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Combined Product of Two RL-graphs and It's Applications

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Abstract. This paper introduces the notion of the combined product of two RL-graphs while it is an RL-graph. It is stated that the combined product has commutative properties, i.e., $G \boxtimes H$ and $H \boxtimes G$ are two isomorphic RL-graphs. Moreover, it is shown under a theorem that two isomorphic RL-graphs G and G' and two isomorphic RL-graphs H and H' have isomorphic combined products $G \boxtimes H$ and $G' \boxtimes H'$. Further, it is investigated the relationships between these graphs and their operations by some notions such as strong RL-graph, regular RL-graph, and totally regular RL-graph. Afterward, it is displayed in a theorem that the combined product of two regular (α-regular, complete, connected) RL-graphs is a regular (α-regular, complete, connected) RL-graph. It is also shown in theorems what properties certain types of RL-graphs combined will have. Also, these notions and theorems are clarified by some examples. The combined product of two RL-graphs has many applications in various fields, such as probability sciences, urban planning, etc. In this article, only two of these applications, which determine the impact of effective factors on people's quality of life and factors effective in raising the production of a factory, are stated and they are clarified by an example.

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Keywords and Phrases: RL-graph, Regular RL-graph, Combined product of two RL-graphs.

1 Introduction

After Euler used graphs to solve the Knigsberg bridge problem, this concept was precisely defined [3]. Since the exodus of graph theory, many researchers have substantially contributed to developing this theory by expressing new concepts and their applications in various fields [1, 2].

Zadeh (1965) [15] proposed the notion of a fuzzy subset as modeled uncertain and vague natural events. Hence, many authors have surveyed its applications. Kaufman initially presented a definition of a fuzzy graph that is more realistic and useful in natural situations [7]. Nevertheless, Rosenfeld laid the foundations for a fuzzy graph applied in various areas, including data mining, communication, clustering, scheduling theory, and planning [14]. In 2021, Mordeson presented many applications of this notion [8].

In 2022, Zahedi and his team introduced the notion of an L-graph, a graph whose membership degree of its vertices and edges comes from a residuated lattice [9]. Besides, they determined various applications of this notion [9, 16]. Furthermore, they defined some operations on these graphs called maximal product and kronecker product, and elaborated their applications in various fields [10, 11, 13]. After that, they constructed the novel L-fuzzy automaton of L-graph, named L-graph automaton, and used these automata to determine an effective solution to a disease like Corona-virus and drugs that have the most similar side effects or health benefits, etc. [9, 12].

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In this study, we introduce the notion of the combined product of two RL-graphs. This operator creates a connection between two unrelated structures and relates the effect of these two structures to each other. Therefore, aims to introduce the combined product of two RL-graphs using a comprehensive, well-defined operation. Additionally, the notions such as strong RL-graph, regular RL-graph, and totally regular RL-graph are notified in detail. Further, we investigated the relationships between these graphs and their operations. Finally, two applications of this operation are presented and elucidated. Accordingly, some examples and theorems are proposed for clarification of suggested notions.

2 Preliminaries

In this section, we prepare some notions of residuated lattice and RL-grpah. Also, some of the concepts used in this paper, such as path graph, cycle graph, complete graph, bipartite graph and complete bipartite graph, are taken from reference [5, 6, 18].

Definition 2.1. [17] *A residuated lattice is an algebra* $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *such that*

1. $L = (L, \wedge, \vee, 0, 1)$ is a lattice (the corresponding order will be denoted by $\leq$) with the smallest element 0 and the greatest element 1,
2. $L = (L, \otimes, 1)$ is a commutative monoid (i.e. $\otimes$ is commutative, associative and $x \otimes 1 = x$ holds),
3. $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ holds (adjointness condition).

Example 2.2. Consider the lattice $L = (P(X), \cap, \cup, \otimes, \rightarrow, \emptyset, X)$, where $X = \{a, b, c, d\}$, $A \otimes B = A \cap B$ and $A \rightarrow B = \begin{cases} X & if\ A \subset B, \\ B & otherwise, \end{cases}$ for every $A, B \subset X$. Then, this lattice L is the residuated lattice.

Proposition 2.3. [4] Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. Then the following properties hold:
(R_1) $1 * x = x$, where $* \in \{\wedge, \otimes, \rightarrow\}$,
(R_2) $x \otimes 0 = 0, 1' = 0, 0' = 1$,
(R_3) $x \otimes y \leq x \wedge y \leq x, y,$ and $y \leq (x \rightarrow y)$,
(R_4) $(x \rightarrow y) \otimes x \leq y$,
(R_5) $x \leq y$ implies $x * z \leq y * z$, where $* \in \wedge, \vee, \otimes$,
(R_6) $z \otimes (x \wedge y) \leq (z \otimes x) \wedge (z \otimes y)$,
(R_7) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$,

Definition 2.4. [9] *$G = (\alpha, \beta)$ is called an RL-graph on a simple graph $G^* = (V, E)$ if $\alpha : V \rightarrow L$ and $\beta : E \rightarrow L$ are functions, with $\beta(st) \leq \alpha(s) \otimes \alpha(t)$ for every $st \in E$.*
Besides, if G^* is a path (cycle, complete, connected) graph, then G is called a path (cycle, complete, connected) RL-graph on G^*.
The degree of a vertex v in RL-graph G, denoted by $d_G(v)$, is the number of edges of G incident with v.

Definition 2.5. [9] *Let $G_1 = (\alpha_1, \beta_1)$ and $G_2 = (\alpha_2, \beta_2)$ be two RL-graphs on $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively, and $c \in L \backslash \{1\}$. Then G_1 and G_2 are isomorphic with threshold c, denoted by $G_1 \cong_c G_2$ if there exists a bijection h from V_1 into V_2 such that the following conditions hold true for all $u, v \in V_1$:*

(i) $uv \in E_1$ if and only if $h(u)h(v) \in E_2$,

(ii) $\alpha_1(u) > c$ if and only if $\alpha_2(h(u)) > c$,

(iii) $\beta_1(uv) > c$ if and only if $\beta_2(h(u)h(v)) > c$.

h is an isomorphism ($\cong$) if and only if h is an isomorphism with threshold c for every $c \in L\backslash\{1\}$.

Definition 2.6. [11] *Let $G = (\alpha, \beta)$ on $G^* = (V, E)$ be an RL-graph that G^* is a regular graph. Then G is called the regular RL-graph. If α has the same value for all vertices of the regular RL-graph G, then G is α-regular RL-graph. Additionally, if β has the same value for all edges of the regular RL-graph G, then G is β-regular RL-graph. Besides, it is a totally regular RL-graph if G is α-regular and β-regular RL-graph.*

3 Combined product of two RL-graphs

In this section, the new operation on RL-graphs, combined product, is stated. Also, this section elaborates that this operation is well defined, i.e., the combined product of two RL-graphs is the RL-graph. It determines the number of vertex and edges of this operation. After that, this notion is clarified by an example. It is stated that the combined product has commutative properties, i.e., $G \boxtimes H$ and $H \boxtimes G$ are two isomorphic RL-graphs. In the following, it is shown that if two RL-graphs G and G' are isomorphic, and two RL-graphs H and H' are isomorphic, then their combined product $G \boxtimes H$ and $G' \boxtimes H'$ are isomorphic RL-graphs. Also, this theorem is clarified by an example. Afterward, we display in a theorem that the combined product of two regular (α-regular, complete, connected) RL-graphs is regular (α-regular, complete, connected) RL-graph, and the combined product of two RL-graphs that at least of them is disconnected. Also, we elaborate on this theorem with an example.

Definition 3.1. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be two RL-graphs. Then a combined product of these RL-graphs is defined by $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$, where*

(*i*) $V = V_1 \times V_2$,

(*ii*) $E = B \cup C \cup D$, where $B = \{(q, q'_i)(q, q'_j)|\ \ q \in V_1, q'_i q'_j \in E_2\}$, $C = \{(q_i, q')(q_j, q')|\ \ q_i q_j \in E_1, q' \in V_2\}$ and $D = \{(q_i, q'_j)(q_k, q'_l)|\ \ q_i q_k \in E_1, q'_j q'_l \in E_2\}$,

(*iii*) $\alpha((q_i, q'_j)) = \alpha_1(q_i) \vee \alpha_2(q'_j)$,

(*iv*) $$\beta((q_i, q'_j)(q_k, q'_l)) = \begin{cases} \alpha_1(q_i) \otimes \beta_2(q'_j q'_l) & if\ (q_i, q'_j)(q_k, q'_l) \in B, \\ \alpha_2(q'_j) \otimes \beta_1(q_i q_k) & if\ (q_i, q'_j)(q_k, q'_l) \in C, \\ \beta_1(q_i q_k) \otimes \beta_2(q'_j q'_l) & if\ (q_i, q'_j)(q_k, q'_l) \in D. \end{cases}$$

Theorem 3.2. *If $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ are two RL-graphs, then their combined product $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$ is the RL-graph.*

Proof. It suffices to show that

$$\begin{aligned} \beta((q_i, q'_j)(q_k, q'_l)) &\leq \alpha((q_i, q'_j)) \otimes \alpha((q_k, q'_l)), \\ &= (\alpha_1(q_i) \vee \alpha_2(q'_j)) \otimes (\alpha_1(q_k) \vee \alpha_2(q'_l)), \end{aligned}$$

for every $(q_i, q'_j)(q_k, q'_l) \in E$.
We know that there exist three cases, for every $(q_i, q'_j)(q_k, q'_l) \in E$.
Case 1. If $(q_i, q'_j)(q_k, q'_l) \in B$, then $q_i = q_k$. Thus,

$$\begin{aligned} \beta((q_i, q'_j)(q_k, q'_l)) &= \alpha_1(q_i) \otimes \beta_2(q'_j q'_l), \text{ By Definiton 3.1}(iv), \\ &\leq \alpha_1(q_i) \otimes (\alpha_2(q'_j) \otimes \alpha_2(q'_l)), \text{ By definition of the } RL\text{-graph } H, \\ &\leq (\alpha_1(q_i) \vee \alpha_2(q'_j)) \otimes (\alpha_2(q'_j) \otimes \alpha_2(q'_l)), \text{ By Proposition 2.3}(R_5), \\ &\leq (\alpha_1(q_i) \vee \alpha_2(q'_j)) \otimes \alpha_2(q'_l), \text{ By Proposition 2.3}(R_3), (R_5), \\ &\leq (\alpha_1(q_i) \vee \alpha_2(q'_j)) \otimes (\alpha_1(q_k) \vee \alpha_2(q'_l)), \text{ By Proposition 2.3}(R_5), \\ &= \alpha((q_i, q'_j)) \otimes \alpha((q_k, q'_l)). \end{aligned}$$

Case 2. If $(q_i, q'_j)(q_k, q'_l) \in C$, then same as above with some modifications, we gain that the following result is

$$\beta((q_i, q'_j)(q_k, q'_l)) \quad \leq \quad \alpha((q_i, q'_j)) \otimes \alpha((q_k, q'_l)).$$

Case 3. If $(q_i, q'_j)(q_k, q'_l) \in D$, then

$$\begin{aligned}
\beta((q_i, q'_j)(q_k, q'_l)) &= \beta_1(q_i q_k) \otimes \beta_2(q'_j q'_l), \text{ By Definiton 3.1(iv)}, \\
&\leq (\alpha_1(q_i) \otimes \alpha_1(q_k)) \otimes (\alpha_2(q'_j) \otimes \alpha_2(q'_l)), \text{ By definition of } RL\text{-graphs } G \text{ and } H, \\
&\leq (\alpha_1(q_i) \otimes \alpha_2(q'_j)) \otimes (\alpha_1(q_k) \otimes \alpha_2(q'_l)), \text{ By Definition 2.1(2)}, \\
&\leq (\alpha_1(q_i) \vee \alpha_2(q'_j)) \otimes (\alpha_1(q_k) \vee \alpha_2(q'_l)), \text{ By Proposition 2.3}(R_5), \\
&= \alpha((q_i, q'_j)) \otimes \alpha((q_k, q'_l)).
\end{aligned}$$

□

Example 3.3. Consider the residuated lattice L in Example 2.2 and two RL-graphs $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$, as in Figure 1, where $V_1 = \{q_1, q_2\}$, $E_1 = \{q_1 q_2\}$, $\alpha_1(q_1) = \{a, b\}$, $\alpha_1(q_2) = \{b, c\}$, $\beta_1(q_1 q_2) = \{b\}$, $V_2 = \{q'_1, q'_2, q'_3\}$, $E_2 = \{q'_2 q'_3\}$, $\alpha_2(q'_1) = \{a, c\}$, $\alpha_2(q'_2) = \{b, d\}$, $\alpha_2(q'_3) = \{a, c, d\}$ and $\beta_2(q'_2 q'_3) = \{d\}$. Then their combined product $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$, as in Figure 1, where $V = \{(q_1, q'_i), (q_2, q'_i) |\ i = 1, 2, 3\}$, $E = \{(q_1, q'_1)(q_2, q'_1), (q_1, q'_2)(q_2, q'_2),$ $(q_1, q'_3)(q_2, q'_3), (q_1, q'_2)(q_1, q'_3), (q_2, q'_2)(q_2, q'_3), (q_1, q'_2)(q_2, q'_3), (q_2, q'_2)(q_1, q'_3)\}$, $\alpha((q_1, q'_1)) = \{a, b, c\}$, $\alpha((q_1, q'_2)) = \{a, b, d\}$, $\alpha((q_1, q'_3)) = \{a, b, c, d\}$, $\alpha((q_2, q'_1)) = \{a, b, c\}$, $\alpha((q_2, q'_2)) = \{b, c, d\}$, $\alpha((q_2, q'_3)) = \{a, b, c, d\}$, $\beta((q_1, q'_1)(q_2, q'_1)) = \emptyset$, $\beta((q_1, q'_2)(q_2, q'_2)) = \{b\}$, $\beta((q_1, q'_3)(q_2, q'_3)) = \emptyset$, $\beta((q_1, q'_2)(q_1, q'_3)) = \emptyset$, $\beta((q_2, q'_2)(q_2, q'_3)) = \emptyset$, $\beta((q_1, q'_2)(q_2, q'_3)) = \emptyset$ and $\beta((q_2, q'_2)(q_1, q'_3)) = \emptyset$.

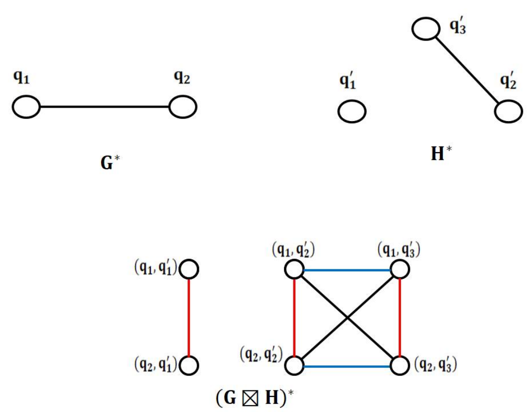

Figure 1: Two graphs G^* and H^*, and their combined product $(G \boxtimes H)^*$

Theorem 3.4. *If $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ are two RL-graphs, then $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$ and $H \boxtimes G = (\alpha', \beta')$ on $(H \boxtimes G)^* = (V', E')$ are two isomorphic RL-graphs.*

Proof. Consider the map $h : V \to V'$ such that $h(q_i, q'_j) = (q'_j, q_i)$. Then, clearly, h is a bijection function. We know that every edge of the RL-graph $G \boxtimes H$ has three cases.
Case 1. For every

$$\begin{aligned}
(q_i, q'_j)(q_i, q'_l) \in E \cap B &\iff q_i \in V_1, \quad q'_j q'_l \in E_2, \text{ By definition of } B, \\
&\iff (q'_j, q_i)(q'_l, q_i) \in E' \cap C', \text{ By definition of } C'.
\end{aligned}$$

Case 2. As same as case 1, with some modifications, we have

$$(q_i, q'_j)(q_k, q'_j) \in E \cap C \iff (q'_j, q_i)(q'_j, q_k) \in E' \cap B'.$$

Case 3. For every

$$\begin{aligned}(q_i, q'_j)(q_k, q'_l) \in E \cap D &\iff q_iq_k \in E_1,\ \ q'_jq'_l \in E_2, \text{ By definition of } D,\\ &\iff (q'_j, q_i)(q'_l, q_k) \in E' \cap D', \text{ By definition of } D'.\end{aligned}$$

For every

$$\alpha((q_i, q'_j)) = \alpha_1(q_i) \vee \alpha_2(q'_j) > c \iff \alpha'((q'_j, q_i)) = \alpha_2(q'_j) \vee \alpha_1(q_i) > c,$$

for every $c \in L \setminus 1$.
According to the three cases above, we can prove that

$$\beta((q_i, q'_j)(q_k, q'_j)) > c \iff \beta'((q'_j, q_i)(q'_l, q_k)) > c,$$

for every $c \in L \setminus 1$. $\square$

Example 3.5. Suppose three RL-graphs G, H and $G \boxtimes H$ in Example 3.3. Then $H \boxtimes G = (\alpha', \beta')$ on $(H \boxtimes G)^* = (V', E')$, as in Figure 2, where $V' = \{(q'_i, q_1), (q'_i, q_2)|\ \ i = 1, 2, 3\}$, $E' = \{(q'_1, q_1)(q'_1, q_2), (q'_2, q_1)(q'_2, q_2), (q'_3, q_1)(q'_3, q_2), (q'_2, q_1)(q'_3, q_1), (q'_2, q_2)(q'_3, q_2), (q'_2, q_1)(q'_3, q_2), (q'_2, q_2)(q'_3, q_1)\}$, $\alpha((q'_1, q_1)) = \{a, b, c\}$, $\alpha((q'_2, q_1)) = \{a, b, d\}$, $\alpha((q'_3, q_1)) = \{a, b, c, d\}$, $\alpha((q'_1, q_2)) = \{a, b, c\}$, $\alpha((q'_2, q_2)) = \{b, c, d\}$, $\alpha((q'_3, q_2)) = \{a, b, c, d\}$, $\beta((q'_1, q_1)(q'_1, q_2)) = \emptyset$, $\beta((q'_2, q_1)(q'_2, q_2)) = \{b\}$, $\beta((q'_3, q_1)(q'_3, q_2)) = \emptyset$, $\beta((q'_2, q_1)(q'_3, q_1)) = \emptyset$, $\beta((q'_2, q_2)(q'_3, q_2)) = \emptyset$, $\beta((q'_2, q_1)(q'_3, q_2)) = \emptyset$ and $\beta((q'_2, q_2)(q'_3, q_1)) = \emptyset$. Clearly, $G \boxtimes H$ and $H \boxtimes G$ are two isomorphic RL-graphs.

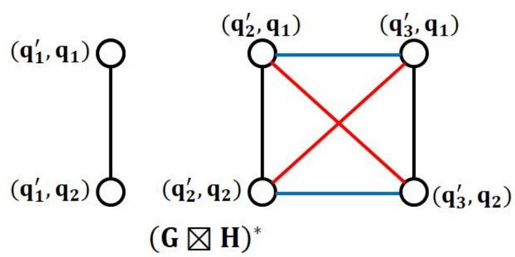

Figure 2: The graph $(H \boxtimes G)^*$

Proposition 3.6. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be two RL-graphs. Also, suppose their combined product $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$. Then*

(i) $|V| = |V_1| \times |V_2|$,

(ii) $|E| = (|E_1| \times |V_2|) + (|E_2| \times |V_1|) + 2(|E_1| \times |E_2|)$.

Proof. (i) The proof is straightforward.
(ii) We know that $|E| = |B| + |C| + |D|$, such that $|C| = |E_2| \times |V_1|$ and $|B| = |E_1| \times |V_2|$. In other hands, we know that $(q_i, q'_j)(q_k, q'_l) \in D \Rightarrow q_iq_k \in E_1, q'_jq'_l \in E_2$, and since the components are even ordered, $|D| = 2(|E_1| \times |E_2|)$. $\square$

Example 3.7. Consider two RL-graphs G and H, and their combined product $G \boxtimes H$ in Example 3.3. Thus,

$$|V| = 6 = |V_1| \times |V_2| \text{ and } |E| = 7 = (|E_1| \times |V_2|) + (|E_2| \times |V_1|) + 2(|E_1| \times |E_2|).$$

Theorem 3.8. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $G' = (\alpha'_1, \beta'_1)$ on $G'^* = (V'_1, E'_1)$ be two isomorphic RL-graphs, and let $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ and $H' = (\alpha'_2, \beta'_2)$ on $H'^* = (V'_2, E'_2)$ be two isomorphic RL-graphs. Then $G \boxtimes H$ on $(G \boxtimes H)^*$ and $G' \boxtimes H'$ on $(G' \boxtimes H')^*$ are two isomorphic RL-graphs.*

Proof. Suppose $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$ and $G' \boxtimes H' = (\alpha', \beta')$ on $(G' \boxtimes H')^* = (V', E')$. Also, since G and G' are two isomorphic RL-graphs, and H and H' are two isomorphic RL-graphs, there exist two bijection functions $h : V_1 \to V'_1$ such that for every $q_i \in V_1$ there exists only one $p_i \in V'_1$ such that $h(q_i) = p_i$, and $h' : V_2 \to V'_2$ such that for every $q'_i \in V_2$ there exists only one $p'_i \in V'_2$ such that $h'(q'_i) = p'_i$. Now, we define the map $g : V \to V'$ such that $g(q_i, q'_j) = (h(q_i), h'(q'_j))$, for every $(q_i, q'_j) \in V$. Clearly, this map is the bijection function. We suppose that $E = B \cup C \cup D$ and $E' = B' \cup C' \cup D'$. So

$$\begin{aligned} (q_i, q'_j)(q_k, q'_l) \in E \cap B \quad &\Leftrightarrow \quad (h(q_i), h'(q'_j))(h(q_k), h'(q'_l)) \in E' \cap B', \text{ By definition of } E \text{ and } E' \\ &\Leftrightarrow \quad g(q_i, q'_j)g(q_k, q'_l) \in E' \cap B'. \end{aligned}$$

Thus, we use the above result and gain the above result when $(q_i, q'_j)(q_k, q'_l) \in E \cap C$ and $(q_i, q'_j)(q_k, q'_l) \in E \cap D$. Clearly,

$$(q_i, q'_j)(q_k, q'_l) \in E \Leftrightarrow g(q_i, q'_j)g(q_k, q'_l) \in E'.$$

And

$$\begin{aligned} \alpha((q_i, q'_j)) = \alpha_1(q_i) \vee \alpha_2(q'_j) \geq c \quad &\Leftrightarrow \quad \alpha'_1(h(q_i)) \vee \alpha'_2(h'(q'_j)) \geq c, \text{ By definitions of } \vee, h \text{ and } h', \\ &\Leftrightarrow \quad \alpha'(g(q_i, q'_j)) \geq c. \end{aligned}$$

We know that the function β has three cases.
Case 1. If $(q_i, q'_j)(p_k, q'_l) \in B$, then

$$\begin{aligned} \beta((q_i, q'_j)(q_k, q'_l)) \geq c \quad &\Leftrightarrow \quad \alpha_1(q_i) \otimes \beta_2(q'_j q'_l) \geq c, \\ &\Leftrightarrow \quad \alpha'_1(h(q_i)) \otimes \beta'_2(h'(q'_j)h'(q'_l)) \geq c, \text{ By definitions of } h \text{ and } h', \\ &\Leftrightarrow \quad \beta'((h(q_i), h'(q'_j))(h(q_k), h'(q'_l))) \geq c, \\ &\Leftrightarrow \quad \beta'(g((q_i, q'_j)(q_k, q'_l))) \geq c. \end{aligned}$$

Case 2. If $(q_i, q'_j)(p_k, q'_l) \in C$, then the same as above,

$$\beta((q_i, q'_j)(q_k, q'_l)) \geq c \Leftrightarrow \beta'(g((q_i, q'_j)(q_k, q'_l))) \geq c.$$

Case 3. If $(q_i, q'_j)(p_k, q'_l) \in D$ such that $\beta((q_i, q'_j)(q_k, q'_l)) \geq c$, then

$$\begin{aligned} \beta_1(q_i q_k) \otimes \beta_2(q'_j q'_l) \geq c \quad &\Leftrightarrow \quad \beta'_1(h(q_i)h(q_k)) \otimes \beta'_2(h'(q'_j)h'(q'_l)) \geq c \\ &\Leftrightarrow \quad \beta'(g((q_i, q'_j)(q_k, q'_l))) \geq c. \end{aligned}$$

□

Example 3.9. Consider the residuated lattice $L = ([0, 1], \wedge, \vee, \otimes, \to, 0, 1)$ and two isomorphic RL-graphs $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $G' = (\alpha'_1, \beta'_1)$ on $G'^* = (V'_1, E'_1)$, as in Figure 3, where $a \otimes b = a \wedge b$, $a \to b = \begin{cases} 1 & if\ a \geq b, \\ b & otherwise, \end{cases}$ $V_1 = \{q_1, q_2, \ldots, q_5\}$, $E_1 = \{q_1q_2, q_2q_3, q_3q_4, q_4q_5, q_1q_5\}$, $\alpha_1(q_i) = 0.8$, for every $1 \leq i \leq 5$, $\beta_1(q_1q_2) = 0.6$, $\beta_1(q_2q_3) = 0.5$, $\beta_1(q_3q_4) = 0.3$, $\beta_1(q_4q_5) = 0.6$, $\beta_1(q_1q_5) = 0.4$, $V'_1 = \{q'_1, q'_2, \ldots, q'_5\}$,

$E'_1 = \{q'_1q'_3, q'_2q'_3, q'_2q'_4, q'_4q'_5, q'_1q'_5\}$, $\alpha'_1(q_i) = 0.8$, for every $1 \leq i \leq 5$, $\beta'_1(q'_1q'_3) = 0.6$, $\beta'_1(q'_1q'_5) = 0.3$, $\beta'_1(q'_2q'_3) = 0.4$, $\beta'_1(q'_4q'_5) = 0.5$ and $\beta'_1(q'_2q'_4) = 0.6$, and there exists the function $h : V_1 \to V'_1$ that is the isomorphism between two RL-graphs G and G', where $h(q_1) = q'_2$, $h(q_2) = q'_4$, $h(q_3) = q'_5$, $h(q_4) = q'_1$ and $h(q_5) = q'_3$. Also, suppose an RL-graph $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$, as in Figure 3, where $V_2 = \{p_1, p_2, p_3\}$, $E_2 = \{p_1p_2, p_2p_3, p_1p_3\}$, $\alpha_2(p_i) = 0.7$, for every $i = 1, 2, 3$, $\beta_2(p_1p_2) = 0.4$, $\beta_2(p_2p_3) = 0.3$ and $\beta_2(p_1p_3) = 0.2$. Given the complexity of the diagram here, we suppose that $H = H'$. Hence, as in Figure 4, $G \boxtimes H = (\alpha, \beta)$

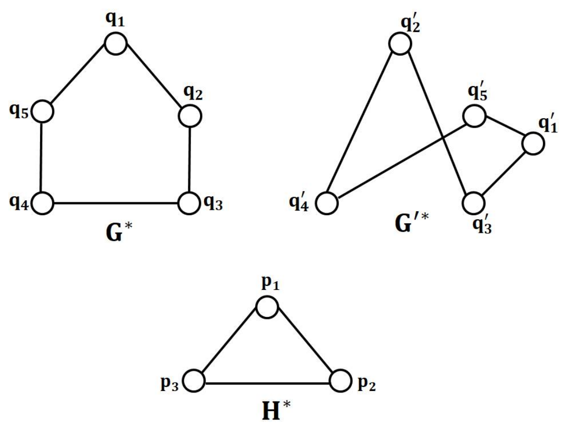

Figure 3: Two isomorphic graphs G^* and G'^*, and the graph H^*

on $(G \boxtimes H)^* = (V, E)$ is a combined product of two RL-graphs, where $V = \{(q_i, p_j)| \ 1 \leq i \leq 5, 1 \leq j \leq 3\}$, $E = \{(q_i, p_1)(q_i, p_2), (q_i, p_2)(q_i, p_3), (q_i, p_1)(q_i, p_3), (q_1, p_j)(q_2, p_j), (q_2, p_j)(q_3, p_j), (q_3, p_j)(q_4, p_j), (q_4, p_j)(q_5, p_j),$ $(q_1, p_j)(q_5, p_j), (q_1, p_1)(q_2, p_2), (q_1, p_1)(q_2, p_3), (q_1, p_1)(q_5, p_2), (q_1, p_1)(q_5, p_3), (q_1, p_2)(q_2, p_1), (q_1, p_2)(q_2, p_3),$ $(q_1, p_2)(q_5, p_1), (q_1, p_2)(q_5, p_3), (q_1, p_3)(q_2, p_1), (q_1, p_3)(q_2, p_2), (q_1, p_3)(q_5, p_1), (q_1, p_3)(q_5, p_2), (q_2, p_1)(q_3, p_2),$ $(q_2, p_1)(q_3, p_3), (q_2, p_2)(q_3, p_1), (q_2, p_2)(q_3, p_3), (q_2, p_3)(q_3, p_1), (q_2, p_3)(q_3, p_2), (q_3, p_1)(q_4, p_2), (q_3, p_1)(q_4, p_3),$ $(q_3, p_2)(q_4, p_1), (q_3, p_2)(q_4, p_3), (q_3, p_3)(q_4, p_1), (q_3, p_3)(q_4, p_2), (q_4, p_1)(q_5, p_2), (q_4, p_1)(q_5, p_3), (q_4, p_2)(q_5, p_1),$ $(q_4, p_2)(q_5, p_3), (q_4, p_3)(q_5, p_1), (q_4, p_3)(q_5, p_2), | \ 1 \leq i \leq 5, 1 \leq j \leq 3\}$, $\alpha((q_i, p_j)) = 0.8$, for every $(q_i, p_j) \in V$,

$$
\begin{array}{lll}
\beta((q_i, p_1)(q_i, p_2)) = 0.4, & \beta((q_i, p_2)(q_i, p_3)) = 0.3, & \beta((q_i, p_1)(q_i, p_3)) = 0.2, \\
\beta((q_1, p_j)(q_2, p_j)) = 0.6, & \beta((q_2, p_j)(q_3, p_j)) = 0.5, & \beta((q_3, p_j)(q_4, p_j)) = 0.3, \\
\beta((q_4, p_j)(q_5, p_j)) = 0.6, & \beta((q_1, p_j)(q_5, p_j)) = 0.4, & \beta((q_1, p_1)(q_2, p_2)) = 0.4, \\
\beta((q_1, p_1)(q_2, p_3)) = 0.2, & \beta((q_1, p_1)(q_5, p_2)) = 0.4, & \beta((q_1, p_1)(q_5, p_3)) = 0.2, \\
\beta((q_1, p_2)(q_2, p_1)) = 0.4, & \beta((q_1, p_2)(q_2, p_3)) = 0.3, & \beta((q_1, p_2)(q_5, p_1)) = 0.4, \\
\beta((q_1, p_2)(q_5, p_3)) = 0.3, & \beta((q_1, p_3)(q_2, p_1)) = 0.2, & \beta((q_1, p_3)(q_2, p_2)) = 0.3, \\
\beta((q_1, p_3)(q_5, p_1)) = 0.2, & \beta((q_1, p_3)(q_5, p_2)) = 0.3, & \beta((q_2, p_1)(q_3, p_2)) = 0.4, \\
\beta((q_2, p_1)(q_3, p_3)) = 0.2, & \beta((q_2, p_2)(q_3, p_1)) = 0.4, & \beta((q_2, p_2)(q_3, p_3)) = 0.3, \\
\beta((q_2, p_3)(q_3, p_1)) = 0.2, & \beta((q_2, p_3)(q_3, p_2)) = 0.3, & \beta((q_3, p_1)(q_4, p_2) = 0.3, \\
\beta((q_3, p_1)(q_4, p_3)) = 0.2, & \beta((q_3, p_2)(q_4, p_1)) = 0.3, & \beta((q_3, p_2)(q_4, p_3)) = 0.3, \\
\beta((q_3, p_3)(q_4, p_1)) = 0.2, & \beta((q_3, p_3)(q_4, p_2)) = 0.3, & \beta((q_4, p_1)(q_5, p_2)) = 0.4, \\
\beta((q_4, p_1)(q_5, p_3)) = 0.2, & \beta((q_4, p_2)(q_5, p_1)) = 0.4, & \beta((q_4, p_2)(q_5, p_3)) = 0.3, \\
\beta((q_4, p_3)(q_5, p_1)) = 0.2, & \beta((q_4, p_3)(q_5, p_2)) = 0.3. &
\end{array}
$$

and $G' \boxtimes H = (\alpha', \beta')$ on $(G' \boxtimes H)^* = (V', E')$, as in Figure 5, is a combined product of RL-graphs $V' = \{(q'_i, p_j)| \ 1 \leq i \leq 5, 1 \leq j \leq 3\}$, $E' = \{(q'_i, p_1)(q'_i, p_2), (q'_i, p_2)(q'_i, p_3), (q'_i, p_1)(q'_i, p_3), (q'_1, p_j)(q'_3, p_j),$ $(q'_1, p_j)(q'_5, p_j), (q'_2, p_j)(q'_3, p_j), (q'_4, p_j)(q'_5, p_j), (q'_2, p_j)(q'_4, p_j), (q'_2, p_1)(q'_4, p_2), (q'_2, p_1)(q'_4, p_3), (q'_2, p_1)(q'_3, p_2),$

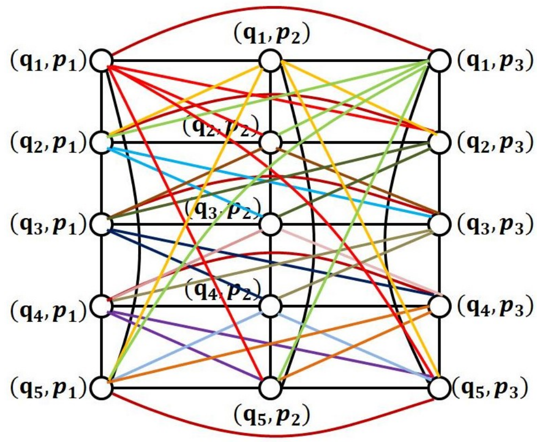

Figure 4: The graph $(G \boxtimes H)^*$

$(q'_2, p_1)(q'_3, p_3), (q'_2, p_2)(q'_4, p_1), (q'_2, p_2)(q'_4, p_3), (q'_2, p_2)(q'_3, p_1), (q'_2, p_2)(q'_3, p_3), (q'_2, p_3)(q'_4, p_1), (q'_2, p_3)(q'_4, p_2),$ $(q'_2, p_3)(q'_3, p_1), (q'_2, p_3)(q'_3, p_2), (q'_4, p_1)(q'_5, p_2), (q'_4, p_1)(q'_5, p_3), (q'_4, p_2)(q'_5, p_1), (q'_4, p_2)(q'_5, p_3), (q'_4, p_3)(q'_5, p_1),$ $(q'_4, p_3)(q'_5, p_2), (q'_5, p_1)(q'_1, p_2), (q'_5, p_1)(q'_1, p_3), (q'_5, p_2)(q'_1, p_1), (q'_5, p_2)(q'_1, p_3), (q'_5, p_3)(q'_1, p_1), (q'_5, p_3)(q'_1, p_2),$ $(q'_1, p_1)(q'_3, p_2), (q'_1, p_1)(q'_3, p_3), (q'_1, p_2)(q'_3, p_1), (q'_1, p_2)(q'_3, p_3), (q'_1, p_3)(q'_3, p_1), (q'_1, p_3)(q'_3, p_2), | \ 1 \leq i \leq 5,$ $1 \leq j \leq 3\}$, $\alpha'((q'_i, p_j)) = 0.8$, for every $(q'_i, p_j) \in V'$,

$$\begin{array}{lll}
\beta'((q'_i, p_1)(q'_i, p_2)) = 0.4, & \beta'((q'_i, p_2)(q'_i, p_3)) = 0.3, & \beta'((q'_i, p_1)(q'_i, p_3)) = 0.2, \\
\beta'((q'_2, p_j)(q'_4, p_j)) = 0.6, & \beta'((q'_4, p_j)(q'_5, p_j)) = 0.5, & \beta'((q'_5, p_j)(q'_1, p_j)) = 0.3, \\
\beta'((q'_1, p_j)(q'_3, p_j)) = 0.6, & \beta'((q'_2, p_j)(q'_3, p_j)) = 0.4, & \beta'((q'_2, p_1)(q'_4, p_2)) = 0.4, \\
\beta'((q'_2, p_1)(q'_4, p_3)) = 0.2, & \beta'((q'_2, p_1)(q'_3, p_2)) = 0.4, & \beta'((q'_2, p_1)(q'_3, p_3)) = 0.2, \\
\beta'((q'_2, p_2)(q'_4, p_1)) = 0.4, & \beta'((q'_2, p_2)(q'_4, p_3)) = 0.3, & \beta'((q'_2, p_2)(q'_3, p_1)) = 0.4, \\
\beta'((q'_2, p_2)(q'_3, p_3)) = 0.3, & \beta'((q'_2, p_3)(q'_4, p_1)) = 0.2, & \beta'((q'_2, p_3)(q'_4, p_2)) = 0.3, \\
\beta'((q'_2, p_3)(q'_3, p_1)) = 0.2, & \beta'((q'_2, p_3)(q'_3, p_2)) = 0.3, & \beta'((q'_4, p_1)(q'_5, p_2)) = 0.4, \\
\beta'((q'_4, p_1)(q'_5, p_3)) = 0.2, & \beta'((q'_4, p_2)(q'_5, p_1)) = 0.4, & \beta'((q'_4, p_2)(q'_5, p_3)) = 0.3, \\
\beta'((q'_4, p_3)(q'_5, p_1)) = 0.2, & \beta'((q'_4, p_3)(q'_5, p_2)) = 0.3, & \beta'((q'_5, p_1)(q'_1, p_2) = 0.3, \\
\beta'((q'_5, p_1)(q'_1, p_3)) = 0.2, & \beta'((q'_5, p_2)(q'_1, p_1)) = 0.3, & \beta'((q'_5, p_2)(q'_1, p_3)) = 0.3, \\
\beta'((q'_5, p_3)(q'_1, p_1)) = 0.2, & \beta'((q'_5, p_3)(q'_1, p_2)) = 0.3, & \beta'((q'_1, p_1)(q'_3, p_2)) = 0.4, \\
\beta'((q'_1, p_1)(q'_3, p_3)) = 0.2, & \beta'((q'_1, p_2)(q'_3, p_1)) = 0.4, & \beta'((q'_1, p_2)(q'_3, p_3)) = 0.3, \\
\beta'((q'_1, p_3)(q'_3, p_1)) = 0.2, & \beta'((q'_1, p_3)(q'_3, p_2)) = 0.3. &
\end{array}$$

We suppose that the function $g : V \to V'$, where $h(q_1, p_i) = (q'_2, p_i)$, $h(q_2, p_i) = (q'_4, p_i)$, $h(q_3, p_i) = (q'_5, p_i)$, $h(q_4, p_i) = (q'_1, p_i)$ and $h(q_5, p_i) = (q'_3, p_i)$, for every $i = 1, 2, 3$. Thus, two combined products, $G \boxtimes H$ and $G' \boxtimes H$ are isomorphic RL-graphs.

Corollary 3.10. *Let G and G' be two isomorphic RL-graphs. Then $G \boxtimes H$ and $G' \boxtimes H$ are two isomorphic RL-graphs and $H \boxtimes G$ and $H \boxtimes G'$ are two isomorphic RL-graphs, for every RL-graph H on H^*.*

Proof. The proof is the same as Theorem 3.8 with some modifications. □

Theorem 3.11. *Let $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$ be two RL-graphs. Then,*

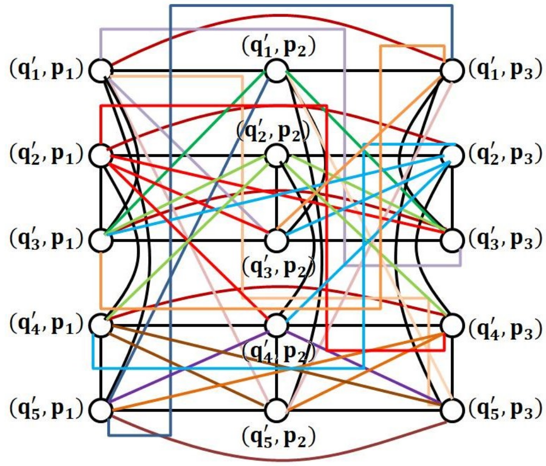

Figure 5: The graph $(G' \boxtimes H)^*$

(i) If G and H are two regular RL-graphs, then their combined product is the regular RL-graph.

(ii) If G and H are α-regular RL-graphs, then their combined product is the α-regular RL-graph.

(iii) If G and H are connected RL-graphs, then their combined product is a connected RL-graph.

(iv) Let at least one of these RL-graphs is disconnected RL-graph. Then their combined product is a disconnected RL-graph.

(v) If G and H are complete RL-graphs, then their combined product is the complete RL-graph.

Proof. (i) Without losing the generality, suppose that G is the k-regular RL-graph and H is the k'-regular RL-graph. Also, consider their combined product $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$. We know that every vertices $(q_i, q'_j) \in V$ connected to the vertex (q_i, q'_l) such that $q'_l q'_j \in E_2$, and (q_k, q'_j) such that $q_i q_k \in E_1$, and (q_k, q'_l) such that $q_i q_k \in E_1$ and $q'_l q'_j \in E_2$. Hence,

$$\begin{aligned} d_{G\boxtimes H}(q_i, q'_j) &= k + k' + (d_G(q_i) \times d_H(q'_j)) \\ &= k + k' + kk'. \end{aligned}$$

So, $G \boxtimes H$ is the regular RL-graph.
(ii) By using part (i) and the definition of combined product, this part is proven.
(iii) Suppose their combined product $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$ and the vertex $(q_i, q'_j) \in V$. Since the RL-graph H is connected, at least one path exists such that this vertex is connected to the vertex (q_i, q'_k) for every $1 \leq i \leq |V_1|$. Also, as the RL-graph G is connected, at least one path exists such that this vertex is connected to the vertex (q_k, q'_j) for every $1 \leq j \leq |V_2|$. Furthermore, at least one path exists such that this vertex is connected to the vertex (q_k, q'_l), because there exists at least one path such that the vertex q_i is connected to the vertex q_k and there exists at least one path such that the vertex q'_j is connected to the vertex q'_l.
(iv) The proof is the same as above with some modifications.
(v) The proof is straight-forward. $\square$

Example 3.12. (a) Suppose two RL-graphs G and H, are in Example 3.9. According to Example 3.9, two RL-graphs, G and H, are 2-regular, and their combined product, $G \boxtimes H$, is an 8-regular RL-graph.

(b) Consider two α-regular RL-graphs, G and H, in Example 3.9. According to Example 3.9, their combined product $G \boxtimes H$ is α-regular RL-graph.

(c) Consider two connected RL-graphs, G and H, in Example 3.9. So, we can see that their combined product $G \boxtimes H$, in Example 3.9, is connected RL-graph.

(d) Suppose two RL-graphs, G and H, in Example 3.3 such that the RL-graph H is disconnected. Thus, according to Example 3.3, their combined product $G \boxtimes H$ is a disconnected RL-graph.

(e) Consider the residuated lattice L in Example 3.3. Also, Suppose two complete and totally regular RL-graphs $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$, as in Figure 6, where $V_1 = \{q_1, q_2\}$, $E_1 = \{q_1q_2\}$, $\alpha_1(q_1) = \alpha_1(q_2) = \{a, b\}$, $\beta_1(q_1q_2) = \{a, b\}$, $V_2 = \{q'_1, q'_2, q'_3, q'_4\}$, $E_2 = \{q'_1q'_2, q'_1q'_3, q'_1q'_4, q'_2q'_3, q'_2q'_4, q'_3q'_4\}$, $\alpha_2(q_i) = \{a, b\}$, for every $1 \leq i \leq 4$ and $\beta_2(q'_iq'_j) = \{a\}$, for every $q_iq_j \in E_2$. Then their combined product is $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$, as in Figure 6, where $V = \{(q_i, q'_j)|\ 1 \leq i \leq 2, 1 \leq j \leq 4\}$, $E = \{(q_i, q'_1)(q_i, q'_2), (q_i, q'_1)(q_i, q'_3), (q_i, q'_1)(q_i, q'_4), (q_i, q'_2)(q_i, q'_3), (q_i, q'_2)(q_i, q'_4), (q_i, q'_3)(q_i, q'_4), (q_1, q'_j)(q_2, q'_j), (q_1, q'_1)(q_2, q'_2), (q_1, q'_1)(q_2, q'_3), (q_1, q'_1)(q_2, q'_4), (q_1, q'_2)(q_2, q'_3), (q_1, q'_3)(q_2, q'_4), (q_1, q'_2)(q_2, q'_4), (q_1, q'_2)(q_2, q'_1), (q_1, q'_3)(q_2, q'_2), (q_1, q'_3)(q_2, q'_1), (q_1, q'_4)(q_2, q'_1), (q_1, q'_4)(q_2, q'_2), (q_1, q'_4)(q_2, q'_3)|\ 1 \leq i \leq 2, 1 \leq j \leq 4\}$, $\alpha(q_i, q_j) = \{a, b\}$, for every $i = 1, 2$ and $1 \leq j \leq 4$, $\beta((q_1, q'_j)(q_2, q'_j)) = \{a, b\}$, for every $1 \leq j \leq 5$, $\beta((q_i, q'_j)(q_k, q'_l)) = \{a\}$, for every $(q_i, q'_j)(q_k, q'_l) \in B \cup D$. Clearly, we can see their combined product is complete and α-regular RL-graph. However, this RL-graph is not a β-regular RL-graph.

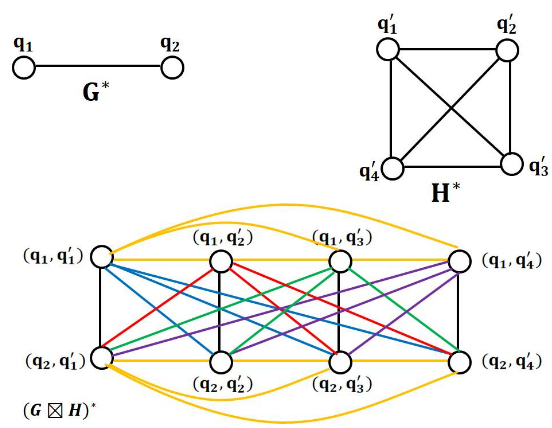

Figure 6: The graphs G^*, H^* and $(G \boxtimes H)^*$

Note 3.13. The above example is shown that the combined product of two β-regular RL-graphs is not β-regular RL-graph. Generally, the combined product of two β-regular RL-graphs is not necessarily a β-regular RL-graph.

Note 3.14. The following example indicates that if G and H are the regular and the α-regular RL-graphs, respectively, then their combined product is not necessarily the α-regular RL-graph.

Example 3.15. Consider the residuated lattice L and α-regular RL-graph H in Example 3.9. Also, suppose the regular RL-graph $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$, as in Figure 7, where $V_1 = \{q_1, q_2, q_3\}$, $E_1 = \{q_1q_2, q_2q_3, q_1q_3\}$, $\alpha_1(q_1) = 0.7$, $\alpha_1(q_2) = 0.9$, $\alpha_1(q_3) = 0.8$, $\beta_1(q_1q_2) = 0.7$, $\beta_1(q_2q_3) = 0.8$ and $\beta_1(q_1q_3) = 0.7$. Thus, their combined product is $G \boxtimes H$ on $(G \boxtimes H)^*$, as in Figure 7, where $V = \{(q_i, p_j)|\ 1 \leq i \leq 3, 1 \leq j \leq 3\}$, $E = \{(q_i, p_1)(q_i, p_2), (q_i, p_1)(q_i, p_3), (q_i, p_2)(q_i, p_3), (q_1, p_i)(q_2, p_i), (q_1, p_i)(q_3, p_i), (q_2, p_i)(q_3, p_i), (q_1, p_1)(q_2, p_2),$

$(q_1,p_1)(q_2,p_3),(q_1,p_1)(q_3,p_2),(q_1,p_1)(q_3,p_3),(q_1,p_2)(q_2,p_1),(q_1,p_2)(q_2,p_3),(q_1,p_2)(q_3,p_1),(q_1,p_2)(q_3,p_3),$ $(q_1,p_3)(q_2,p_1),(q_1,p_3)(q_2,p_2),(q_1,p_3)(q_3,p_1),(q_1,p_3)(q_3,p_2),(q_2,p_1)(q_3,p_2),(q_2,p_1)(q_3,p_3),(q_2,p_2)(q_3,p_1),$ $(q_2,p_2)(q_3,p_3),(q_2,p_3)(q_3,p_1),(q_2,p_3)(q_3,p_2)|\ 1 \leq i \leq 3\}$, $\alpha((q_1,p_i)) = 0.7$, $\alpha((q_2,p_i)) = 0.9$, $\alpha((q_3,p_i)) = 0.8$,

$$\begin{array}{lll}
\beta((q_i,p_1)(q_i,p_2)) = 0.4, & \beta((q_i,p_2)(q_i,p_3)) = 0.3, & \beta((q_i,p_1)(q_i,p_3)) = 0.2,\\
\beta((q_1,p_i)(q_2,p_i)) = 0.7, & \beta((q_1,p_i)(q_3,p_i)) = 0.7, & \beta((q_2,p_i)(q_3,p_i)) = 0.7,\\
\beta((q_1,p_1)(q_2,p_2)) = 0.4, & \beta((q_1,p_1)(q_2,p_3)) = 0.2, & \beta((q_1,p_1)(q_3,p_2)) = 0.4,\\
\beta((q_1,p_1)(q_3,p_3)) = 0.2, & \beta((q_1,p_2)(q_2,p_1)) = 0.4, & \beta((q_1,p_2)(q_2,p_3)) = 0.3,\\
\beta((q_1,p_2)(q_3,p_1)) = 0.4, & \beta((q_1,p_2)(q_3,p_3)) = 0.3, & \beta((q_1,p_3)(q_2,p_1)) = 0.2,\\
\beta((q_1,p_3)(q_2,p_2)) = 0.3, & \beta((q_1,p_3)(q_3,p_1)) = 0.2, & \beta((q_1,p_3)(q_3,p_2)) = 0.3,\\
\beta((q_2,p_1)(q_3,p_2)) = 0.4, & \beta((q_2,p_1)(q_3,p_3)) = 0.2, & \beta((q_2,p_2)(q_3,p_1)) = 0.4,\\
\beta((q_2,p_2)(q_3,p_3)) = 0.3, & \beta((q_2,p_3)(q_3,p_1)) = 0.2, & \beta((q_2,p_3)(q_3,p_2)) = 0.3,
\end{array}$$

for every $1 \leq i \leq 3$. Clearly, we can see that $G \boxtimes H$ is not the α-regular RL-graph.

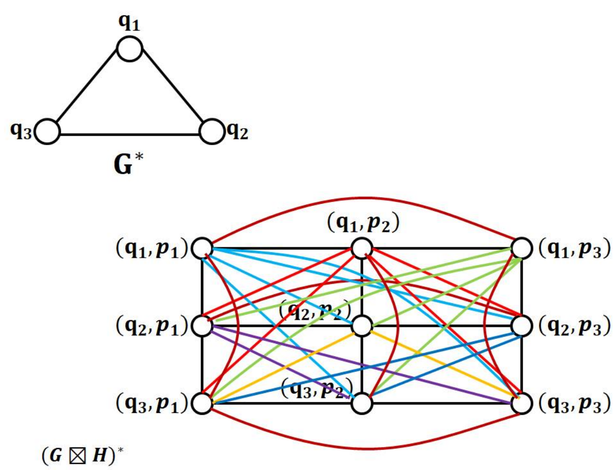

Figure 7: The graphs G^* and $(G \boxtimes H)^*$

Example 3.16. Suppose two RL-graphs in Example 3.3. Clearly, G is the complete RL-graph, but H is not a complete RL-graph. According to Example 3.3, we can see that their combined product is not a complete RL-graph.

Note 3.17. Using the above example, we can say that if one of the two RL-graphs, G and H, is not a complete RL-graph, then their combined product is not necessarily a complete RL-graph.

4 Application

In articles [9, 16], we showed that RL-graphs, have many applications in various fields, such as categorizing books in a library, finding drugs with the most therapeutic similarities, etc. Now we know that the combined product of two RL-graphs also has many applications in various fields, and in this section, we will discuss two of them in detail and make it clearer with an example.

In general, the effective factors in raising the quality of a specific issue can be categorized into two separate groups, then these issues can be connected with this operator (combined product), and we can check the effects of each one.

Application 4.1. (a) Each person's quality of life has different factors, which can be divided into two groups: individual and social. If we consider that individual aspects include financial status, personal health, education, family environment and social relationships, and social aspects include the unemployment rate, inflation rate, informing people to create a better life, medical services, educational services, and community security. We model the group of individual factors with RL-graph G in such a way that we place the influencing factors in the vertices and connect the factors that are dependent on each other with an edge. The amount of each vertex is the quality of these factors, and the amount of each edge is equal to the effect of these factors on each other. In the same way, let's model the group of social factors in the form of RL-graph H. Then, by using the combined product of these two RL-graphs, we get the impact of each individual factor on the social and the impact of new factors that combine the individual and social factors on each other.

Let $L = (\{1, \ldots, 10\}, \wedge, \vee, \otimes, \to, 1, 10)$ be residuated lattice, where $a \otimes b = a \wedge b$ and $a \to b = \begin{cases} 1 & if\ a \le b, \\ b & otherwise, \end{cases}$ for every $a,\ b \in \{0, 1, \ldots, 10\}$. Then individual and social aspects are modeled by two RL-graphs $G = (\alpha_1, \beta_1)$ on $G^* = (V_1, E_1)$ and $H = (\alpha_2, \beta_2)$ on $H^* = (V_2, E_2)$, as in Figure 8, respectively, where $V_1 = \{\text{financial status}(q_1), \text{personal health}(q_2), \text{education}(q_3), \text{family environment}(q_4), \text{social relationships}(q_5)\}$, $E_1 = \{q_1q_2, q_1q_3, q_1q_4, q_1q_5, q_2q_3, q_2q_4, q_3q_4, q_4q_5\}$, $\alpha_1(q_i) =$ The amount of quality of (q_i), for every $1 \le i \le 5$, $\beta_1(q_iq_j) = \alpha_1(q_i) \otimes \alpha_1(q_j)$, for every $q_iq_j \in E_1$, $V_2 = \{\text{unemployment rate}(q'_1), \text{inflation rate}(q'_2), \text{informing people to create a better life}(q'_3), \text{medical services}(q'_4), \text{educational services}(q'_5), \text{community security}(q'_6)\}$, $E_2 = \{q'_1q'_2, q'_1q'_4, q'_1q'_5, q'_1q'_6, q'_2q'_4, q'_2q'_5, q'_2q'_6, q'_3q'_4, q'_3q'_5, q'_3q'_6, q'_4q'_6, q'_5q'_6\}$, $\alpha_2(q'_i) =$ The amount of quality of (q'_i), for every $1 \le i \le 6$ and $\beta_2(q'_iq'_j) = \alpha_2(q'_i) \otimes \alpha_2(q'_j)$, for every $q'_iq'_j \in E_2$. Thus, the combined product of these RL-graphs is $G \boxtimes H = (\alpha, \beta)$ on $(G \boxtimes H)^* = (V, E)$, as in Figures 9, 10, 11, (It should be noted that since the shape of this graph is complex, we have placed it in three different Figures 9, 10, 11, and the overlapping of these three figures form the main graph. Of course, because Figure 11 was complex, we only drew a part of it.) where $V = \{(q_i, q'_j)|1 \le i \le 5, 1 \le j \le 6\}$, $E = \{(q_i,q'_1)(q_i,q'_2), (q_i,q'_1)(q_i,q'_4), (q_i,q'_1)(q_i,q'_5), (q_i,q'_1)(q_i,q'_6),$
$(q_i,q'_2)(q_i,q'_4), (q_i,q'_2)(q_i,q'_5), (q_i,q'_2)(q_i,q'_6), (q_i,q'_3)(q_i,q'_4), (q_i,q'_3)(q_i,q'_5), (q_i,q'_3)(q_i,q'_6), (q_i,q'_4)(q_i,q'_6),$
$(q_i,q'_5)(q_i,q'_6), (q_1,q'_j)(q_2,q'_j), (q_1,q'_j)(q_3,q'_j), (q_1,q'_j)(q_4,q'_j), (q_1,q'_j)(q_5,q'_j), (q_2,q'_j)(q_3,q'_j), (q_2,q'_j)(q_4,q'_j),$
$(q_3,q'_j)(q_4,q'_j), (q_4,q'_j)(q_5,q'_j), (q_1,q'_1)(q_2,q'_2), (q_1,q'_1)(q_2,q'_4), (q_1,q'_1)(q_2,q'_5), (q_1,q'_1)(q_2,q'_6), (q_1,q'_1)(q_3,q'_2),$
$(q_1,q'_1)(q_3,q'_4), (q_1,q'_1)(q_3,q'_5), (q_1,q'_1)(q_3,q'_6), (q_1,q'_1)(q_4,q'_2), (q_1,q'_1)(q_4,q'_4), (q_1,q'_1)(q_4,q'_5), (q_1,q'_1)(q_4,q'_6),$
$(q_1,q'_1)(q_5,q'_2), (q_1,q'_1)(q_5,q'_4), (q_1,q'_1)(q_5,q'_5), (q_1,q'_1)(q_5,q'_6), (q_1,q'_2)(q_2,q'_4), (q_1,q'_2)(q_2,q'_5), (q_1,q'_2)(q_2,q'_6),$
$(q_1,q'_2)(q_3,q'_4), (q_1,q'_2)(q_3,q'_5), (q_1,q'_2)(q_3,q'_6), (q_1,q'_2)(q_4,q'_4), (q_1,q'_2)(q_4,q'_5), (q_1,q'_2)(q_4,q'_6), (q_1,q'_2)(q_5,q'_4),$
$(q_1,q'_2)(q_5,q'_5), (q_1,q'_2)(q_5,q'_6), (q_1,q'_3)(q_2,q'_4), (q_1,q'_3)(q_2,q'_5), (q_1,q'_3)(q_2,q'_6), (q_1,q'_3)(q_3,q'_4), (q_1,q'_3)(q_3,q'_5),$
$(q_1,q'_3)(q_3,q'_6), (q_1,q'_3)(q_4,q'_4), (q_1,q'_3)(q_4,q'_5), (q_1,q'_3)(q_4,q'_6), (q_1,q'_3)(q_5,q'_4), (q_1,q'_3)(q_5,q'_5), (q_1,q'_3)(q_5,q'_6),$
$(q_1,q'_4)(q_2,q'_6), (q_1,q'_4)(q_3,q'_6), (q_1,q'_4)(q_4,q'_6), (q_1,q'_4)(q_5,q'_6), (q_1,q'_5)(q_2,q'_6), (q_1,q'_5)(q_3,q'_6), (q_1,q'_5)(q_4,q'_6),$
$(q_1,q'_5)(q_5,q'_6), (q_2,q'_1)(q_3,q'_2), (q_2,q'_1)(q_3,q'_4), (q_2,q'_1)(q_3,q'_5), (q_2,q'_1)(q_3,q'_6), (q_2,q'_2)(q_3,q'_4), (q_2,q'_2)(q_3,q'_5),$
$(q_2,q'_2)(q_3,q'_6), (q_2,q'_3)(q_3,q'_4), (q_2,q'_3)(q_3,q'_5), (q_2,q'_3)(q_3,q'_6), (q_2,q'_4)(q_3,q'_6), (q_2,q'_5)(q_3,q'_6), (q_2,q'_1)(q_4,q'_2),$
$(q_2,q'_1)(q_4,q'_4), (q_2,q'_1)(q_4,q'_5), (q_2,q'_1)(q_4,q'_6), (q_2,q'_2)(q_4,q'_4), (q_2,q'_2)(q_4,q'_5), (q_2,q'_2)(q_4,q'_6), (q_2,q'_3)(q_4,q'_4),$
$(q_2,q'_3)(q_4,q'_5), (q_2,q'_3)(q_4,q'_6), (q_2,q'_4)(q_3,q'_6), (q_2,q'_5)(q_4,q'_6), (q_3,q'_1)(q_4,q'_2), (q_3,q'_1)(q_4,q'_4), (q_3,q'_1)(q_4,q'_5),$
$(q_3,q'_1)(q_4,q'_6), (q_3,q'_2)(q_4,q'_4), (q_3,q'_2)(q_4,q'_5), (q_3,q'_2)(q_4,q'_6), (q_3,q'_3)(q_4,q'_4), (q_3,q'_3)(q_4,q'_5), (q_3,q'_3)(q_4,q'_6),$
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$(q_1,q'_2)(q_2,q'_1), (q_1,q'_4)(q_2,q'_1), (q_1,q'_5)(q_2,q'_1), (q_1,q'_6)(q_2,q'_1), (q_1,q'_4)(q_2,q'_2), (q_1,q'_5)(q_2,q'_2), (q_1,q'_6)(q_2,q'_2),$
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$(q_1,q'_6)(q_3,q'_3), (q_1,q'_6)(q_3,q'_5), (q_1,q'_6)(q_3,q'_4), (q_1,q'_2)(q_4,q'_1), (q_1,q'_4)(q_4,q'_1), (q_1,q'_5)(q_4,q'_1), (q_1,q'_6)(q_4,q'_1),$
$(q_1,q'_4)(q_4,q'_2), (q_1,q'_5)(q_4,q'_2), (q_1,q'_6)(q_4,q'_2), (q_1,q'_4)(q_4,q'_3), (q_1,q'_5)(q_4,q'_3), (q_1,q'_6)(q_4,q'_3), (q_1,q'_6)(q_4,q'_5),$

$(q_1, q'_6)(q_4, q'_4), (q_1, q'_2)(q_5, q'_1), (q_1, q'_4)(q_5, q'_1), (q_1, q'_5)(q_5, q'_1), (q_1, q'_6)(q_5, q'_1), (q_1, q'_4)(q_5, q'_2), (q_1, q'_5)(q_5, q'_2),$ $(q_1, q'_6)(q_5, q'_2), (q_1, q'_4)(q_5, q'_3), (q_1, q'_5)(q_5, q'_3), (q_1, q'_6)(q_5, q'_3), (q_1, q'_6)(q_5, q'_5), (q_1, q'_6)(q_5, q'_4), (q_2, q'_2)(q_3, q'_1),$ $(q_2, q'_4)(q_3, q'_1), (q_2, q'_5)(q_3, q'_1), (q_2, q'_6)(q_3, q'_1), (q_2, q'_4)(q_3, q'_2), (q_2, q'_5)(q_3, q'_2), (q_2, q'_6)(q_3, q'_2), (q_2, q'_4)(q_3, q'_3),$ $(q_2, q'_5)(q_3, q'_3), (q_2, q'_6)(q_3, q'_3), (q_2, q'_6)(q_3, q'_5), (q_2, q'_6)(q_3, q'_4), (q_2, q'_2)(q_4, q'_1), (q_2, q'_4)(q_4, q'_1), (q_2, q'_5)(q_4, q'_1),$ $(q_2, q'_6)(q_4, q'_1), (q_2, q'_4)(q_4, q'_2), (q_2, q'_5)(q_4, q'_2), (q_2, q'_6)(q_4, q'_2), (q_2, q'_4)(q_4, q'_3), (q_2, q'_5)(q_4, q'_3), (q_2, q'_6)(q_4, q'_3),$ $(q_2, q'_6)(q_4, q'_5), (q_2, q'_6)(q_4, q'_4), (q_3, q'_2)(q_4, q'_1), (q_3, q'_4)(q_4, q'_1), (q_3, q'_5)(q_4, q'_1), (q_3, q'_6)(q_4, q'_1), (q_3, q'_4)(q_4, q'_2),$ $(q_3, q'_5)(q_4, q'_2), (q_3, q'_6)(q_4, q'_2), (q_3, q'_4)(q_4, q'_3), (q_3, q'_5)(q_4, q'_3), (q_3, q'_6)(q_4, q'_3), (q_3, q'_6)(q_4, q'_5), (q_3, q'_6)(q_4, q'_4),$ $(q_4, q'_2)(q_5, q'_1), (q_4, q'_4)(q_5, q'_1), (q_4, q'_5)(q_5, q'_1), (q_4, q'_6)(q_5, q'_1), (q_4, q'_4)(q_5, q'_2), (q_4, q'_5)(q_5, q'_2), (q_4, q'_6)(q_5, q'_2),$ $(q_4, q'_4)(q_5, q'_3), (q_4, q'_5)(q_5, q'_3), (q_4, q'_6)(q_5, q'_3), (q_4, q'_6)(q_5, q'_5), (q_4, q'_6)(q_5, q'_4) \mid 1 \leq i \leq 5, 1 \leq j \leq 6\}$, $\alpha((q_i, q'_j)) = \alpha_1(q_i) \otimes \alpha_2(q'_j)$ and $\beta((q_i, q'_j)(q_i, q'_l)) = \alpha_1(q_i) \otimes \beta_2(q'_j q'_l)$, for every $1 \leq i \leq 5$ and $q'_j q'_l \in E_2$, $\beta((q_i, q'_j)(q_k, q'_j)) = \alpha_2(q'_j) \otimes \beta_1(q_i q_k)$, for every $1 \leq j \leq 6$ and $q_i q_k \in E_1$, and $\beta((q_i, q'_j)(q_k, q'_l)) = \beta_1(q_i q_k) \otimes \beta_2(q'_j q'_l)$, for every $q_i q_k \in E_1$ and $q'_j q'_l \in E_2$. With this modeling, the effect of each individual factor on each social factor can be easily obtained by α. Also, the effects of interdependent individual factors on interdependent social factors are obtained by β.

(b) Factors effective in raising a factory's production can be divided into two groups: internal factors (efficient managers, good communication between managers and employees, skilled workers, etc.) and external factors (raw materials, availability of updated equipment in the market, etc.). After that, these factors are modeled by two RL-graphs. Thus, the effect of each internal factor on each external factor can be easily obtained by α. Also, the effects of interdependent internal factors on interdependent external factors are obtained by β.

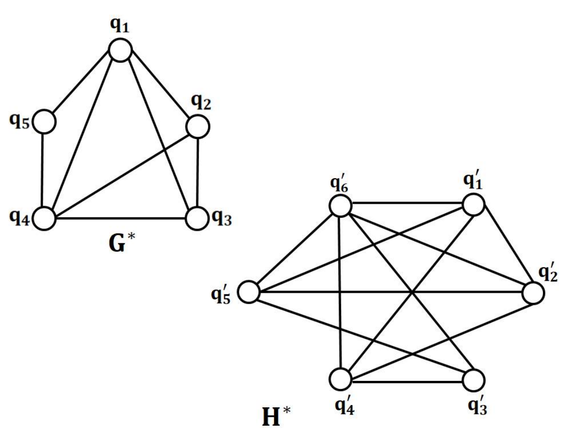

Figure 8: The graphs G^* and H^*

5 Conclusion

In this study, two RL-graphs have established the notion of combined product RL-graphs. Some theorems and examples have also been presented to identify the close relationship between two RL-graphs and their combined product. The material presented in the mathematical sciences has always helped improve human life, so they have always used these concepts to solve their problems. So we can say that different notions can use as utilities that may apply in many fields. Accordingly, using this combined product of two RL-graphs, we can relate two groups unrelated to each other and predict how much their work efficiency will change if these two groups merge. By obtaining this information, more accurate decisions can make. We are willing to investigate this topic in more detail in our future work, gain more insights into these structures, and measure

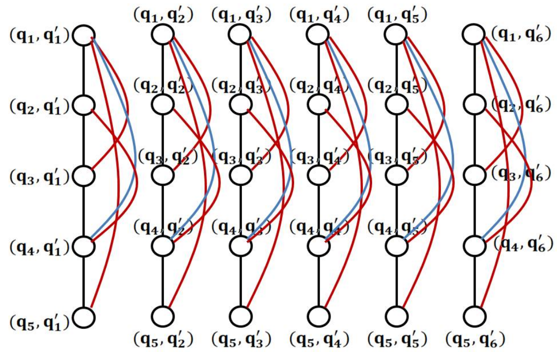

Figure 9: The graph $(G \boxtimes H)^*$

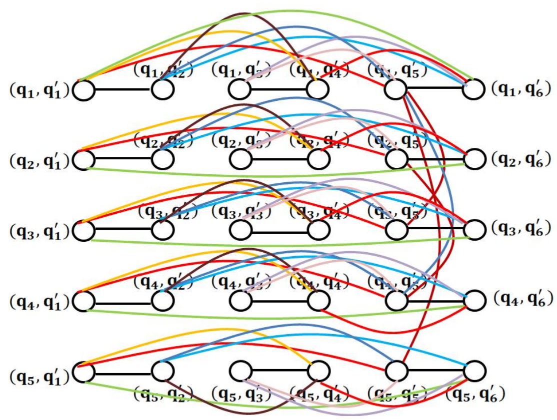

Figure 10: The graph $(G \boxtimes H)^*$

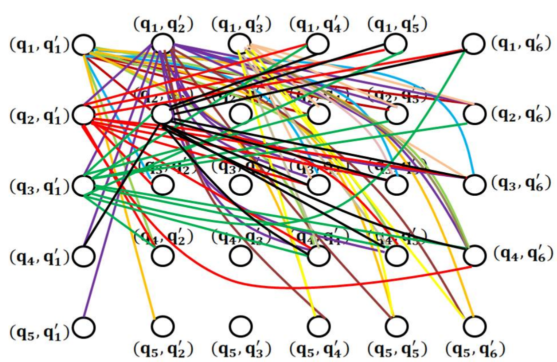

Figure 11: The graph $(G \boxtimes H)^*$

their complexity. We also decided to compare this model with other models and show which is the best. Furthermore, we intend to create a deep relationship between graphs and automata by combining product and study and identify these relationships in detail. In addition, we search for more associations between these structures for application in the computer network.

Conflict of Interest: The authors declare no conflict of interest.

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