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Intuitionistic Hesitant Fuzzy Algorithm for Multi-Objective Structural Model Using Various Membership Functions

Sanjoy Biswas*, **Samir Dey**

Abstract. In real life, structural problems can be described in linear and nonlinear forms. This nonlinear structural problem is very challenging to solve when its all parameters are imprecise in nature. Intuitionistic fuzzy sets were proposed to manage circumstances in which experts have some membership and non-membership value to judge an option. Hesitant fuzzy sets were used to manage scenarios in which experts pause between many possible membership values while evaluating an alternative. A new growing area of a generalized fuzzy set theory called intuitionistic hesitant fuzzy set (IHFS) provides useful tools for dealing with uncertainty in structural design problem that is observed in the actual world. In this article, we have developed a procedure to solve non-linear structural problem in an intuitionistic hesitant fuzzy (IHF) environment. The concept of an intuitionistic hesitant fuzzy set is introduced to provide a computational basis to manage the situations in which experts assess an alternative in possible membership values and non-membership values. This important feature is not available in the intuitionistic fuzzy optimization technique. Here we have discussed the solution procedure of intuitionistic hesitant fuzzy optimization technique dedicatedly for linear, exponential, and hyperbolic types of membership and non-membership functions. Some theoretical development based on these functions has been discussed. A numerical illustration is given to justify the effectiveness and efficiency of the proposed method in comparison with fuzzy multi-objective nonlinear programming method and intuitionistic fuzzy multi-objective nonlinear programming method. Finally, based on the proposed work, conclusions and future research directions are addressed.

AMS Subject Classification 2020: 49K35; 90C29; 90C70
Keywords and Phrases: Multi objective structural problem, Hesitant fuzzy set, Intuitionistic fuzzy optimization, Intuitionistic-hesitant fuzzy optimization, Pareto optimal solution.

1 Introduction

In structural and civil engineering, structural optimization is a key idea. Although the concept of structural optimization is well-established.It is frequently treated in a single objective form, with the objective being (the weight function). In addition to the minimization of the weight function, this optimization also involves satisfying one or more constants consequently. But in the real world, there are multiple competing objectives. The Multiple objective structural optimizations (MOSOs) methodology was used to address multiple competing objectives. Due to the growing technological demand for structural optimization, the MOSO is becoming a more and more important research area in the last ten years.
The development of fuzzy optimum structural design methods was required because the input data and the parameters in structural design problems are frequently/imprecise. The fuzzy set (FS) theory was first developed by Zadeh [17] to deal with erroneous and imperfect data. The decision-making problem was later addressed by Zadeh [29] and Bellman and Zadel [6] using the fuzzy set theory. Later on, Zimmermann [30]

*Corresponding Author: Sanjoy Biswas, Email: sanjoy.biswas@gnit.ac.in, ORCID: 0000-0002-5438-7154
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proposed a fuzzy programming approach (FPA) for several objective optimization problems. The FS theory is also used in the structural model. Many researchers (see [27, 12, 11, 23]) have given remarkable contributions in the field of structural optimization under fuzzy environments. Also Dey et.al. [15] developed a methodology using different norm(Yager, Hamacher, Dombi) under the fuzzy environment in the context of structural design. Here they have optimized three objective functions simultaneously in three bar truss structural model. Numerous extensions of fuzzy sets emerged as a result of the growing use of FS in structural problems when the available information is ambiguous.

1.1 Literature Review

The intuitionistic fuzzy set (IFS), one of the generalizations of FS theory, was introduced by professor Atanassov [4] in 1986.IFS plays an important role when imprecise information cannot be expressed by conventional fuzzy sets. It is a more advanced version of FS. In IFS, we usually consider the degree of acceptance, degree of rejection and hesitancy such that the sum of degrees of membership should be less than or equal to one, whereas we consider the degree of acceptance only in FS. P. P. Angelov [3] introduced optimization for the first time in a widespread intuitionistic fuzzy environment(IFEv) in 1997. The field of intuitionistic fuzzy optimization (IFO) is still unexplored. There has been little research work done on IFO in terms of structural optimization. The methodology of Multi-objective linear programming(MOLP) under IFEv was developed by Jana and Roy [18] to find an optimal solution to the transportation problem. Luo et. al.[19] had discussed multi-criteria decision making (MCDM) problems based on the inclusion degree of IFS in 2008. In 2015, Dey et al. [13] used multi-objective intuitionistic fuzzy optimization approach to solve three bar truss structural model. Farther, M. Sarkar et al. [21] proposed a new computational algorithm based on t-norm and t-conorm in the intuitionistic fuzzy environment to solve a welded beam design problem. Ahmadini et al. [1] proposed intuitionistic fuzzy goal programming with preference relations to solve a multi-objective problem in 2021.M. Akram, et al.[2] introduced interval-valued Fermatean fuzzy set(IVFFS) which is the extension of Fermatean fuzzy sets(FFSs) and applied IVFFS in the fractional transportation problem. Further, M.K. Sharma, et al.[22] originally solved multi-objective transportation problem (MOTP) in Fermatean fuzzy environment. They also anticipated a new score function to convert the Fermatean fuzzy data into Crisp data to solve MOTP.In an IFS, the degree of acceptance, degree of rejection, and degree of hesitation of an element may not be a specific number in some situations. As a result, it has been extended to interval-valued intuitionistic fuzzy sets [5].

The concept of hesitant fuzzy set (HFS), which is an extension of regular FS, was first introduced by Torra [25] and Torra and Narukawa [24]. This is a useful tool because it allows for more possible degrees of an element to be in a set which is a sub-interval of [0,1]. In the literature survey, we have seen Meany researchers have implemented the concept of HFS in different fields of research. In 2016, Xu et al. [26] developed a computational programming technique based on HFS for a hybrid multi-criteria group decision making (MCGDM) model. L. Dymova [16] created a user-friendly computer application using a fuzzy multiple-criteria decision-making (MCDM) technique. In 2018, Bharati [7] developed a multi-objective hesitant fuzzy optimization technique. He also published some research articles on interval-valued intuitionistic hesitant fuzzy sets (see [8, 9]), and hesitant intuitionistic fuzzy sets [10] between 2021 and 2022. Xia et. al. [28] introduced hesitant fuzzy on decision making. In 2022, M. Ranjbar et al. [20] introduced the ranking of hesitant fuzzy numbers and new arithmetic operations based on the extension principle.

1.2 Motivation for that research

According to the literature review, numerous methods have been developed to solve multi-objective optimization problems (MOOPs) in fuzzy and intuitionistic fuzzy environments. Dey et al. [14] used fuzzy and intuitionistic fuzzy approaches to solve multi-objective three-bar truss structural model. This method can satisfy the objective(s) with a bigger degree than the analogous fuzzy optimization problem and the crisp one, but there is no space for the decision maker's point of view. In real life, the decision makers priority plays an important role in any decision-making. Therefore, it is necessary to develop a new decision-making method based on IHF decision-making set that assigns a set of potential values for each objective functions membership and non-membership in IHF environment.

1.3 Contribution of the work

Many scholars are working continuously to find the best solution to multi objective structural optimization problems (MOSOPs). A large amount of the literature is composed of fuzzy-based optimization approaches that use the generalized concept of a FS to solve MOSOPs. Many researchers optimize the MOSOPs using an intuitionistic fuzzy-based optimization technique. The study focused on IHFS under different membership and non-membership functions. After that, MOSOP can be solved by using the proposed IHF approach. However, the following are the few major aspects that guarantee a significant contribution to the field of multi objective optimization techniques.

- The intuitionistic hesitant fuzzy (IHF) is a recent extension of fuzzy sets that are explained in a structural model of three bar truss.
- In this paper, we present an IHF set theory that provides an opportunity for the decision-maker to select the best result or reject the worse result in comparison to others.
- Instead of a single fixed degree, a set of possible degrees of acceptance and degrees of rejection are defined to address the uncertainty and hesitancy of MONLPP.
- The intuitionistic hesitant Pareto optimal is also introduced in this paper.
- We have developed a multi-objective structural model under an intuitionistic hesitant fuzzy environment. A computational algorithm for intuitionistic hesitant fuzzy optimization has been developed to solve multi-objective structural models.
- The HIFS might be a useful tool to deal with any real-life situation in the context of uncertainty and hesitation.

1.4 Framework of the article

The rest of the manuscript is organized as follows: Sect. 2, we have explained the multi-objective structural optimization model. In Sect. 3 recalls some basic concepts of FS, IFS, IHFS. For the practical perspective, a computational algorithm was proposed to solve MOOP using the intuitionistic hesitant fuzzy optimization technique (IHFOT) in Sect. 4. In Sect. 5, stepwise solution procedures are described for the solution of multi-objective structural model using IHFOT. An illustrative example is examined in Sect. 6 that shows the applicability and validity of the proposed algorithm efficiently. Finally, conclusions are highlighted based on the present work in Sect. 7.

2 Mathematical form of Multi-Objective Structural Problem (MOSP)

In the structural model, the basic parameters of a bar truss structure system (such as elastic modulus, material density, height possible stress etc.) are identified, and the goal is to find the optimum cross section area of the bar truss so that we can find the lightest weight of the structure and smallest node displacement

under loading condition.
The multi-objective problem in structure model is written as follows:

$$\begin{aligned} \text{Minimize} \quad & WG(C) \\ \text{Minimize} \quad & d(C) \\ \text{Such that} \quad & T[C] \leq [T_0] \\ & C \in [C_{min}, C_{max}] \end{aligned} \tag{1}$$

Where n number design parameters $C = [C_1, C_2, C_3, ..., C_n]^T$ are considered.The design parameters are the cross section of the truss bar,the total structural weight is $WG(C) = \sum_{i=1}^{n} \delta_i C_i L_i$, $d(C)$ is the deflection of loaded joint, L_i, C_i and δ_i were the lengths of the bar, cross section area and density of the i^{th}group bars respectively. Under different conditions, the stress constraint=$T(C)$and maximum possible stress of the group bars=$[T_0]$, cross section area (minimum)= C_{min} and cross section area (maximum) = C_{max} respectively.

3 Preliminaries

In this section, we talked about several fundamental ideas related to intuitionistic fuzzy logic.

Definition 3.1. (see [4]) *(Intuitionistic Fuzzy Set(IFS)) Let* $E = \{x_1, x_2, ..., x_n\}$ *be the collection of finite objects then the IFS* $\overline{Y}$ *in* E *is defined as:* $\overline{Y} = \{(x_j, \gamma_{\overline{Y}(x_j)}, \lambda_{\overline{Y}(x_j)}) : x_j \in E\}$*, where the function* $\gamma_{\overline{A}(x_j)} : E \to [0,1]$*define the degree of membership function and* $\lambda_{\overline{Y}(x_j)} : x_j : E \to [0,1]$ *define the degree of non-membership function of an element* $x_j \in E$ *respectively, with the condition* $0 \leq \gamma_{\overline{Y}}(x_j) + \lambda_{\overline{Y}(x_j))} \leq 1 \quad \forall x_j \in E$*.For each* $\overline{Y} \in E$ *the amount* $\overline{\pi}_{\overline{Y}}(x_j) = 1 - \gamma_{\overline{A}(x_j)} - \lambda_{\overline{Y}(x_j)}$ *is called Atanassovs intuitionistic index of the element* $x_j \in E$ *or degree of indeterminacy (uncertainty) of* x_j *of the measure of hesitation.*

Definition 3.2. (see [4]) *(*(α, β)*-cut) A subset* (α, β)*-cut of* E *generated by an IFS, where* (α, β)*are fixed numbers such that* $\alpha + \beta \leq 1$ *is defined by* $\overline{Y}_{\alpha,\beta}(x_j) = \{x_j \in E : \gamma_{\overline{Y}(x_j)} \geq \alpha, \lambda_{\overline{Y}(x_j))} \leq \beta\}$ *.Thus* (α, β) *of an IFS to be denoted by* $\{\overline{Y}_{\alpha,\beta}(x_j)$ *as a crisp set of the element* x_j *which belong to* $\overline{Y}_{\alpha,\beta}(x_j)$ *at least to the degree* α *and at most to the degree* β*.*

Definition 3.3. (see [25, 24]) *(Hesitant Fuzzy Set(HFS)) Torra in 2009 and Torra and Narukawa in 2010, created a new tool called hesitant fuzzy sets (HFSs) and which allow the membership degree to the set of various possible values. The HFS can be stated as follows:*
Let E *be the fixed set then a HFS on* E *is expressed as* $\overline{Y} = \{(x_j, h_{\overline{h}(x_j)}) : x_j \in E\}$*, where is set of possible membership degrees of the element* $x_j \in E$ *in [0,1].Also, we call* $h_{\overline{Y}}(x_j)$*, a hesitant fuzzy element. Further, Xia and Xu [?] applied it in their works of research.*

Definition 3.4. [10] *(Intuitionistic Hesitant Fuzzy Set) When making a decision, a decision-maker may hesitate to determine the exact degrees of membership and non-membership between 0 and 1. In such a scenario, the IHFS, which is a generalized version of FS where the membership and nonmember ship degrees of an element to a specific set can be represented by multiple distinct values between 0 and 1. The IHFS is perfect at dealing with circumstances in which decision maker disagreement or hesitate to make a decision. Let there be a fixed set*E *; a IHFS* $\overline{Y}$ *on* E *is represented as* $\overline{Y} = \{(x_j, h_{\overline{h}(x_j)} : x_j \in E\}$ *where* $h_{\overline{Y}}(x_j)$ *is set of some values of IHFSs in [0,1] , denoting the possible membership degree and non-membership degree of the element* $x_j \in E$ *. Let* I_{h_1}, I_{h_2} *be two IHFSs and* $h_1 \in I_{h_1}, h_2 \in I_{h_2}$*.Then the complement of IHFS* I_h*, union and intersection of* I_{h_1}, I_{h_2} *are defined as follows respectively.*

- $I_h^c = \{h^c : h \in I_h\}$

- $I_{h_1} \cup I_{h_2} = \{max(h_1, h_2) : h_1 \in I_{h_1}, h_2 \in I_{h_2} \quad where \quad h_1 \cup h_2 = \{max(\gamma_{h_1}, \gamma_{h_2}), min(\lambda_{h_1}, \lambda_{h_1})\}\}$
- $I_{h_1} \cap I_{h_2} = \{min(h_1, h_2) : h_1 \in I_{h_1}, h_2 \in I_{h_2} \quad where \quad h_1 \cap h_2 = \{min(\gamma_{h_1}, \gamma_{h_2}), max(\lambda_{h_1}, \lambda_{h_1})\}\}$

Definition 3.5. [8] *(Pareto-optimal solution) An ideal solution derived from a single objective may or may not satisfy all of the conflicting objectives at the same time. However, it is difficult to find Pareto-optimal solutions, which optimize all objectives while satisfying all constraints. Mathematically, Suppose*Λ *be the collection of feasible solution for (1) of MOSOP. Then a point* x^* *is considered to be a Pareto optimal or efficient solution of (1) iff there exists no* $x \in \Lambda$ *such that* $\Theta_\sigma(x^*) \geq \Theta_\sigma(x)$ *for all* σ *and* $\Theta_\sigma(x^*) > \Theta_\sigma(x)$ *for at least one* σ. *And a point* $x^* \in \Lambda$ *is called a weak Pareto optimal solution of (1). iff there exists no* $x \in \Lambda$ *such that* $\Theta_\sigma(x^*) \geq \Theta_\sigma(x)$ *for all* σ

Definition 3.6. [10] *(Pareto-optimal solutions of IHF) The Pareto-optimal solutions for the IHF optimization can be defined as follows:*
A solution $X_0 \in \Omega$ *is said to be Pareto-optimal solution for (1) if there does not exist another* $X \in \Omega$ *such that* $\Theta_\sigma(X) \geq \Theta_\sigma(X_0)$ *with* $\gamma_\sigma^{IF}\Theta_\sigma(X) \geq \gamma_\sigma^{IF}\Theta_\sigma(X_0), \lambda_\sigma^{IF}\Theta_\sigma(X) \leq \lambda_\sigma^{IF}(\Theta_\sigma(X_0))$, *and* $\Theta_{\sigma_0}(X) > \Theta_{\sigma_0}(X_0)$ *with* $\gamma_{\sigma_0}^{IF}(\Theta_{\sigma_0}(X)) > \gamma_{\sigma_0}^{IF}(\Theta_{\sigma_0}(X_0))$ *and* $\lambda_{\sigma_0}^{IF}(\Theta_{\sigma_0}(X)) < \lambda_{\sigma_0}^{IF}(\Theta_{\sigma_0}(X_0))$ *for at least one* $\sigma_0 = \{1, 2, ..., \Sigma\}$.

Definition 3.7. *(Intuitionistic Hesitant fuzzy Non Linear Programming (IHFNLP)) Most real-world problems involve the optimization of more than one objectives at the same time. The best compromise solution is the most promising solution set that efficiently satisfies each objective. Therefore, a Multi-Objective Non-Linear Programming (MONLP) with P objectives should be greater than or equal to some value* $\lesssim g_p^0(x), p = 1, 2, ..., P$ *may be taken in the following form:*

$$\begin{aligned} Minimize \quad & \Theta(x) = [\Theta_1, \Theta_2, ..., \Theta_\sigma]^T \\ subject\ to \quad & \Theta_\sigma(x) \lesssim g_\sigma^0(x), \sigma = 1, 2, ..., \Sigma \\ & g_j(x) \leq 0, x_j \geq 0 for j = 1, 2, .., m \\ & x = \{x_1, x_2, ..., x_n\} \end{aligned} \tag{2}$$

Where $g_\sigma^0(x)$ *is goal for* σ^{th} *objective and* $\lesssim$ *is uncertain form of* $\leq$.

4 Problem formulation and solution algorithm

4.1 Intuitionistic Hesitant Fuzzy algorithm to Solve MONLPP

A MONLP with σ objective may be taken in the following form:

$$\begin{aligned} \text{Minimize} \quad & \Theta(x) = [\Theta_1(x), \Theta_2(x), ..., \Theta_\sigma(x)]^T \\ \text{subject to} \quad & \{x \in R^n : g_j(x) \leq or = or \geq b_j \quad \text{for} \quad j = 1, 2, ..., m\} \\ & L_i \leq x_i \leq U_i \quad (i = 1, 2, ..., n) \end{aligned} \tag{3}$$

Zimmermann [30] showed that fuzzy programmin technique (FPT) can be used to solve the MOOP. To solve MONLPP, following steps are used.
Step 1 Solve the MONLP (3) as a single objective function from the set of σ objectives and solve it as a single objective subject to the given constrains and ignoring the others objective function. Determine the value of objective functions and basic feasible solutions.
Step 2 Calculate the values of the remaining $(\sigma-1)$ objectives at the basic feasible solutions that are obtained from **Step 1**.
Step 3 Repeat the **Step 1** and **Step 2** for the remaining $(\sigma - 1)$ objective functions.

Table 1: Please write your table caption here

Minimum	Θ_1	Θ_2	Θ_3	—	Θ_σ	X
Minimum Θ_1	Θ_1^*			—		X_1
Minimum Θ_2		Θ_2^*		—		X_2
Minimum Θ_3			Θ_3^*	—		X_2
—	—	—	—	—	—	—
Minimum Θ_Σ				—	Θ_σ^*	X_σ
Maximum	Θ_1'	Θ_2'	Θ_3'	—	Θ_σ'	X_σ'

Step 4 From the result of **Step 1**, **Step 2** and **Step 3**, obtained the corresponding tabulated values of objective functions from a Table 1. and these are known as positive ideal solution.
Step 5 From **Step 4**, obtain the lower bounds and upper bounds for each objective functions, where Θ_σ^* and $\Theta_\sigma^{'}$ are maximum, minimum values of Θ_Σ respectively.
Step 6 Here, we denote and define upper and lower bounds by $U_\sigma^\gamma = max\{Z_p X_p\}$ and $p = 1, 2, 3, ..., P$ for respectively for each uncertain and imprecise objective functions of MONLPPs
Step 7 Set upper bound or upper tolerance level and lower bound or lower tolerance limit for the σ^{th} objective function Θ_σ for hesitant degree of acceptance and rejection based on the set of solutions obtained in **Step 4**. For hesitant membership function: Upper and lower tolerance level for hesitant membership functions are

$$U_\sigma^\gamma = max\{\Theta_\sigma(X_p)\} \quad \text{and} \quad L_\sigma^\gamma = min\{\Theta_\sigma(X_p)\}, 1 \le p \le P, \sigma = 1, 2, ..., \Sigma$$

For hesitant non-membership function: Upper and lower tolerance level for hesitant membership functions are

$$U_\sigma^\lambda = U_\sigma^\gamma, L_\sigma^\lambda = L_\sigma^\gamma + \epsilon_\sigma$$

where $0 \le \epsilon_\sigma \le (U_\sigma - L_\sigma)$ is predetermined real numbers prescribed by decision-makers.
Step 8 In this step, we can define uncertainty and imprecise objectives of different hesitant membership functions as linear, exponential and hyperbolic more elaborately under IHF environment. Each of them is defined for the hesitant membership and a hesitant non-membership functions, which seems to be more realistic.

4.1.1 Linear-type intuitionistic hesitant membership functions approach (LTIHMFA)

The truth membership function of linear type $\gamma_\sigma^{Lf_i}(\Theta_\sigma(x))$ and a falsity membership function of linear type $\lambda_\sigma^{Lf_1}(\Theta_\sigma(x))$ functions under IHF environment can be explained in the following way:

For truth hesitant fuzzy membership functions:

$$\gamma_\sigma^{Lf_1}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \le L_\sigma^\gamma \\ \phi_1\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\gamma \le \Theta_\sigma(x)) \le U_\sigma^\gamma \\ 0 & \text{if } \Theta_p\sigma(x)) > U_\sigma^\gamma \end{cases}$$

$$\gamma_\sigma^{Lf_2}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \le L_\sigma^\gamma \\ \phi_2\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\gamma \le \Theta_\sigma(x)) \le U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

.....

$$\gamma_\sigma^{Lf_n}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \le L_\sigma^\gamma \\ \phi_n\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\gamma \le \Theta_\sigma(x)) \le U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

For Falsity hesitant fuzzy membership functions

$$\lambda_\sigma^{Lf_1}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \le L_\sigma^\lambda \\ \zeta_1\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\lambda \le \Theta_\sigma(x)) \le U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

$$\lambda_\sigma^{Lf_2}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \le L_\sigma^\lambda \\ \zeta_2\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\lambda \le \Theta_\sigma(x)) \le U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

.....

$$\lambda_\sigma^{Lf_n}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \le L_\sigma^\lambda \\ \zeta_n\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) & \text{if } L_\sigma^\lambda \le \Theta_\sigma(x)) \le U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

The mathematical expression for objective functions defined as follows

$$\begin{aligned} &\text{Max } \min_{\sigma=1,2,\ldots,\Sigma} \gamma_\sigma^{Lf_i}(\Theta_\sigma(x))^\dagger) \\ &\text{Min } \max_{\sigma=1,2,\ldots,\Sigma} \lambda_\sigma^{Lf_i}(\Theta_\sigma(x))^\dagger) \\ &i = 1, 2, ..., n \end{aligned} \tag{4}$$

Subject to all constraints of (3).
Also assume that $\gamma_\sigma^{Lf_i}(\Theta_\sigma(x))^\dagger) \ge \nu_i$ and $\lambda_r^{Lf_i}(\Theta_\sigma(x))^\dagger) \le \eta_i$ $i = 1, 2, ..., n$ for all σ. Where the parameter $\dagger > 0$
Using auxiliary parameters ν_i and η_i, the problem (4) can be transformed into the following problem (5)

$$\text{LTIHMFA} \quad Max\left(\sum_i \nu_i - \sum_i \eta_i\right)$$

Subject to

$$\begin{aligned}
&(\Theta_\sigma(x))^\dagger + \frac{\nu_1}{\phi_1}\left((U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger\right) \le (U_\sigma^\gamma)^\dagger,\\
&(\Theta_\sigma(x))^\dagger + \frac{\nu_2}{\phi_2}\left((U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger\right) \le (U_\sigma^\gamma)^\dagger,\\
&.........,\\
&(\Theta_\sigma(x))^\dagger + \frac{\nu_3}{\phi_3}\left((U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger\right) \le (U_\sigma^\gamma)^\dagger;\\
&(\Theta_\sigma(x))^\dagger - \frac{\eta_1}{\zeta_1}\left((U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger\right) \le (L_{\sigma^\lambda})^\dagger,\\
&(\Theta_\sigma(x))^\dagger - \frac{\eta_2}{\zeta_2}\left((U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger\right) \le (L_\sigma^\lambda)^\dagger,\\
&.........,\\
&(\Theta_\sigma(x))^\dagger - \frac{\eta_3}{\zeta_3}\left((U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger\right) \le (L_\sigma^\lambda)^\dagger;
\end{aligned} \tag{5}$$

$\nu_i \ge \eta_i; \nu_i + \eta_i \le 1$ and $\eta_i, \nu_i, \phi_i, \zeta_i \in [0,1]$ $\forall i = 1, 2, ..., n$all the constraints of (3).

Theorem 4.1. *There is only one optimal solution (x^*, ν^*, η^*) of (5) that is also an efficient solution to the problem (3) where $\nu^* = (\nu_1^*, \nu_2^*, ..., \nu_n^*)$ and $\eta^* = (\eta_1^*, \eta_2^*, ..., \eta_n^*)$*

Proof. Assume that (x^*, ν^*, η^*) be the only optimal solution of (5) that it is an inefficient solution to the problem (3). Then there exist different feasible alternative $x'(x' \ne x^*)$ of problem (3), so that $\Theta_\sigma(x^*) \le \Theta_\sigma(x') \forall \sigma = 1, 2, ..., \Sigma$ and $\Theta_\sigma(x^*) < \Theta_\sigma(x')$ for at least one σ .

$$\begin{aligned}
\text{Therefore, we have} \quad &\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \le \phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \quad \forall \quad \sigma = 1, 2, ..., \Sigma\\
\text{and} \quad &\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} < \phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \quad \text{for atleast one} \quad \sigma,\\
\text{where} \quad &0 \le \phi \le 1
\end{aligned}$$

$$\begin{aligned}
\text{Thus,} \quad &Max_{\forall\sigma}\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - ((L_\sigma^\gamma)^\dagger}\right) \le Max_{\forall\sigma}\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\\
\text{and} \quad &Max_\sigma\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right) < Max_\sigma\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma)^\dagger - (L_\sigma)^\dagger}\right) \quad \text{for at least one } \sigma.
\end{aligned}$$

$$\begin{aligned}
\text{Similarly,} \quad &Min_{\forall\sigma}\left(\zeta\frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right) \ge Min_{\forall\sigma}\left(\zeta\frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\\
\text{and} \quad &Min_\sigma\left(\zeta\frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right) > Min_\sigma\left(\zeta\frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right) \quad \text{for at least one } \sigma.\\
&\text{where} \quad 0 \le \zeta \le 1
\end{aligned}$$

Now suppose that,

$$\nu' = Max_\sigma\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right), \nu^* = Max_\sigma\left(\phi\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right),$$
$$\eta' = Min_\sigma\left(\zeta\frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right), \text{and} \quad \eta^* = Min_\sigma\left(\zeta\frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right) \quad \text{for at least one } \sigma.$$

Then, $\nu^* \leq (<)\nu'$ and $\eta^* \geq (>)\eta'$ which gives $(\nu^* - \eta^*) < (\nu' - \eta')$ that implies the solution is not optimal which contradicts that $x'(x' \neq x^*)$ is the only one optimal solution of (5). Hence, it is an effective solution of (5). Hence the proof is now complete. □

4.1.2 Exponential-type intuitionistic hesitant membership functions approach (ETIHMFA)

The truth membership function of exponential type $\gamma_\sigma^{Ef_i}(\Theta_\sigma(x))$ and a falsity membership function of exponential type $\lambda_\sigma^{Ef_i}(\Theta_\sigma(x))$ functions under IHF environment can be explained in the following way:

For truth hesitant fuzzy membership functions:

$$\gamma_\sigma^{Ef_1}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\gamma \\ \phi_1\left[1 - exp\left\{-\psi\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}\right] & \text{if } L_\sigma^\gamma \leq \Theta_\sigma(x)) \leq U_\sigma^\gamma \\ 0 & \text{if } \Theta_p\sigma(x)) > U_\sigma^\gamma \end{cases}$$
$$\gamma_\sigma^{Ef_2}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\gamma \\ \phi_2\left[1 - exp\left\{-\psi\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}\right] & \text{if } L_\sigma^\gamma \leq \Theta_\sigma(x)) \leq U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

.....

$$\gamma_\sigma^{Ef_n}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\gamma \\ \phi_n\left[1 - exp\left\{-\psi\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}\right] & \text{if } L_\sigma^\gamma \leq \Theta_\sigma(x)) \leq U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

For Falsity hesitant fuzzy membership functions

$$\lambda_\sigma^{Ef_1}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\lambda \\ \zeta_1\left[1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\right\}\right] & \text{if } L_\sigma^\lambda \leq \Theta_\sigma(x)) \leq U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$
$$\lambda_\sigma^{Ef_2}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\lambda \\ \zeta_2\left[1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\right\}\right] & \text{if } L_\sigma^\lambda \leq \Theta_\sigma(x)) \leq U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

.....

$$\lambda_\sigma^{Ef_n}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\lambda \\ \zeta_n\left[1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\right\}\right] & \text{if } L_\sigma^\lambda \leq \Theta_\sigma(x)) \leq U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

Where ψ denotes the ambiguity degree or shape parameter assigned by the decision-maker.
Using by problem (4), we consider that $\gamma_\sigma^{Ef_i}(\Theta_\sigma(x)) \geq \nu_i$ and $\lambda_\sigma^{Ef_i}(\Theta_\sigma(x)) \leq \eta_i$ for $i = 1, 2, ..., n$ and $\forall\sigma$, where the parameter $\dagger > 0$.
The auxiliary parameters ν_i and η_i allow the problem (4) to be changed into (6)

$$
\begin{aligned}
&\text{ETIHMFA} \quad Max\left(\sum_i \nu_i - \sum_i \eta_i\right) \\
&\text{Subject to} \\
&\phi_1\left[1 - exp\left\{-\psi\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}\right] \geq \nu_1, \\
&\phi_2\left[1 - exp\left\{-\psi\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}\right] \geq \nu_2, \\
&........., \\
&\phi_n\left[1 - exp\left\{-\psi\left(\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}\right] \geq \nu_n; \\
&\zeta_1\left[1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\right\}\right] \leq \eta_1, \\
&\zeta_2\left[1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\right\}\right] \leq \eta_2, \\
&........., \\
&\zeta_n\left[1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger - (L_\sigma^\lambda)^\dagger}\right)\right\}\right] \leq \eta_n;
\end{aligned} \tag{6}
$$

$\nu_i \geq \eta_i; \nu_i + \eta_i \leq 1$ and $\eta_i, \nu_i, \phi_i, \zeta_i \in [0, 1]$ $\forall i = 1, 2, ..., n$all the constraints of (3).

Theorem 4.2. *There is only one optimal solution* (x^*, ν^*, η^*) *of (6) that is also an efficient solution to the problem (3) where* $\nu^* = (\nu_1^*, \nu_2^*, ..., \nu_n^*)$ *and* $\eta^* = (\eta_1^*, \eta_2^*, ..., \eta_n^*)$

Proof. Assume that (x^*, ν^*, η^*) be the only optimal solution of (6) that it is an inefficient solution to the problem (3). Then there exist different feasible alternative $x'(x' \neq x^*)$ of problem (3), so that $\Theta_\sigma(x^*) \leq \Theta_\sigma(x')\, \forall\sigma = 1, 2, ..., \Sigma$ and $\Theta_\sigma(x^*) < \Theta_\sigma(x')$ for at least one σ

$$
\begin{aligned}
\text{Therefore, we have} \quad &\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \leq \frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \quad \forall \quad \sigma = 1, 2, ..., \Sigma \\
\text{and} \quad &\frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} < \frac{(U_\sigma^\gamma)^\dagger - (\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger} \quad \text{for atleast one} \quad \sigma,
\end{aligned}
$$

$$
\text{Now,} \quad 1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\} \leq 1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}
$$

$\forall \quad \sigma = 1, 2, ..., \Sigma$

$$
\text{and} \quad 1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x^*))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\} < 1 - exp\left\{-\psi\left(\frac{(\Theta_\sigma(x'))^\dagger - (L_\sigma^\gamma)^\dagger}{(U_\sigma^\gamma)^\dagger - (L_\sigma^\gamma)^\dagger}\right)\right\}
$$

for atleast one σ,

Thus,

$$Max_{\forall\sigma}\phi\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Bigg)\Bigg\}\Bigg] \leq Max_{\forall\sigma}\phi\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Bigg)\Bigg\}\Bigg]$$

$$\text{and}\quad Max_{\sigma}\phi\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Bigg)\Bigg\}\Bigg] < Max_{\sigma}\phi\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Bigg)\Bigg\}\Bigg]$$

$$\text{for at least one } \sigma, \quad 0 \leq \phi \leq 1$$

$$Min_{\forall\sigma}\zeta\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(\Theta_\sigma(x^*))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Bigg)\Bigg\}\Bigg] \geq Min_{\forall\sigma}\zeta\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(\Theta_\sigma(x'))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Bigg)\Bigg\}\Bigg])$$

$$\text{and}\quad Min_{\sigma}\zeta\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(\Theta_\sigma(x^*))^\dagger-(L_\sigma^\gamma)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Bigg)\Bigg\}\Bigg] > Min_{\sigma}\zeta\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(\Theta_\sigma(x'))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Bigg)\Bigg\}\Bigg]$$

$$\text{for at least one } \sigma, \quad 0 \leq \zeta \leq 1$$

Now suppose that,

$$\nu' = Max_{\sigma}\phi\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x'))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Bigg)\Bigg\}\Bigg]),$$

$$\nu^* = Max_{\sigma}\phi\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(U_\sigma^\gamma)^\dagger-(\Theta_\sigma(x^*))^\dagger}{(U_\sigma^\gamma)^\dagger-(L_\sigma^\gamma)^\dagger}\Bigg)\Bigg\}\Bigg]),$$

$$\eta' = Min_{\sigma}\zeta\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(\Theta_\sigma(x'))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Bigg)\Bigg\}\Bigg],$$

$$\text{and}\quad \eta^* = Min_{\sigma}\zeta\Bigg[1-exp\Bigg\{-\psi\Bigg(\frac{(\Theta_\sigma(x^*))^\dagger-(L_\sigma^\lambda)^\dagger}{(U_\sigma^\lambda)^\dagger-(L_\sigma^\lambda)^\dagger}\Bigg)\Bigg\}\Bigg] \quad \text{for at least one } \sigma.$$

Then, $\nu^* \leq (<)\nu'$ and $\eta^* \geq (>)\eta'$ which gives $(\nu^*-\eta^*) < (\nu'-\eta')$ that implies the solution is not optimal which contradicts that $x'(x' \neq x^*)$ is the only one optimal solution of (6). Hence, it is an effective solution of (6). Hence the proof is now complete. □

4.1.3 Hyperbolic type intuitionistic hesitant membership functions approach (HTIHMFA)

The truth membership function of hyperbolic type $\gamma_\sigma^{Hf_i}(\Theta_\sigma(x))$ and a falsity membership function of hyperbolic type $\lambda_\sigma^{Hf_i}(\Theta_\sigma(x))$ membership functions under IHF environment can be explained in the following way: For truth hesitant fuzzy membership functions:

$$\gamma_\sigma^{Hf_1}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\gamma \\ \phi_1\Bigg[\dfrac{1}{2}+\dfrac{1}{2}tanh\Bigg\{\Bigg(\dfrac{(U_\sigma^\gamma)^\dagger+(L_\sigma^\gamma)^\dagger}{2}-(\Theta_\sigma(x))^\dagger\Bigg)\tau_\sigma\Bigg\}\Bigg] & \text{if } L_\sigma^\gamma \leq \Theta_\sigma(x)) \leq U_\sigma^\gamma \\ 0 & \text{if } \Theta_p\sigma(x)) > U_\sigma^\gamma \end{cases}$$

$$\gamma_\sigma^{Hf_2}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \leq L_\sigma^\gamma \\ \phi_2\Bigg[\dfrac{1}{2}+\dfrac{1}{2}tanh\Bigg\{\Bigg(\dfrac{(U_\sigma^\gamma)^\dagger+(L_\sigma^\gamma)^\dagger}{2}-(\Theta_\sigma(x))^\dagger\Bigg)\tau_\sigma\Bigg\}\Bigg] & \text{if } L_\sigma^\gamma \leq \Theta_\sigma(x)) \leq U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

.....

$$\gamma_\sigma^{Hf_n}(\Theta_\sigma(x)) = \begin{cases} 1 & \text{if } \Theta_\sigma(x) \le L_\sigma^\gamma \\ \phi_n\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x))^\dagger\right)\tau_\sigma\right\}\right] & \text{if } L_\sigma^\gamma \le \Theta_\sigma(x)) \le U_\sigma^\gamma \\ 0 & \text{if } \Theta_\sigma(x)) > U_\sigma^\gamma \end{cases}$$

For Falsity hesitant fuzzy membership functions

$$\lambda_\sigma^{Hf_1}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \le L_\sigma^\lambda \\ \zeta_1\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] & \text{if } L_\sigma^\lambda \le \Theta_\sigma(x)) \le U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

$$\lambda_\sigma^{Hf_2}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \le L_\sigma^\lambda \\ \zeta_2\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] & \text{if } L_\sigma^\lambda \le \Theta_\sigma(x)) \le U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

.....

$$\lambda_\sigma^{Hf_n}(\Theta_\sigma(x)) = \begin{cases} 0 & \text{if } \Theta_\sigma(x) \le L_\sigma^\lambda \\ \zeta_n\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] & \text{if } L_\sigma^\lambda \le \Theta_\sigma(x)) \le U_\sigma^\lambda \\ 1 & \text{if } \Theta_\sigma(x)) > U_\sigma^\lambda \end{cases}$$

Where $\tau_\sigma = \dfrac{6}{U_\sigma - L_\sigma}$ denotes the ambiguity degree or shape parameter assigned by the decision-maker. Assume that $\gamma_\sigma^{Hf_i}(\Theta_\sigma(x)) \ge \nu_i$ and $\lambda_\sigma^{Hf_i}(\Theta_\sigma(x)) \le \eta_i$ for $i = 1, 2, ..., n$ and $\forall\sigma$, where the parameter $\dagger > 0$. The auxiliary parameters ν_i and η_i allow the problem (4) to be changed into (7)

$$\text{HTIHMFA} \quad Max\left(\sum_i \nu_i - \sum_i \eta_i\right)$$

Subject to

$$\begin{aligned}
&\phi_1\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x))^\dagger\right)\tau_\sigma\right\}\right] \ge \nu_1, \\
&\phi_2\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x))^\dagger\right)\tau_\sigma\right\}\right] \ge \nu_2, \\
&........., \\
&\phi_n\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x))^\dagger\right)\tau_\sigma\right\}\right] \ge \nu_n; \\
&\zeta_1\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] \le \eta_1, \\
&\zeta_2\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] \le \eta_2, \\
&........., \\
&\zeta_n\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] \le \eta_n;
\end{aligned} \tag{7}$$

$\nu_i \geq \eta_i; \nu_i + \eta_i \leq 1$ and $\eta_i, \nu_i, \phi_i, \zeta_i \in [0,1]$ $\forall i = 1,2,...,n$ Where $\tau_\sigma = \dfrac{6}{U_\sigma - L_\sigma}$ all the constraints of (3).

Theorem 4.3. *There is only one optimal solution (x^*, ν^*, η^*) of (7) that is also an efficient solution to the problem (3) where $\nu^* = (\nu_1^*, \nu_2^*, ..., \nu_n^*)$ and $\eta^* = (\eta_1^*, \eta_2^*, ..., \eta_n^*)$*

Proof. Assume that (x^*, ν^*, η^*) be the only optimal solution of (7) that it is an inefficient solution to the problem (3). Then there exist different feasible alternative $x'(x' \neq x^*)$ of problem (3), so that $\Theta_\sigma(x^*) \leq \Theta_\sigma(x')$ $\forall \sigma = 1,2,....,\Sigma$ and $\Theta_\sigma(x^*) < \Theta_\sigma(x')$ for at least one σ .
Therefore, we have

$$tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\} \leq tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\} \quad \forall$$
$$\sigma = 1,2,....,\Sigma$$
$$\text{and} \quad tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\} < tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\}$$
$$\text{for atleast one} \quad \sigma,$$

$$\begin{aligned}
& Max_{\forall\sigma}\phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\}\right] \\
& \leq Max_{\forall\sigma}\phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\}\right] \\
\text{and} \quad & Max_\sigma\phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\}\right] \\
& < Max_\sigma\phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\gamma)^\dagger + (L_\sigma^\gamma)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\}\right] \\
& \text{for at least one } \sigma, \quad 0 \leq \phi \leq 1
\end{aligned}$$

Similarly,

$$\begin{aligned}
& Min_{\forall\sigma}\zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x^*))^\dagger - \frac{(U_\sigma)^\dagger + (L_\sigma)^\dagger}{2}\right)\tau_\sigma\right\}\right] \\
& \geq Min_{\forall\sigma}\zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x'))^\dagger - \frac{(U_\sigma)^\dagger + (L_\sigma)^\dagger}{2}\right)\tau_\sigma\right\}\right] \\
\text{and} \quad & Min_\sigma\zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x^*))^\dagger - \frac{(U_\sigma)^\dagger + (L_\sigma)^\dagger}{2}\right)\tau_\sigma\right\}\right] \\
& > Min_\sigma\zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x'))^\dagger - \frac{(U_\sigma)^\dagger + (L_\sigma)^\dagger}{2}\right)\tau_\sigma\right\}\right] \\
& \text{for at least one } \sigma, \quad 0 \leq \zeta \leq 1
\end{aligned}$$

Table 2: Tabulation value of objective functions

	$WG(C_1, C_2)$	$d(C_1, C_2)$
C^1	$WG(C^1)$	$d(C^1)$
C^2	$WG(C^2)$	$d(C^2)$

Now suppose that,

$$\begin{aligned}
\nu' &= Max_\sigma \phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2} - (\Theta_\sigma(x'))^\dagger\right)\tau_\sigma\right\}\right],\\
\nu^* &= Max_\sigma \phi\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left(\frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2} - (\Theta_\sigma(x^*))^\dagger\right)\tau_\sigma\right\}\right],\\
\eta' &= Min_\sigma \zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x'))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right],\\
\text{and} \quad \eta^* &= Min_\sigma \zeta\left[\frac{1}{2} + \frac{1}{2}tanh\left\{\left((\Theta_\sigma(x^*))^\dagger - \frac{(U_\sigma^\lambda)^\dagger + (L_\sigma^\lambda)^\dagger}{2}\right)\tau_\sigma\right\}\right] \quad \text{for at least one } \sigma.
\end{aligned}$$

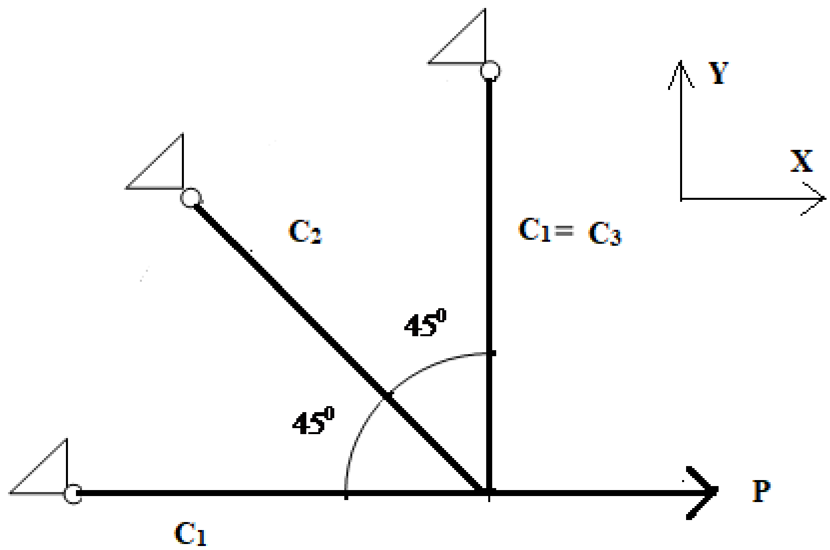
Then, $\nu^* \leq (<)\nu'$ and $\eta^* \geq (>)\eta'$ which gives $(\nu^* - \eta^*) < (\nu' - \eta')$ that implies the solution is not optimal which contradicts that $x'(x' \neq x^*)$ is the only one optimal solution of (7). Hence, it is an effective solution of (7). Hence the proof is now complete. □

5 Proposed Algorithm

5.1 Computation Algorithm for MOSP using IHF programming technique

Step 1 Solve the first goal function in the collection of objectives, (1) treating it as a single objective while taking into account the specified constraints. Evaluate the values of the objective functions and decision variables.
Step 2 Calculate the values of the remaining objectives based on the values of these decision variables.
Step 3 For the remaining objective functions, repeat **Step 1** and **Step 2**.
Step 4 As per the **Step 3**, obtained the corresponding tabulated values of objective functions from a Table 2 as follows:
Step 5 The upper and lower limits are $U_1 = max\left\{WG(C^1), WG(C^2)\right\}$,
$L_1 = min\left\{WG(C^1), WG(C^2)\right\}$ for weight function $WG(C)$, where $WG(C) \in [L_1, U_1]$ and the upper limit and lower limit of objective are $U_2 = max\left\{d(C^1), d(C^2)\right\}, L_2 = min\left\{d(C^1), d(C^2)\right\}$ for deflection function $d(C)$, where $d(C) \in [L_1, U_1]$ are identified.
Step 6 Now the IHF programming approach for MOSOP with linear (or exponential or hyperbolic) truth intuitionistic membership and falsity intuitionistic membership functions gives equivalent nonlinear programming problem as

$$\begin{aligned}
&Max\left(min\gamma_\sigma^{If_i}(WG(C))\right); Max\left(min\gamma_\sigma^{If_i}(d(C))\right);\\
&Min\left(max\lambda_\sigma^{If_i}(WG(C))\right); Min\left(max\lambda_\sigma^{If_i}(d(C))\right)\\
&\text{Subject to,} \quad [T(C)] = [T_0]\\
&C \in [C_{min}, C_{max}], \quad If_i = Lf_i, Ef_i, Hf_i; \quad i = 1, 2, .., n]\\
&\text{where} \quad x \in E = \{x \in \Re : g_j \leq or \geq b_j \quad j = 1, 2, ...m\} \quad \text{and} \quad L_i \leq x_i \leq U_i
\end{aligned} \tag{8}$$



bar truss.png bar truss.bb

Figure 1: The three-bar planar truss's design

Now, using arithmetic aggregation operator the above equation (8) can be formulated as follows:

$$\begin{aligned} & Max\Im = \frac{\nu_1+\nu_2+.....+\nu_n}{n} - \frac{\eta_1+\eta_2+.....+\eta_n}{n} \\ \text{Subject to,}\quad & \gamma_\sigma^{If_1}(WG(C)) \geq \nu_1, \gamma_\sigma^{If_2}(WG(C)) \geq \nu_2, ..., \gamma_\sigma^{If_n}(WG(C)) \geq \nu_n; \\ & \lambda_\sigma^{If_1}(WG(C)) \leq \eta_1, \lambda_\sigma^{If_2}(WG(C)) \leq \eta_2, ..., \lambda_\sigma^{If_n}(WG(C)) \leq \eta_n; \\ & \gamma_\sigma^{If_1}(d(C)) \geq \nu_1, \gamma_\sigma^{If_2}(d(C)) \geq \nu_2, ..., \gamma_\sigma^{If_n}(d(C)) \geq \nu_n; \\ & \lambda_\sigma^{If_1}(d(C)) \leq \eta_1, \lambda_\sigma^{If_2}(d(C)) \leq \eta_2, ..., \lambda_\sigma^{If_n}(d(C)) \leq \eta_n \\ & [T(C)] = [T_0]; C \in [C_{min}, C_{max}], \quad If_i = Lf_i, Ef_i, Hf_i; C \geq 0; \\ & \nu_i, \eta_i \in [0,1]; \nu_i + \eta_i \leq 1` \quad i = 1,2,..,n \end{aligned} \tag{9}$$

Step 8 An appropriate mathematical programming algorithm can easily solve the above non-linear programming problem (9).

6 Numerical solution of a three-bar truss MOSOP

In Figure (1), a well-known three-bar planar truss structure is taken into consideration to minimize vertical deflection $d(C_1, C_2)$ along x and y axes at the loading point of a statistically loaded three-bar planar truss subjected to stress $T_i(C_1, C_2)$ constraints on each of the truss members $i = 1, 2, 3$ and reduce structural weight $WG(C_1, C_2)$. The MOSOP can be stated in the following manner:

Table 3: Tabulation value of objective functions

	$WG\left(C_1, C_2\right)$	$d_x\left(C_1, C_2\right)$	$d_x\left(C_1, C_2\right)$
C^1	2.187673	20	5.8578664
C^2	15	3	1
C^3	10.1	3.960784	0.03921569

$$\begin{aligned}
\text{Minimize} \quad & WG\left(C_1, C_2\right)=\delta L\left(2 \sqrt{2} C_1+C_2\right), \\
\text{Minimize} \quad & d_x\left(C_1, C_2\right)=\frac{P L\left(2 C_1+C_2\right)}{E\left(2 C_1^2+2 C_1 C_2\right)}, \\
\text{Minimize} \quad & d_y\left(C_1, C_2\right)=\frac{P L C_2}{E\left(2 C_1^2+2 C_1 C_2\right)}, \\
\text{subject to,} \; & T_1\left(C_1, C_2\right)=\frac{P L\left(2 C_1+C_2\right)}{\left(2 C_1^2+2 C_1 C_2\right)} \leq\left[T_1^T\right], \\
& T_2\left(C_1, C_2\right)=\frac{P}{\left(\sqrt{2} C_1+C_2\right)} \leq\left[T_2^T\right], \\
& T_3\left(C_1, C_2\right)=\frac{P C_2}{\left(2 C_1^2+2 C_1 C_2\right)} \leq\left[T_3^C\right], \quad C_i^{min} \leq C_i \leq C_i^{max} \quad i=1,2
\end{aligned} \tag{10}$$

Where, applied load=P;material density= δ,$L=$ Length of each bar, maximum limit of tensile stress for bar 1 and 2 $=T_i^T$ for $i=1,2$,maximum limit of compressive stress for bar 3=T_3^C, Youngs modulus $=E$,$C_1=$Bar 1 and Bar 3 cross sections and $C_2=$ Bar 2 cross section. d_x and d_y are the deflection of loaded along x and $y$axes respectively.

The input data for MOSOP (10) is given as follows:
$P=20KN, \quad \delta=100KN/m^3, \quad L=1m, \left[T_1^T\right]=20KN/m^2, \quad \left[T_2^T\right]=10KN/m^2$
and $\left[T_3^C\right]=20KN/m^2, \quad E=2\times 10^8 KN/m^2, \quad 0.1\times 10^{-4}m^2 \leq C_1, C_2 \leq 0.5\times 10^{-4}m^2$

Solution According to step 2 the corresponding tabulated values of objective functions obtained from Table 3 as follows:

Here, $WG_U^\gamma=WG_U^\lambda=15, \quad WG_L^\gamma=2.187673, WG_L^\lambda=WG_L^\gamma+\epsilon_1$, where $0 \leq \epsilon_1 \leq (15-2.187673)$, $(d_x)_U^\gamma=(d_x)_U^\lambda=20, (d_x)_L^\gamma=3, (d_x)_L^\lambda+\epsilon_2,$ where $0 \leq \epsilon_2 \leq (20-3)$, $(d_y)_U^\gamma=(d_y)_U^\lambda=5.857864, (d_y)_L^\gamma=0.03921569, (d_y)_L^\lambda+\epsilon_3,$ where $0 \leq \epsilon_3 \leq (5.857864-0.03921569)$ Using the Linear type hesitant membership functions approach (LTHMFA) (5) the problem (10)
equivalent to the following (11)

$$Max \Im=\frac{1}{3}\left(\Sigma_{i=1}^3 \nu_i-\Sigma_{i=1}^3 \eta_i\right)$$

Subject to,
For 1st objective

$$
\begin{aligned}
&(2C_1+C_2)^{\dagger}+\left((15)^{\dagger}-(2.187673)^{\dagger}\right)\times\frac{\nu_1}{0.98}\le(15)^{\dagger},\\
&(2C_1+C_2)^{\dagger}+\left((15)^{\dagger}-(2.187673)^{\dagger}\right)\times\frac{\nu_2}{0.99}\le(15)^{\dagger},\\
&(2C_1+C_2)^{\dagger}+\left((15)^{\dagger}-(2.187673)^{\dagger}\right)\times\nu_3\le(15)^{\dagger}\\
&(2C_1+C_2)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\le\left((15)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\right)\times\frac{\eta_1}{0.98},\\
&(2C_1+C_2)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\le\left((15)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\right)\times\frac{\eta_2}{0.99},\\
&(2C_1+C_2)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\le\left((15)^{\dagger}-(2.187673)^{\dagger}-(\epsilon_1)^{\dagger}\right)\times\eta_3
\end{aligned}
$$

For 2nd objective

$$
\begin{aligned}
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}+\left((20)^{\dagger}-(3)^{\dagger}\right)\times\frac{\nu_1}{0.98}\le(20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}+\left((20)^{\dagger}-(3)^{\dagger}\right)\times\frac{\nu_2}{0.99}\le(20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}+\left((20)^{\dagger}-(3)^{\dagger}\right)\times\nu_3\le(20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\le\left((20)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\right)\times\frac{\eta_1}{0.98},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\le\left((20)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\right)\times\frac{\eta_2}{0.99},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\le\left((20)^{\dagger}-(3)^{\dagger}-(\epsilon_2)^{\dagger}\right)\times\eta_3
\end{aligned}
\tag{11}
$$

For 3rd objective

$$
\begin{aligned}
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}+\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}\right)\times\frac{\nu_1}{0.98}\le(5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}+\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}\right)\times\frac{\nu_2}{0.99}\le(5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}+\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}\right)\times\nu_3\le(5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\le\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\right)\times\frac{\eta_1}{0.98},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\le\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\right)\times\frac{\eta_2}{0.99},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\le\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\right)\times\eta_3,\\
&\nu_i\ge\eta_i,\nu_i+\eta_i\le1,\nu_i,\eta_i\in[0,1];`\quad i=1,2,3\quad\text{and all the constraints of (10).}
\end{aligned}
$$

Using the Exponential type hesitant membership functions approach (ETHMFA) (6) the problem (10) equivalent to the following (12)

$$Max\Im = \frac{1}{3}\left(\Sigma_{i=1}^{3}\nu_i - \Sigma_{i=1}^{3}\eta_i\right)$$

Subject to,
For 1st objective

$$\begin{aligned}
&(2C_1+C_2)^{\dagger} - \left((15)^{\dagger} - (2.187673)^{\dagger}\right) \times ln\left(1-\frac{\nu_1}{0.98}\right)/\psi \leq (15)^{\dagger},\\
&(2C_1+C_2)^{\dagger} - \left((15)^{\dagger} - (2.187673)^{\dagger}\right) \times ln\left(1-\frac{\nu_2}{0.99}\right)/\psi \leq (15)^{\dagger},\\
&(2C_1+C_2)^{\dagger} - \left((15)^{\dagger} - (2.187673)^{\dagger}\right) \times ln\left(1-\nu_3\right)/\psi \leq (15)^{\dagger}\\
&(2C_1+C_2)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger} \leq \left((15)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger}\right) \times \left\{-ln\left(1-\frac{\eta_1}{0.98}\right)\right\}/\psi,\\
&(2C_1+C_2)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger} \leq \left((15)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger}\right) \times \left\{-ln\left(1-\frac{\eta_2}{0.99}\right)\right\}/\psi,\\
&(2C_1+C_2)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger} \leq \left((15)^{\dagger} - (2.187673)^{\dagger} - (\epsilon_1)^{\dagger}\right) \times \{-ln\left(1-\eta_3\right)\}/\psi
\end{aligned}$$

For 2nd objective

$$\begin{aligned}
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((20)^{\dagger} - (3)^{\dagger}\right) \times ln\left(1-\frac{\nu_1}{0.98}\right)/\psi \leq (20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((20)^{\dagger} - (3)^{\dagger}\right) \times ln\left(1-\frac{\nu_2}{0.99}\right)/\psi \leq (20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((20)^{\dagger} - (3)^{\dagger}\right) \times ln\left(1-\nu_3\right)/\psi \leq (20)^{\dagger},\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger} \leq \left((20)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger}\right) \times \left\{-ln\left(1-\frac{\eta_1}{0.98}\right)\right\}/\psi,\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger} \leq \left((20)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger}\right) \times \left\{-ln\left(1-\frac{\eta_2}{0.99}\right)\right\}/\psi,\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger} \leq \left((20)^{\dagger} - (3)^{\dagger} - (\epsilon_2)^{\dagger}\right) \times \{-ln\left(1-\eta_3\right)\}/\psi
\end{aligned} \tag{12}$$

For 3rd objective

$$\begin{aligned}
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((5.857864)^{\dagger} - (0.03921569)^{\dagger}\right) \times ln\left(1-\frac{\nu_1}{0.98}\right)/\psi \leq (5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((5.857864)^{\dagger} - (0.03921569)^{\dagger}\right) \times ln\left(1-\frac{\nu_2}{0.99}\right)/\psi \leq (5.857864)^{\dagger},\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger} - \left((5.857864)^{\dagger} - (0.03921569)^{\dagger}\right) \times ln\left(1-\nu_3\right)/\psi \leq (5.857864)^{\dagger},
\end{aligned}$$

$$\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\leq\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\right)$$
$$\times\left\{-ln\left(1-\frac{\eta_1}{0.98}\right)\right\}/\psi,$$
$$\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\leq\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\right)$$
$$\times\left\{-ln\left(1-\frac{\eta_2}{0.99}\right)\right\}/\psi,$$
$$\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\leq\left((5.857864)^{\dagger}-(0.03921569)^{\dagger}-(\epsilon_3)^{\dagger}\right)$$
$$\times\{-ln\,(1-\eta_3)\}/\psi,$$
$$\nu_i\geq\eta_i,\nu_i+\eta_i\leq 1,\nu_i,\eta_i\in[0,1];` \quad i=1,2,3 \quad \text{and all the constraints of } (10).$$

Using the Hyperbolic type hesitant membership functions approach (HTHMFA) (7) the problem (10) equivalent to the following (13)
For 1st objective

$$Max\Im=\frac{1}{3}\left(\Sigma_{i=1}^{3}\nu_i-\Sigma_{i=1}^{3}\eta_i\right)$$

Subject to,
For 1st objective

$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}+tanh^{-1}\left(\frac{2\nu_1}{0.98}-1\right)\leq\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}+tanh^{-1}\left(\frac{2\nu_2}{0.99}-1\right)\leq\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}+tanh^{-1}(2\nu_3-1)\leq\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}-tanh^{-1}\left(\frac{2\eta_1}{0.98}-1\right)\leq\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}+(\epsilon_1)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}-tanh^{-1}\left(\frac{2\eta_2}{0.99}-1\right)\leq\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}+(\epsilon_1)^{\dagger}\right),$$
$$(2C_1+C_2)^{\dagger}\,\tau_{WG(C)}-tanh^{-1}\left(2\eta_3-1\right)\leq\frac{\tau_{WG(C)}}{2}\left((15)^{\dagger}+(2.187673)^{\dagger}+(\epsilon_1)^{\dagger}\right)$$

Table 4: The input values for MOSOP (10)

P (KN)	δ (KN/m^3)	$L(m)$	$[T_1^T]$ (KN/m^2)	$[T_2^T]$ (KN/m^2)	$[T_3^C]$ (KN/m^2)	E (KN/m^2)	$C_i^{min}.C_i^{max}$ $(10^{-4}m^2)$
20	100	1	20	10	20	2×10^7	$C_1^{min}=0.1, C_1^{max}=5.0,$ $C_2^{min}=0.1, C_2^{max}=5.0$

For 2nd objective

$$\begin{aligned}
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}+tanh^{-1}\left(\frac{2\nu_1}{0.98}-1\right)\leq\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}+tanh^{-1}\left(\frac{2\nu_2}{0.99}-1\right)\leq\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}+tanh^{-1}\left(2\nu_3-1\right)\leq\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}-tanh^{-1}\left(\frac{2\eta_1}{0.98}-1\right)\leq\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}+(\epsilon_2)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}-tanh^{-1}\left(\frac{2\eta_2}{0.99}-1\right)\leq\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}+(\epsilon_2)^{\dagger}\right),\\
&\left(\frac{20(2C_1+C_2)}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_x(C)}-tanh^{-1}\left(2\eta_3-1\right)\leq\frac{\tau_{d_x(C)}}{2}\left((20)^{\dagger}+(3)^{\dagger}+(\epsilon_2)^{\dagger}\right)
\end{aligned}\tag{13}$$

For 3rd objective

$$\begin{aligned}
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}+tanh^{-1}\left(\frac{2\nu_1}{0.98}-1\right)\leq\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}\right),\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}+tanh^{-1}\left(\frac{2\nu_2}{0.99}-1\right)\leq\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}\right),\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}+tanh^{-1}\left(2\nu_3-1\right)\leq\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}\right),
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}-tanh^{-1}\left(\frac{2\eta_1}{0.98}-1\right)\leq\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}+(\epsilon_3)^{\dagger}\right),\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}-tanh^{-1}\left(\frac{2\eta_2}{0.99}-1\right)\leq\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}+(\epsilon_3)^{\dagger}\right),\\
&\left(\frac{20C_2}{2C_1^2+2C_1C_2}\right)^{\dagger}\tau_{d_y(C)}-tanh^{-1}\left(2\eta_3-1\right)\leq\frac{\tau_{d_y(C)}}{2}\left((5.857864)^{\dagger}+(0.03921569)^{\dagger}+(\epsilon_3)^{\dagger}\right)
\end{aligned}$$

where $\tau_{WG(C)}=\dfrac{6}{15-2.187673}\quad\tau_{d_x(C)}=\dfrac{6}{20-3}$ and $\tau_{d_y(C)}=\dfrac{6}{5.857864-0.03921569}$,
$\nu_i\geq\eta_i,\nu_i+\eta_i\leq 1,\nu_i,\eta_i\in[0,1];`\quad i=1,2,3$ and all the constraints of (10).

Comparison of optimal solution of MOSOP (10) using several methods.

The Pareto optimal solution of MOSOP model (10) using fuzzy, intuitionistic fuzzy, and intuitionistic hesitant fuzzy multi-objective nonlinear programming techniques is given in Table 5. Here we get the best

Table 5: Optimal values on Structural Weight and Deflections for † = 1

Membership function	Various algorithm MONLP	$C_1 \times 10^{-4} m^2$	$C_2 \times 10^{-4} m^2$	$WG(C_1, C_2)$	$d_x(C_1, C_2)$	$d_y(C_1, C_2)$
Linear-type	Fuzzy multi-objective nonlinear programming[14]	2.677489	0.1000000	5.454979	7.335216	0.1344683
Linear-type	Intuitionistic fuzzy multi-objective nonlinear programming $\epsilon_1 = 0.76873962$, $\epsilon_2 = 1.7$, $\epsilon_3 = 0.2480392$ [14]	2.613073	0.1000000	5.326147	7.512768	0.1410545
Linear type	Proposed Method $\epsilon_1 = 0.76873962$, $\epsilon_2 = 1.7$, $\epsilon_3 = 0.2480392$	2.576483	0.1000000	5.252965	7.617507	0.1450135
Exponential -type	Proposed Method $\epsilon_1 = 0.76873962$, $\epsilon_2 = 1.7$, $\epsilon_3 = 0.2480392$	2.677490	0.1000000	5.454980	7.335215	0.1344682
Hyperbolic -type	Proposed Method $\epsilon_1 = 0.76873962$, $\epsilon_2 = 1.7$, $\epsilon_3 = 0.2480392$	2.471704	0.1000000	5.043407	7.934265	0.13573195

solution for different tolerances $\epsilon_1, \epsilon_2, \epsilon_3$ for non-membership function of objective functions. The Table 5 shows that the proposed intuitionistic hesitant fuzzy optimization technique gives a better Pareto optimal solution from the perspective of structural optimization.

7 Conclusion and Future Implication

To demonstrate the performance of the stated algorithm, a numerical example is given and compare their results with the existing studies[14]. It is concluded from this study that the proposed work gives more reasonable ways to handle the hesitant fuzzy information to solve practical problems.
In the future, we shall lengthen the methodology of intuitionistic hesitant fuzzy optimization technique to the diverse fuzzy environment as well as different fields of application such as transportation, networking, portfolio management, and emerging decision problems.

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Sanjoy Biswas
Department of Applied Science and Humanities (Mathematics)
Guru Nanak Institute of Technology
Sodepur,Kolkata 700114, West Bengal, India
E-mail: sanjoy.biswas@gnit.ac.in

Samir Dey
Department of Mathematics
JIS University
81, Nilgunj Road, Agarpara, Kolkata 700109, West Bengal, India
E-mail: samir.dey@jisuniversity.ac.in