Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.2, No.2, (2023), 113-126
DOI: http://doi.org/10.30495/TFSS.2023.1972768.1055

Article Type: Original Research Article

Intuitionistic Fuzzy Modular Spaces

Tayebe Lal Shateri*

Abstract. After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalizations of the notion of fuzzy sets. There are many viewpoints on the notion of metric space in fuzzy topology. One of the most important problems in fuzzy topology is obtaining an appropriate concept of fuzzy metric space. This problem has been investigated by many authors from different points of view. Atanassov gives the concept of the intuitionistic fuzzy set as a generalization of the fuzzy set. Park introduced the notion of intuitionistic fuzzy metric space as a natural generalization of fuzzy metric spaces due to George and Veeramani. This paper introduces the concept of intuitionistic fuzzy modular space. Afterward, a Hausdorff topology induced by a δ-homogeneous intuitionistic fuzzy modular is defined and some related topological properties are also examined. After giving the fundamental definitions and the necessary examples, we introduce the definitions of intuitionistic fuzzy boundedness, intuitionistic fuzzy compactness, and intuitionistic fuzzy convergence, and obtain several preservation properties and some characterizations concerning them. Also, we investigate the relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. Finally, we prove some known results of metric spaces including Baires theorem and the Uniform limit theorem for intuitionistic fuzzy modular spaces.

AMS Subject Classification 2020: 42C15; 06D22
Keywords and Phrases: Fuzzy set, Modular space, Fuzzy modular space, Intuitionistic fuzzy modular space.

1 Introduction

The notion of fuzzy sets was introduced by Zadeh [20] in 1965 and there are many viewpoints on the notion of metric space in fuzzy topology. The concept of fuzzy topology may have very important applications in quantum particle physics, [3, 4]. One of the most important problems in fuzzy topology is obtaining an appropriate concept of a fuzzy metric space. This problem has been investigated by many authors from different points of view. Kramosil and Michálek [11] introduced the concept of a fuzzy metric space, which can be regarded as a generalization of the probabilistic metric space. Afterward, Grabiec [5] defined the fuzzy metric space's completeness and extended the Banach contraction theorem to the complete fuzzy metric spaces. Next, George and Veeramani [6] modified the definition of the Cauchy sequence introduced by Grabiec. Atanassov [1] gave the concept of an intuitionistic fuzzy set as a generalization of a fuzzy set. Park [17] introduced the notion of an intuitionistic fuzzy metric space as a natural generalization of a fuzzy metric space due to George and Veeramani. He proved Baire's theorem and the Uniform limit theorem for these spaces. For more details on intuitionistic fuzzy metric space and related results, we refer the reader to [2, 18].
The concept of a modular space was founded by Nakano [14] and developed by Luxemburg [12]. Then, Musielak and Orlicz [13] redefined and generalized the notion of modular space. A real function ρ on an

*Corresponding Author: Tayebe Lal Shateri, Email: t.shateri@hsu.ac.ir, ORCID: 0000-0002-5767-8619
Received: 15 November 2022; **Revised**: 29 May 2023; **Accepted**: 10 June 2023; **Available Online**: 10 June 2023; **Published Online**: 7 November 2023.

How to cite: Shateri TL. Intuitionistic Fuzzy Modular Spaces. *Trans. Fuzzy Sets Syst.* 2023; 2(2): 113-126. DOI: http://doi.org/10.30495/TFSS.2023.1972768.1055

arbitrary vector space $\mathcal{X}$ is called a *modular* if for arbitrary $x, y \in \mathcal{X}$ the following conditions hold:
(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.
A *modular space* $\mathcal{X}_\rho$ is defined as $\mathcal{X}_\rho = \{x \in \mathcal{X} : \quad \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$.

Based on the definition of modular space, Kozlowski [8, 9] introduced a modular function space. In the sequel, Kozlowski and Lewicki [10] considered the problem of analytic extension of measurable functions in modular function spaces and discussed some extension properties by means of polynomial approximation. Afterward, Kilmer and Kozlowski [7] studied the existence of best approximations in modular function spaces by elements of sublattices. Nourouzi [15] proposed probabilistic modular spaces based on the theory of modular spaces and in [16] he extended the well-known Baire's theorem to probabilistic modular spaces by using a special condition. Shen and Chen [19] introduced the notion of fuzzy modular space by using continuous t-norm and continuous t-conorm.

The concept of intuitionistic fuzzy modular space is first proposed in this paper. By using some ideas of [17, 19] we introduce the concept of an intuitionistic fuzzy modular space and give a Hausdorff topology in this space. We investigate some topological properties and the existence of a relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. The paper is organized as follows.
First, we recall the fundamental definitions and the necessary examples of an intuitionistic fuzzy metric space. In section 2, following the idea of fuzzy modular spaces and the definition of an intuitionistic fuzzy metric space, we give a new concept named intuitionistic fuzzy modular space and give two examples to show that there does not exist a direct relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. In section 3, a Hausdorff topology induced by a δ-homogeneous intuitionistic fuzzy modular is defined, and several theorems on μ-ν-completeness of the intuitionistic fuzzy modular space are given. Finally, the well-known Baire's theorem and the uniform limit theorem are extended to intuitionistic fuzzy modular spaces.

Definition 1.1. [18] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm if it satisfies the following conditions:

1. $*$ is commutative and associative;
2. $*$ is continuous;
3. $a * 1 = a$ for every $a \in [0,1]$;
4. $a * b \leq c * d$ whenever $a \leq c, b \leq d$, and $a, b, c, d \in [0,1]$.

The common examples of a t-norm are as follows:
(1) $a * b = ab$ (2) $a *_M b = \min\{a, b\}$ (3) $a * b = \max\{0, a + b - 1\}$.

Definition 1.2. [18] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-conorm if it satisfies the following conditions:

1. $\diamond$ is commutative and associative;
2. $\diamond$ is continuous;
3. $a \diamond 0 = a$ for every $a \in [0,1]$;
4. $a \diamond b \leq c \diamond d$ whenever $a \leq c, b \leq d$ and $a, b, c, d \in [0,1]$.

Some examples of a t-conorm are as follows.
(1) $a \diamond b = a + b - ab$ (2) $a \diamond_M b = \max\{a, b\}$ (3) $a \diamond b = \min\{1, a + b\}$.
By properties of t-norm and t-conorm, we get the following lemma.

Lemma 1.3. *[6] (i) If $a, b \in (0, 1)$ with $a > b$, then there exist $c, d \in (0, 1)$ such that $a * c \leq b$ and $a \geq d \diamond b$.*
(ii) If $a \in (0, 1)$, then there exist $c, d \in (0, 1)$ such that $c * c \geq a$ and $a \geq d \diamond d$.

Now, we recall the definition of an intuitionistic fuzzy metric space.

Definition 1.4. [17] A 5-tuple $(\mathcal{X}, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if $\mathcal{X}$ is a real or complex vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $\mathcal{X}^2 \times (0, \infty)$ such that for all $x, y, z \in \mathcal{X}$ and $s, t > 0$ the followings hold:

1. $M(x, y, t) + N(x, y, t) \leq 1$,
2. $M(x, y, t) > 0$,
3. $M(x, , y, t) = 1$ if and only if $x = y$,
4. $M(x, y, t) = M(y, x, t)$,
5. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
6. $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous,
7. $N(x, y, t) > 0$,
8. $N(x, y, t) = 0$ if and only if $x = y$,
9. $N(x, y, t) = N(y, x, t)$,
10. $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,
11. $N(x, y, .) : (0, \infty) \to (0, 1]$ is continuous.

(M, N) is called an intuitionistic fuzzy metric on $\mathcal{X}$.

Example 1.5. [17, Example 2.8] Let $(\mathcal{X}, d)$ be a metric space. Denote $a * b = ab$ and $a \diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $\mathcal{X}^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}$$

for all $h, k, m, n \in \mathbb{R}^+$. Then $(\mathcal{X}, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space.

2 Intuitionistic Fuzzy Modular Spaces

In this section, we introduce the concept of an intuitionistic fuzzy modular space by using continuous t-norm and continuous t-conorm. We investigate the relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric.

Definition 2.1. A 5-tuple $(\mathcal{X}, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy modular space if $\mathcal{X}$ is a real or complex vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, ν are fuzzy sets on $\mathcal{X} \times (0, \infty)$ such that for all $x, y, z \in \mathcal{X}$, $s, t > 0$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ followings hold:

1. $\mu(x,t)+\nu(x,t)\leq 1$,
2. $\mu(x,t)>0$,
3. $\mu(x,t)=1$ if and only if $x=0$,
4. $\mu(x,t)=\mu(-x,t)$,
5. $\mu(\alpha x+\beta y,s+t)\geq \mu(x,s)*\mu(y,t)$,
6. $\mu(x,.):(0,\infty)\to(0,1]$ is continuous,
7. $\nu(x,t)>0$,
8. $\nu(x,t)=0$ if and only if $x=0$,
9. $\nu(x,t)=\nu(-x,t)$,
10. $\nu(\alpha x+\beta y,s+t)\leq \nu(x,s)\diamond\nu(y,t)$,
11. $\nu(x,.):(0,\infty)\to(0,1]$ is continuous.

Then (μ,ν) is called an intuitionistic fuzzy modular or intuitionistic $\mathfrak{F}$-modular on $\mathcal{X}$. The 5-tuple $(\mathcal{X},\mu,\nu,*,\diamond)$ is called δ-homogeneous, where $\delta\in(0,1]$, if for each $x\in\mathcal{X}$, $t>0$ and $\lambda\in\mathbb{R}-\{0\}$,

$$\mu(\lambda x,t)=\mu\Big(x,\frac{t}{|\lambda|^{\delta}}\Big),\ \nu(\lambda x,t)=\nu\Big(x,\frac{t}{|\lambda|^{\delta}}\Big).$$

Remark 2.2. (i) If $(\mathcal{X},\mu,*)$ is a $\mathfrak{F}$-modular space, then $(\mathcal{X},\mu,1-\mu,*,\diamond)$ is an intuitionistic $\mathfrak{F}$-modular space such that for any $a,b\in[0,1]$, $a\diamond b=1-((1-a)*(1-b))$.
(ii) In intuitionistic $\mathfrak{F}$-modular space $(\mathcal{X},\mu,\nu,*,\diamond)$, for all $x,y\in\mathcal{X}$, $\mu(x,y,.)$ is non-decreasing and $\nu(x,y,.)$ is non-increasing.

Example 2.3. Let $(\mathcal{X},\rho)$ be a modular space. Take $a*b=ab$ and $a\diamond b=\min\{1,a+b\}$, for all $a,b\in[0,1]$, and define fuzzy sets μ_ρ and ν_ρ on $\mathcal{X}\times(0,\infty)$ as follows:

$$\mu_\rho(x,t)=\frac{ht^n}{ht^n+m\rho(x)},\ \nu_\rho(x,t)=\frac{\rho(x)}{kt^n+m\rho(x)},$$

for all $h,k\in\mathbb{R}^+$ and $m,n\in\mathbb{N}$. Then $(\mathcal{X},\mu,\nu,*,\diamond)$ is an intuitionistic $\mathfrak{F}$-modular space. We investigate condition (5) in Definition 2.1. For this let $\alpha,\beta\geq 0$ with $\alpha+\beta=1$, since ρ is modular, we have

$$\rho(\alpha x+\beta y)\leq\rho(x)+\rho(y), \tag{1}$$

for all $x,y\in\mathcal{X}$. Hence

$$\begin{aligned}\mu(x,s)*\mu(y,t)&=\frac{hs^n}{hs^n+m\rho(x)}*\frac{ht^n}{ht^n+m\rho(y)}\\&=\frac{h^2s^nt^n}{(hs^n+m\rho(x))(ht^n+m\rho(y))}\\&\leq\frac{hs^nt^n}{hs^nt^n+m(t^n\rho(x)+s^n\rho(y))}.\end{aligned}$$

Without loss of generality, we assume that $t \leq s$. Then by using (1) we get

$$\begin{aligned} \mu(x,s) * \mu(y,t) &\leq \frac{hs^n}{hs_n + m\rho(\alpha x + \beta y)} \\ &\leq \frac{h(s+t)^n}{h(s+t)^n + m\rho(\alpha x + \beta y)} \\ &= \mu(\alpha x + \beta y, s+t). \end{aligned}$$

Remark 2.4. In Example 2.3, by taking $h = k = m = n = 1$, we get

$$\mu_\rho(x,t) = \frac{t}{t+\rho(x)},\ \nu_\rho(x,t) = \frac{\rho(x)}{t+\rho(x)}.$$

This intuitionistic $\mathfrak{F}$-modular space is called the standard intuitionistic $\mathfrak{F}$-modular space.

It should be noted that, in general, an intuitionistic fuzzy modular and an intuitionistic fuzzy metric do not necessarily induce mutually a metric when the triangular norm is the same one. In essence, the intuitionistic fuzzy modular and intuitionistic fuzzy metric can be viewed as two different characterizations for the same set. Next, we give two examples to show that there does not exist a direct relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. In fact, the intuitionistic fuzzy modular and the intuitionistic fuzzy metric can be viewed as two different characterizations for the same set. The following examples are the motivation of [19, Example 8, Example 9] in the view of intuitionistic fuzzy modular spaces.

Example 2.5. Let $\mathcal{X} = \mathbb{R}$ and put $\rho(x) = |x|$, then ρ is modular on $\mathcal{X}$. Put $a * b = \min\{a, b\}$, and $a \diamond b = 1 - ((1-a) * (1-b))$ or $a \diamond b = \max\{a, b\}$. For every $t \in (0, \infty)$ and $x \in \mathcal{X}$, define $\mu(x,t) = \frac{t}{t+|x|}$. Then [19, Example 8] implies that $(\mathcal{X}, \mu, *)$ is an $\mathfrak{F}$-modular space and so by Remark 2.2, $(\mathcal{X}, \mu, 1-\mu, *, \diamond)$ is an intuitionistic $\mathfrak{F}$-modular space.
However, if we set

$$M(x,y,t) = \mu(x-y,t) = \frac{t}{t+|x-y|}, \text{ and } \quad N(x,y,t) = \frac{|x-y|}{t+|x-y|},$$

then [17, Remark 2.11] implies that (M, N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$.

Example 2.6. Let $\mathcal{X} = \mathbb{R}$. Take t-norm $a * b = \min a, b$ and t-conorm $a \diamond b = a + b - ab$. For every $x, y \in \mathcal{X}$ and $t \in (0, \infty)$, we define

$$M(x,y,t) = \begin{cases} 1, & x = y \\ \frac{1}{2}, & x \neq y, x, y \in \mathbb{Z} \\ \frac{1}{4}, & x \in \mathbb{Z}, y \in \mathbb{R}\backslash\mathbb{Z} \ \ or \ \ x \in \mathbb{R}\backslash\mathbb{Z}, y \in \mathbb{Z} \\ \frac{1}{4}, & x \neq y, x, y \in \mathbb{R}\backslash\mathbb{Z}, \end{cases}$$

and

$$M(x,y,t) = \begin{cases} 1, & x = y \\ \frac{1}{2}, & x \neq y, x, y \in \mathbb{Z} \\ \frac{1}{4}, & x \in \mathbb{Z}, y \in \mathbb{R}\backslash\mathbb{Z} \ \ or \ \ x \in \mathbb{R}\backslash\mathbb{Z}, y \in \mathbb{Z} \\ \frac{1}{4}, & x \neq y, x, y \in \mathbb{R}\backslash\mathbb{Z} \end{cases}$$

It can easily be shown that $(M, N, *, \diamond)$ is an intuitionistic fuzzy metric on $\mathcal{X}$.
Now, set

$$\mu(x,t) = \begin{cases} 1, & x = 0 \\ \frac{1}{2}, & x \in \mathbb{Z}\backslash\{0\} \\ \frac{1}{4}, & x \in \mathbb{R}\backslash\mathbb{Z} \end{cases}, \text{ and } \quad \nu(x,t) = \begin{cases} 0, & x = 0 \\ \frac{1}{4}, & x \in \mathbb{Z}\backslash\{0\} \\ \frac{1}{2}, & x \in \mathbb{R}\backslash\mathbb{Z}. \end{cases}$$

If we take $\alpha = \frac{\sqrt{2}}{2}$, $\beta = 1-\alpha$, $x \neq y$, and $x, y \in \mathbb{Z}$, then $\alpha x + \beta y \in \mathbb{R}\backslash\mathbb{Z}$. Hence for each $s, t > 0$, we have $\mu(\alpha x + \beta y, s+t) = \frac{1}{4}$, but

$$\mu(x,s) * \mu(y,t) = \frac{1}{2}.$$

Also $\nu(\alpha x + \beta y, s+t) = \frac{1}{2}$, but

$$\nu(x,s) \diamond \nu(y,t) = \frac{1}{4}.$$

Therefore (μ, ν) is not an intuitionistic fuzzy modular on $\mathcal{X}$.

3 Topology Induced by an δ-homogeneous Intuitionistic Fuzzy Modular Space

In this section, we define a topology induced by a δ-homogeneous intuitionistic $\mathfrak{F}$-modular and investigate some topological properties in δ-homogeneous intuitionistic $\mathfrak{F}$-modular space. The results obtained in this section are an extension of the results presented in [19] to intuitionistic fuzzy modular spaces.

Definition 3.1. Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be an intuitionistic $\mathfrak{F}$-modular space and let $x \in \mathcal{X},\ r \in (0,1)$ and $t > 0$. Then the μ-ν-ball with center x and radius r with respect to t is defined as

$$B(x,r,t) = \{y \in X : \mu(x-y,t) > 1-r,\ \nu(x-y,t) < r\}.$$

Let $E \subseteq \mathcal{X}$. An element $x \in E$ is called a μ-ν-interior point of E if there exist $r \in (0,1)$ and $t > 0$ such that $B(x,r,t) \subseteq E$. We say that E is a μ-ν-open set in $\mathcal{X}$ if and only if every element of E is a μ-ν-interior point. Note that each open set in an intuitionistic $\mathfrak{F}$-modular space is not a μ-ν-ball in general.

Example 3.2. Let $\mathcal{X} = \mathbb{R}$ and let ρ, μ, $*$ and $\diamond$ be as in Example 2.5.
Consider $V = \{x \in \mathbb{R} : 0 < x < 1\} \cup \{x \in \mathbb{R} : 1 < x < 2\}$. Then V is an open set in $(\mathbb{R}, \mu, 1-\mu, *, \diamond)$, but it is not a μ-$(1-\mu)$-ball. In fact, the μ-$(1-\mu)$-ball in $(\mathbb{R}, \mu, 1-\mu, *, \diamond)$ with center x and radius r is as follows.

$$\begin{aligned} B(x,r,t) &= \{y \in \mathbb{R} : \frac{t}{t+|x-y|} > 1-r,\ \frac{|x-y|}{t+|x-y|} < r\} \\ &= \{y \in \mathbb{R} : |x-y| < \frac{r}{1-r} t\}. \end{aligned}$$

Theorem 3.3. *Each μ-ν-ball in a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space is an open set.*

Proof. Let $B(x,r,t)$ be a μ-ν-ball and $y \in B(x,r,t)$. Then

$$\mu(x-y,t) > 1-r,\ \nu(x-y,t) < r.$$

Put $t = 2t_1$. Since $\mu(x-y,.)$ and $\nu(x-y,.)$ are continuous, there exists $\varepsilon y > 0$ such that

$$\mu(x-y, \frac{t_1 - \varepsilon_y}{2^{\delta-1}}) > 1-,\ \nu(x-y, \frac{t_1-\varepsilon_y}{2^{\delta-1}}) < r.$$

For some $\varepsilon > 0$ with $\frac{t_1-\varepsilon}{2^{\delta-1}} > 0$ and $\frac{\varepsilon}{2^{\delta-1}} \in (0, \varepsilon_y)$, put $r_0 = \mu(x-y, \frac{t_1-\varepsilon}{2^{\delta-1}})$. Since $r_0 > r-1$, there exists $s \in (0,1)$ such that $r_0 > 1-s > 1-r$, by Lemma 1.3, we can choose $r_1 \in (0,1)$ such that

$$r_0 *\ r_0 \geq 1-s,\ (1-r_0) \diamond (1-r_0) \leq s.$$

Put $r_3 = \max\{r_1, r_2\}$. We show that $B(y, 1-r_3, \frac{\varepsilon}{2^{\delta-1}}) \subseteq B(x, r, 2t_1)$. Suppose that $z \in B(y, 1-r_3, \frac{\varepsilon}{2^{\delta-1}})$ then

$$\mu(y-z, \frac{\varepsilon}{2^{\delta-1}}) > r_3, \ \nu(y-z, \frac{\varepsilon}{2^{\delta-1}}) < 1-r_3.$$

Therefore

$$\begin{aligned}\mu(x-z,t) &= \mu(x-z, 2t_1) \geq \mu(2(x-y), 2(t_1-\varepsilon)) * \mu(2(y-z), 2\varepsilon)\\ &= \mu(x-y, \frac{t_1-\varepsilon}{2^{\delta-1}}) * \mu(y-z, \frac{\varepsilon}{2^{\delta-1}})\\ &\geq r_0 * r_1 \geq 1-s > 1-r,\end{aligned}$$

and

$$\begin{aligned}\nu(x-z,t) &= \nu(x-z, 2t_1) \leq \nu(2(x-y), 2(t_1-\varepsilon)) \diamond \nu(2(y-z), 2\varepsilon)\\ &= \nu(x-y, \frac{t_1-\varepsilon}{2^{\delta-1}}) \diamond \nu(y-z, \frac{\varepsilon}{2^{\delta-1}})\\ &< (1-r_0) \diamond (1-r_3) \leq (1-r_0) \diamond (1-r_2) \leq s < r.\end{aligned}$$

Therefore $z \in B(x, r, t)$ and hence $B(y, 1-r_3, \frac{\varepsilon}{2^{\delta-1}}) \subseteq B(x, r, t)$. □

Now, we define a topology on a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space.

Definition 3.4. Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space. Define

$$\tau_{(\mu,\nu)} = \{V \subseteq \mathcal{X} : \forall x \in V, \exists t > 0, r \in (0,1); B(x, r, t) \subseteq V\}.$$

Then $\tau_{(\mu,\nu)}$ is a topology on $\mathcal{X}$.

Remark 3.5. Since the family of μ-ν-balls $\{B(x, \frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at x, the topology $\tau_{(\mu,\nu)}$ is first countable.

Example 3.6. Let $\mathcal{X} = \mathbb{R}$ and let ρ, μ, $*$ and $\diamond$ be as in Example 2.5. Then the set of all $\{(a,b) : a, b \in \mathbb{R}\}$ induces a topology on $(\mathbb{R}, \mu, 1-\mu, *, \diamond)$.

Theorem 3.7. *Every δ-homogeneous intuitionistic $\mathfrak{F}$-modular space is Hausdorff.*

Proof. Let x, y be two distinct points in δ-homogeneous intuitionistic $\mathfrak{F}$-modular space $(\mathcal{X}, \mu, \nu, *, \diamond)$. Then for all $t > 0$, $0 < \mu(x-y,t) < 1$, $0 < \nu(x-y,t) < 1$. Put $r_1 = \mu(x-y,t)$, $r_2 = \nu(x-y,t)$ and $r = \max\{r_1, r_2\}$. For $r_0 \in (r, 1)$, there are r_3, r_4 such that

$$r_3 * r_3 \geq r_0, \ \ (1-r_4) \diamond (1-r_4) \leq 1-r_0.$$

Put $r_5 = \max\{r_3, r_4\}$. Then $B(x, 1-r_5, \frac{t}{2^{\delta+1}}) \cap B(y, 1-r_5, \frac{t}{2^{\beta+1}}) = \emptyset$. Otherwise, if there exists $z \in B(x, 1-r_5, \frac{t}{2^{\delta+1}}) \cap B(y, 1-r_5, \frac{t}{2^{\delta+1}})$, then

$$\begin{aligned}r_1 = \mu(x-y,t) &\geq \mu(2(x-z), \frac{t}{2}) * \ \mu(2(z-y), \frac{t}{2})\\ &= \mu(x-z, \frac{t}{2^{\delta+1}}) * \mu(z-y, \frac{t}{2^{\delta+1}})\\ &\geq r_5 * r_5 \geq r_3 * \ r_3 \geq r_0 > r_1,\end{aligned}$$

and

$$\begin{aligned} r_2 = \nu(x-y,t) &\leq \nu(2(x-z),\frac{t}{2}) \diamond \nu(2(z-y),\frac{t}{2}) \\ &= \nu(x-z,\frac{t}{2^{\delta+1}}) \diamond \nu(z-y,\frac{t}{2^{\delta+1}}) \\ &\leq (1-r_5) \diamond (1-r_5) \leq (1-r_4) \diamond (1-r_4) \leq 1-r_0 < r_2, \end{aligned}$$

which is a contradiction. Therefore $(\mathcal{X},\mu,\nu,*,\diamond)$ is Hausdorff. □

In the following, we give further properties of a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space.

Definition 3.8. Let $(\mathcal{X},\mu,\nu,*,\diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space.

1. A subset $\mathcal{A}$ of $\mathcal{X}$ is said to be μ-ν-bounded if there are $t>0$ and $r\in(0,1)$ such that for all $x\in\mathcal{A}$, $\mu(x,t)>1-r$ and $\nu(x,t)<r$.
2. A subset $\mathcal{A}$ of $\mathcal{X}$ is said to be μ-ν-compact if every μ-ν-open cover of $\mathcal{A}$ has a finite subcover.
3. A sequence $\{x_n\}$ in $\mathcal{X}$ is said to be μ-ν-convergent to $x\in\mathcal{X}$ if for every $r\in(0,1)$ and $t>0$ there exists $n_0\in\mathbb{N}$ such that for each $n>n_0$, $x_n\in B(x,r,t)$.

Example 3.9. Let $\mathcal{X}=\mathbb{R}$ and let ρ, μ, $*$ and $\diamond$ be as in Example 2.5.
(i) Consider $V=\{x\in\mathbb{R}:0<x<1\}$. Then V is a bounded set in $(\mathbb{R},\mu,1-\mu,*,\diamond)$.
(ii) Each finite set in $(\mathbb{R},\mu,1-\mu,*,\diamond)$ is μ-ν-compact.
(iii) The sequence $\{\frac{1}{n}\}$ is μ-ν-convergent to 0 in $(\mathbb{R},\mu,1-\mu,*,\diamond)$ by choosing n_0 such that $1-t<\frac{1}{n_0}<r$.

Theorem 3.10. *Every μ-ν-compact subset of a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space $(\mathcal{X},\mu,\nu,*,\diamond)$, is μ-ν-bounded.*

Proof. Let $\mathcal{A}$ be a μ-ν-compact subset of $(\mathcal{X},\mu,\nu,*,\diamond)$. Fix $t>0$ and $r\in(0,1)$, then the family $\{B(x,r,\frac{t}{2^{\delta+1}}):x\in A\}$ is an open cover of $\mathcal{A}$, since $\mathcal{A}$ is compact there exist $x_1,\cdots,x_n\in\mathcal{A}$ such that $\mathcal{A}\subset\cup_{i=1}^{n}B(x_i,r,\frac{t}{2^{\delta+1}})$. Hence for each $x\in\mathcal{A}$ there exists i such that $x\in B(x_i,r,\frac{t}{2^{\delta+1}})$. Thus

$$\mu(x-x_i,\frac{t}{2^{\delta+1}})>1-r,\ \ \nu(x-x_i,\frac{t}{2^{\delta+1}})<r.$$

Put $\alpha_1=\min\{\mu(x_i,\frac{t}{2^{\delta+1}}):1\leq i\leq n\}$ and $\alpha_2=\max\{\nu(x_i,\frac{t}{2^{\delta+1}}):1\leq i\leq n\}$, it is clear that $\alpha_1,\alpha_2>0$, hence for some $s_1,s_2\in(0,1)$ we have

$$\begin{aligned} \mu(x,t)=\mu((x-x_i)+x_i,t) &\geq \mu(2(x-x_i),\frac{t}{2})*\mu(2x_i,\frac{t}{2}) \\ &= \mu(x-x_i,\frac{t}{2^{\delta+1}})*\mu(X_I,\frac{t}{2^{\delta+1}}) \geq (1-r)*\alpha_1 > 1-s_1, \end{aligned}$$

and

$$\begin{aligned} \nu(x,t)=\nu((x-x_i)+x_i,t) &\leq \nu(2(x-x_i),\frac{t}{2})\diamond\nu(2x_i,\frac{t}{2}) \\ &= \nu(x-x_i,\frac{t}{2^{\delta+1}})\diamond\nu(X_i,\frac{t}{2^{\delta+1}}) \leq r\diamond\alpha_2 < s_2. \end{aligned}$$

Taking $s=\max\{s_1,s_2\}$ we conclude $\mu(x,t)>1-s$ and $\nu(x,t)<s$, consequently $\mathcal{A}$ is μ-ν-bounded. □

Theorem 3.11. *Let $(\mathcal{X},\mu,\nu,*,\diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space and $\{x_n\}$ a sequence in $\mathcal{X}$. Then $x_n\to x$ if and only if $\mu(x_n-x,t)\to 1$, $\nu(x_n-x,t)\to 0$.*

Proof. Fix $t > 0$. Assume that $x_n \to x$, then for $r \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $x_n \in B(x,r,t)$, so $\mu(x_n - x, t) > 1 - r$, $\nu(x_n - x, t) < r$. Hence

$$\mu(x_n - x, t) \to 1,\ \nu(x_n - x, t) \to 0.$$

Conversely, for each $t > 0$, let $\mu(x_n - x, t) \to 1$, $\nu(x_n - x, t) \to 0$. Then for $r \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $1 - \mu(x_n - x, t) < r$, $\nu(x_n - x, t) < r$. Therefore $\mu(x_n - x, t) > 1 - r$ and $\nu(x_n - x, t) < r$, for all $n \geq n_0$, that is, $x_n \in B(x,r,t)$ and so $x_n \to x$. □

In the following, we give some related results of completeness of an intuitionistic $\mathfrak{F}$-modular space.

Definition 3.12. Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be an intuitionistic $\mathfrak{F}$-modular space.

1. A sequence $\{x_n\}$ in $\mathcal{X}$ is called μ-ν-Cauchy if for every $\varepsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, t) > 1 - r$ and $\nu(x_n - x_m, t) < r$ for all $m, n \geq n_0$.
2. $\mathcal{X}$ is called μ-ν-complete if every μ-ν-Cauchy sequence is μ-ν-convergent.

Theorem 3.13. *Let $(\mathcal{X}, \mu, \nu, *_M, \diamond_M)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space. Then every μ-ν-convergent sequence in $\mathcal{X}$ is a μ-ν-Cauchy sequence.*

Proof. Let $\{x_n\}$ be μ-ν-convergent to $x \in \mathcal{X}$. Then for every $\varepsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x, \frac{t}{2^{\delta+1}}) > 1 - \varepsilon$ and $\nu(x_n - x, \frac{t}{2^{\delta+1}}) < \varepsilon$ for all $n \geq n_0$. For all $m, n \geq n_0$ we get

$$\begin{aligned}\mu(x_m - x_n, t) &\geq \mu(2(x_m - x), \frac{t}{2}) *\ \mu(2(x_n - x), \frac{t}{2})\\ &\geq \mu(x_m - x, \frac{t}{2^{\delta+1}}) * \mu(x_n - x, \frac{t}{2^{\delta+1}})\\ &> (1-\varepsilon) *_M (1-\varepsilon) = 1 - \varepsilon,\end{aligned}$$

and

$$\begin{aligned}\nu(x_m - x_n, t) &\leq \nu(2(x_m - x), \frac{t}{2}) \diamond \nu(2(x_n - x), \frac{t}{2})\\ &\leq \nu(x_m - x, \frac{t}{2^{\delta+1}}) \diamond\ \nu(x_n - x, \frac{t}{2^{\delta+1}}) < \varepsilon \diamond_M \varepsilon = \varepsilon.\end{aligned}$$

□

Remark 3.14. (1) Theorem 3.13 shows that in an intuitionistic $\mathfrak{F}$-modular space, a μ-ν-convergent sequence is not necessarily a μ-ν-Cauchy sequence, and the δ-homogeneity and the choice of t-norm and t-conorm are essential.
(2) From Definition 3.12, it is clear that each μ-ν-closed subspace of μ-ν-complete $\mathfrak{F}$-modular space is μ-ν-complete.

Theorem 3.15. *Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space and Y a subset of $\mathcal{X}$. If every μ-ν-Cauchy sequence of Y is μ-ν-convergent in $\mathcal{X}$, then every μ-ν-Cauchy sequence of $\bar{Y}$ is μ-ν-convergent in $\mathcal{X}$, where $\bar{Y}$ denotes the μ-ν-closure of Y.*

Proof. Let $\{x_n\}$ be a μ-ν-Cauchy sequence of $\bar{Y}$, then for each $n \in \mathbb{N}$ and $t > 0$, there exists $y_n \in Y$ such that $\mu(x_n - y_n, \frac{t}{4^{\delta+1}}) > 1 - \frac{1}{n+1}$ and $\nu(x_n - y_n, \frac{t}{4^{\delta+1}}) < \frac{1}{n+1}$. Since $\mu(x,.)$ is non-decreasing and $\nu(x,.)$ is non-increasing, we have $\mu(x_n - y_n, \frac{t}{2^{\delta+1}}) > 1 - \frac{1}{n+1}$ and $\nu(x_n - y_n, \frac{t}{2^{\delta+1}}) < \frac{1}{n+1}$. Moreover for each $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mu(x_n - x_m, \frac{t}{4^{\delta+1}}) > 1 - r$ and $\nu(x_n - x_m, \frac{t}{4^{\delta+1}}) < r$ for all $m, n \geq n_0$.

That is, $\mu(x_n - x_m, \frac{t}{4^{\delta+1}}) \to 1$ and $\nu(x_n - x_m, \frac{t}{4^{\delta+1}}) \to 0$. Now we show that $\{y_n\}$ is a μ-ν-Cauchy sequence in Y. For all $m, n \geq n_0$ we have

$$\begin{aligned}
\mu(y_n - y_m, t) &\geq \mu(2(y_n - x_n), \frac{t}{2}) * \mu(2(x_n - y_n), \frac{t}{2}) \\
&\geq \mu(2(y_n - x_n), \frac{t}{2}) * \mu(4(x_n - x_m), \frac{t}{4}) * \mu(4(x_m - y_m), \frac{t}{4}) \\
&= \mu(y_n - x_n, \frac{t}{2^{\delta+1}}) * \mu(x_n - x_m, \frac{t}{4^{\delta+1}}) * \mu(x_m - y_m, \frac{t}{4^{\delta+1}}) \\
&> (1 - \frac{1}{n+1}) * (1 - r) * (1 - \frac{1}{m+1}).
\end{aligned}$$

Since $*$ is continuous $\mu(y_n - y_m, t) \to 1$, Furthermore

$$\begin{aligned}
\nu(y_n - y_m, t) &\leq \nu(2(y_n - x_n), \frac{t}{2}) \diamond \nu(2(x_n - y_n), \frac{t}{2}) \\
&\leq \nu(2(y_n - x_n), \frac{t}{2}) \diamond \nu(4(x_n - x_m), \frac{t}{4}) \diamond \nu(4(x_m - y_m), \frac{t}{4}) \\
&= \nu(y_n - x_n, \frac{t}{2^{\delta+1}}) \diamond \nu(x_n - x_m, \frac{t}{4^{\delta+1}}) \diamond \nu(x_m - y_m, \frac{t}{4^{\delta+1}}) \\
&< \frac{1}{n+1} \diamond r \diamond \frac{1}{m+1}.
\end{aligned}$$

Hence $\nu(y_n - y_m, t) \to 0$, that is, $\{y_n\}$ is Cauchy in Y, so it is μ-ν-convergent to $x \in \mathcal{X}$. Thus for each $\varepsilon > 0$ and $t > 0$ there exists $n_1 \in \mathbb{N}$ such that $\mu(x - y_n, \frac{t}{2^{\delta+1}}) > 1 - \varepsilon$ and $\nu(x - y_n, \frac{t}{2^{\delta+1}}) < \varepsilon$ for all $n \geq n_1$. Therefore

$$\begin{aligned}
\mu(x_n - x, t) &\geq \mu(2(x_n - y_n), \frac{t}{2}) * \ \mu(2(y_n - x_n), \frac{t}{2}) \\
&= \mu(x_n - y_n, \frac{t}{2^{\delta+1}}) * \mu(x_n - y_n, \frac{t}{2^{\delta+1}}) \\
&> (1 - \varepsilon) * (1 - \frac{1}{n+1}),
\end{aligned}$$

consequently, $\mu(x_n - x, t) \to 1$. Similarly we have

$$\nu(x_n - x, t) \leq \nu(x_n - y_n, \frac{t}{2^{\delta+1}}) \diamond \nu(x_n - y_n, \frac{t}{2^{\delta+1}}) < \varepsilon * \frac{1}{n+1}.$$

Hence $\nu(x_n - x, t) \to 0$, and so the Cauchy sequence $\{x_n\}$ in $\bar{Y}$ converges to $x \in \mathcal{X}$. This completes the proof. $\square$

From Theorem 3.15 we get the following result.

Corollary 3.16. *Let $(\mathcal{X}, \mu, \nu, *, \diamond)$ be a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space and let Y be a dense subset of $\mathcal{X}$. If every μ-ν-Cauchy sequence of Y is μ-ν-convergent in $\mathcal{X}$, then $\mathcal{X}$ is μ-ν-complete.*

Now we extend the well-known Baire's theorem to δ-homogeneous intuitionistic $\mathfrak{F}$-modular spaces.

Theorem 3.17. *Let $\{U_n\}_{n\in\mathbb{N}}$ be a sequence of μ-ν-dense open subsets in δ-homogeneous intuitionistic μ-ν-complete $\mathfrak{F}$-modular space $(\mathcal{X}, \mu, \nu, *_M, \diamond_M)$. Then $\bigcap_{n=1}^{\infty} U_n$ is μ-ν-dense in $\mathcal{X}$.*

Proof. Consider the μ-ν-ball $B(x, r, t)$ and let $y \in B(x, r, t)$. Then $\mu(x - y, 2t) > 1 - r$ and $\nu(x - y, 2t) < r$. Since $\mu(x - y, .)$ and $\nu(x - y, .)$ are continuous, there exists $\varepsilon_y > 0$ such that $\mu(x - y, \frac{t-\varepsilon}{2^{\delta-1}}) > 1 - r$ and

$\nu(x-y,\frac{t-\varepsilon}{2^{\delta-1}})<r$ for some $\varepsilon>0$ with $\frac{t-\varepsilon}{2^{\delta-1}}>0$ and $\frac{\varepsilon}{2^{\delta-1}}\in(0,\varepsilon_y)$. We claim that $\overline{B(y,r',\frac{\varepsilon}{4^\delta})}\subseteq B(x,r,2t)$. Choose $r'\in(0,1)$ and $z\in\overline{B(y,r',\frac{\varepsilon}{4^\delta})}$, then there exists a sequence $\{z_n\}$ in $\overline{B(y,r',\frac{\varepsilon}{4^\delta})}$ which is μ-ν-converges to z, so we have

$$\begin{aligned}\mu(z-y,\frac{\varepsilon}{2^{\delta-1}})&\geq\mu(2(z-z_n),\frac{\varepsilon}{2^\delta})*_M\mu(2(z_n-y),\frac{\varepsilon}{2^\delta})\\&=\mu(z-z_n,\frac{\varepsilon}{4^\delta})*_M\mu(z_n-y,\frac{\varepsilon}{4^\delta})>1-r,\end{aligned}$$

and

$$\begin{aligned}\nu(z-y,\frac{\varepsilon}{2^{\delta-1}})&\leq\nu(2(z-z_n),\frac{\varepsilon}{2^\delta})\diamond_M\nu(2(z_n-y),\frac{\varepsilon}{2^\delta})\\&=\nu(z-z_n,\frac{\varepsilon}{4^\delta})\diamond_M\nu(z_n-y,\frac{\varepsilon}{4^\delta})<r.\end{aligned}$$

Therefore we have

$$\begin{aligned}\mu(x-z,2t)&=\mu(2(z-y),2\varepsilon)*_M\mu(2(x-y),2(t-\varepsilon))\\&=\mu(z-y,\frac{\varepsilon}{2^{\delta-1}})*_M\mu(x-y,\frac{t-\varepsilon}{2^{\delta-1}})\\&\geq(1-r)*_M(1-r)=1-r,\end{aligned}$$

and

$$\begin{aligned}\nu(x-z,2t)&=\mu(2(z-y),2\varepsilon)\diamond_M\nu(2(x-y),2(t-\varepsilon))\\&=\nu(z-y,\frac{\varepsilon}{2^{\delta-1}})\diamond_M\nu(x-y,\frac{t-\varepsilon}{2^{\delta-1}})\\&\leq r\diamond_M r=r.\end{aligned}$$

So the claim is true and hence if V is a nonempty μ-ν-open set of $\mathcal{X}$, then $V\cap U_1$ is nonempty and μ-ν-open. Suppose $x_1\in V\cap U_1$, so there exist $r_1\in(0,1)$ and $t_1>0$ such that $B(x_1,r_1,\frac{t_1}{2^{\delta-1}})\subseteq V\cap U_1$. Choose $r_1'<r_1$ and $t_1'=\min\{t_1,1\}$ such that $B(x_1,r_1',\frac{t_1'}{2^{\delta-1}})\subseteq V\cap U_1$. Since U_2 is μ-ν-dense in $\mathcal{X}$, we have $B(x_1,r_1',\frac{t_1'}{2^{\delta-1}})\cap U_2\neq\emptyset$. Let $x_2\in B(x_1,r_1',\frac{t_1'}{2^{\delta-1}})\cap U_2$, hence there exist $r_2\in(0,\frac{1}{2})$ and $t_2>0$ such that $B(x_2,r_2,\frac{t_2}{2^{\delta-1}})\subseteq B(x_1,r_1',\frac{t_1'}{2^{\delta-1}})\cap U_2$. Choose $r_2'<r_2$ and $t_2'=\min\{t_2,\frac{1}{2}\}$ such that $\overline{B(x_2,r_2',\frac{t_2'}{2^{\delta-1}})}\subseteq V\cap U_2$. By induction, we can obtain a sequence $\{x_n\}$ in $\mathcal{X}$ and two sequences $\{r_n'\}$, $\{t_n'\}$ such that $0<r_n'<\frac{1}{n}$, $0<t_n'<\frac{1}{n}$ and $\overline{B(x_n,r_n',\frac{t_n'}{2^{\delta-1}})}\subseteq V\cap U_n$. We show that $\{x_n\}$ is μ-ν-Cauchy. Get $t>0$ and $r\in(0,1)$, then we can choose $k\in\mathbb{N}$ such that $2t_k'<t$ and $r_k'<r$. Since $x_m,x_n\in B(x_k,r_k',\frac{t_k'}{2^{\delta-1}})$, for $m,n\geq k$, we get

$$\begin{aligned}\mu(x_m-x_n,2t)&\geq\mu(x_m-x_n,4t_k')\\&\geq\mu(2(x_m-x_k),2t_k')*_M\mu(2(x_k-x_n),2t_k')\\&=\mu(x_m-x_k,\frac{t_k'}{2^{\delta-1}})*_M\mu(x_k-x_n,\frac{t_k'}{2^{\delta-1}})\\&\geq(1-r_k)*_M(1-r_k)>1-r,\end{aligned}$$

and

$$\begin{aligned}\nu(x_m-x_n,2t)&\leq\nu(x_m-x_n,4t_k')\\&\leq\nu(2(x_m-x_k),2t_k')\diamond_M\nu(2(x_k-x_n),2t_k')\\&=\nu(x_m-x_k,\frac{t_k'}{2^{\delta-1}})\diamond_M\nu(x_k-x_n,\frac{t_k'}{2^{\delta-1}})\\&\leq r_k\diamond_M r_k<r.\end{aligned}$$

Therefore $\{x_n\}$ is a μ-ν-Cauchy sequence. Since $\mathcal{X}$ is μ-ν-complete, there exists $x \in \mathcal{X}$ such that $x_n \to x$. For all $n \geq k$, $x_n \in B(x_k, r'_k, \frac{t'_k}{2^{\delta-1}})$ and hence $x \in \overline{B(x_k, r'_k, \frac{t'_k}{2^{\delta-1}})} \subseteq V \cap U_k$. This implies that $V \cap (\bigcap_{n=1}^{\infty} U_n) \neq \emptyset$. Therefore $\bigcap_{n=1}^{\infty} U_n$ is μ-ν-dense in $\mathcal{X}$. $\square$

Finally, we give the uniform limit theorem in δ-homogeneous intuitionistic $\mathfrak{F}$-modular spaces. Let $\mathcal{X}$ be a nonempty set and let $(\mathcal{Y}, \mu, \nu, *, \diamond)$ be an intuitionistic $\mathfrak{F}$-modular space. A sequence $\{f_n\}$ of mappings from $\mathcal{X}$ to $\mathcal{Y}$ is called μ-ν-converges uniformly to a mapping $f : \mathcal{X} \to \mathcal{Y}$ if, for $t > 0$ and $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $\mu(f_n(x) - f(x), t) > 1 - r$ and $\nu(f_n(x) - f(x), t) < r$, for all $n \geq n_0$ and $x \in \mathcal{X}$.

Theorem 3.18. *Let $\{f_n\}$ be a sequence of continuous mappings from a topological space $\mathcal{X}$ to a δ-homogeneous intuitionistic $\mathfrak{F}$-modular space $(\mathcal{Y}, \mu, \nu, *, \diamond)$. If $\{f_n\}$ μ-ν-convergent uniformly to $f : \mathcal{X} \to \mathcal{Y}$, then f is continuous.*

Proof. Let V be a μ-ν-open set of $\mathcal{Y}$ and $x_0 \in f^{-1}(V)$, so there exist $t > 0$ and $r \in (0, 1)$ such that $B(f(x_0), r, t) \subset V$. For $r \in (0, 1)$, we can choose $s \in (0, 1)$ such that $*(1-s)*(1-s) > 1-r$. Since $\{f_n\}$ μ-ν-converges uniformly to f, for $s \in (0, 1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mu(f_n(x) - f(x), \frac{t}{4^{\delta+1}}) > 1 - s$ and $\nu(f_n(x) - f(x), \frac{t}{4^{\delta+1}}) < s$ for all $n \geq n_0$ and $x \in \mathcal{X}$. Furthermore, each f_n is continuous. Then there exists a neighborhood U of x_0 such that $f_n(U) \subset B(f_n(x_0), s, \frac{t}{4^{\delta+1}})$. Therefore $\mu(f_n(x) - f(x_0), \frac{t}{4^{\delta+1}}) > 1 - s$ and $\nu(f_n(x) - f(x_0), \frac{t}{4^{\delta+1}}) < s$ for all $n \geq n_0$ and $x \in U$ and so we have

$$\begin{aligned}
\mu(f(x) - f_n(x_0), t) &\geq \mu(2(f(x) - f_n(x)), \frac{t}{2}) * \mu(2(f_n(x) - f(x_0)), \frac{t}{2}) \\
&= \mu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) * \mu(2(f_n(x) - f(x_0)), \frac{t}{2^{\delta+1}}) \\
&\geq \mu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) * \mu(2(f_n(x) - f_n(x_0)), \frac{t}{2^{\delta+2}}) * \mu(2(f_n(x_0) - f(x_0)), \frac{t}{2^{\delta+2}}) \\
&= \mu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) * \mu(f_n(x) - f_n(x_0), \frac{t}{4^{\delta+1}}) * \mu(f_n(x_0) - f(x_0), \frac{t}{4^{\delta+1}}) \\
&\geq (1-s) * (1-s) * (1-s) > 1 - r.
\end{aligned}$$

and

$$\begin{aligned}
\nu(f(x) - f_n(x_0), t) &\leq \nu(2(f(x) - f_n(x)), \frac{t}{2}) \diamond \nu(2(f_n(x) - f(x_0)), \frac{t}{2}) \\
&= \nu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) \diamond \nu(2(f_n(x) - f(x_0)), \frac{t}{2^{\delta+1}}) \\
&\leq \nu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) \diamond \nu(2(f_n(x) - f_n(x_0)), \frac{t}{2^{\delta+2}}) \diamond \nu(2(f_n(x_0) - f(x_0)), \frac{t}{2^{\delta+2}}) \\
&= \nu(f(x) - f_n(x), \frac{t}{2^{\delta+1}}) \diamond \nu(f_n(x) - f_n(x_0), \frac{t}{4^{\delta+1}}) \diamond \nu(f_n(x_0) - f(x_0), \frac{t}{4^{\delta+1}}) \\
&\leq s \diamond s \diamond s < r.
\end{aligned}$$

This implies that $f(x) \in B(f(x_0), r, t) \subset V$, therefore $f(U) \subseteq V$, hence f is continuous. $\square$

4 Conclusion

In this paper, we have proposed the concept of an intuitionistic fuzzy modular space based on the modular space and continuous t-norm and t-conorm, which can be regarded as a generalization of a modular space in the intuitionistic fuzzy sense. We first deal with the problem of whether there is a relationship between an intuitionistic fuzzy modular and an intuitionistic fuzzy metric. In the sequel, we have defined a Hausdorff topology induced by a δ-homogeneous fuzzy modular and examined some related topological properties.

Finally, we have extended the well-known Baire's theorem and the uniform limit theorem to δ-homogeneous intuitionistic fuzzy modular spaces.

Conflict of Interest: The author declares no conflict of interest.

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Tayebe Lal Shateri
Department of Mathematics and Computer Sciences
Hakim Sabzevari University
Sabzevar, Iran
E-mail: t.shateri@hsu.ac.ir