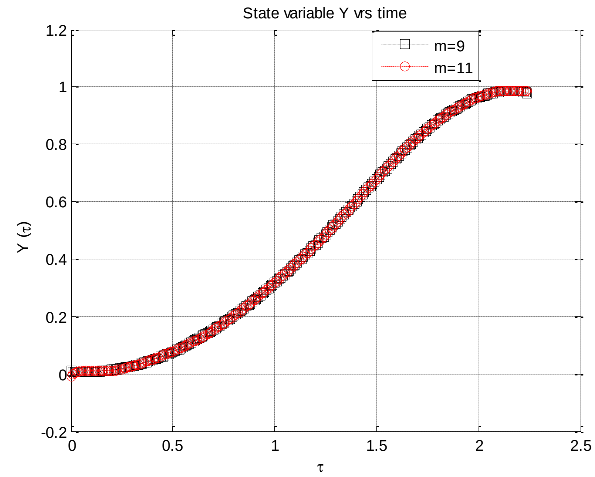
m	Ini or fina	value	|Error|
9	v (-1)	-0.0052075	5.207500e-003
	v (1)	-0.0321069	3.210690e-002
	u (-1)	-0.0130064	1.300640e-002
	u (1)	1.0282297	2.822970e-002
	y (-1)	0.0088051	8.805100e-003
	y (1)	0.9766552	2.334480e-002
11	v (-1)	-0.0046831	4.683100e-003
	v (1)	0.0212223	2.122200e-002
	u (-1)	-0.0065543	6.554300e-003
	u (1)	1.0281744	2.817439e-002
	y (-1)	-0.0083327	8.332700e-003
	y (1)	0.9862312	1.376880e-002



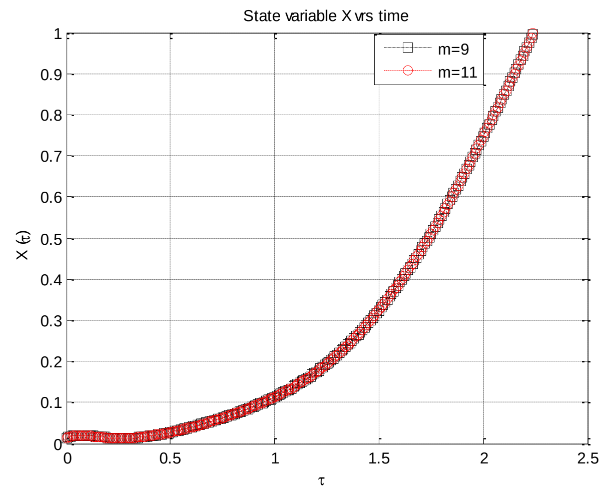
(a)



(b)



(c)



(d)

Fig. 4 State variables for the point-mass SLV problem are illustrated. The comparisons between ninth-order and eleven-order Chebyshev series expansion are illustrated.

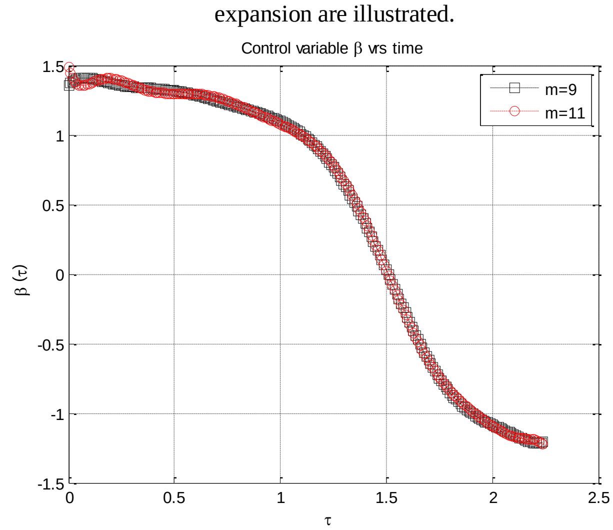


Fig. 5 Control variable for the point-mass SLV problem. The comparison between the ninth-order and eleven-order Chebyshev series expansion is illustrated.

Also, there was an excellent agreement between our results and Ref[31] solutions in the magnitude of the final time. In the Ref[31], the final time or cost function was found to be 2.2347 second and in this research, it was 2.21131 second for m=11 and 2.21145 second for m=9. Also, by using figures in [31]'s work, it could be seen that there was excellent agreement between our time histories (for state and control variables) and [31]'s work.

5 CONCLUSION

In this research, single objective trajectory optimization of a point-mass SLV was conducted using Chebyshev series expansion approach. It was a minimum flight time problem. The main feature of using Chebyshev (or Legender) series expansion was that it converted the differential expressions arising from the system dynamics and the performance index into algebraic equations. Thus, the trajectory optimization of a point-mass SLV was reduced to a problem of solving a system of algebraic equations. Hence, Example1 denoted that we could get more accurate results by increasing the order of series expansion. Moreover, the effectiveness of the Chebyshev approximation with respect to Legender approximation was proved. Also, the suggested strategy is not complicated and can be implemented without too much fuss.

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