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Lattices of (Generalized) Fuzzy Ideals in Double Boolean Algebras

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Abstract. This paper develops the notion of fuzzy ideal and generalized fuzzy ideal on double Boolean algebra (dBa). According to Rudolf Wille, a double Boolean algebra $\underline{D} := (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ is an algebra of type $(2, 2, 1, 1, 0, 0)$, which satisfies a set of properties. This algebraic structure aimed to capture the equational theory of the algebra of protoconcepts. We show that collections of fuzzy ideals and generalized fuzzy ideals are endowed with lattice structures. We further prove that (by isomorphism) lattice structures obtained from fuzzy ideals and generalized fuzzy ideals of a double Boolean algebra D can entirely be determined by sets of fuzzy ideals and generalized fuzzy ideals of the Boolean algebra $D_{\sqcup}$.

AMS Subject Classification 2020: 03B15, 03B16, 03B35, 03B38.
Keywords and Phrases: Double Boolean algebras, Fuzzy ideals, Fuzzy primary ideal.

1 Introduction

Nowadays, fuzzy logic is used in numerous applications such as facial pattern recognition, air conditioners, washing machines, vacuum cleaners, anti-skid braking systems, transmission systems and unmanned helicopters knowledge-based systems for multi-objective optimization of power systems. In Machine Learning, fuzzy logic can be applied in some models such as MLP (Multi Layers Perceptron) model which is a fully connected class of feed-forward artificial neural network (ANN). In forecasting, fuzzification is incorporated at the input layers by considering the degree of participation of each of the features in the prediction model [2]. Fuzzy layers can also be seen as a circuit design as it is an application of Boolean algebra and therefore, we strongly believe that the way of connecting layers can be related to a lattice structure. Lattices can also appear in analysis of cellular traffic for finding anomalies in the performance and provisioning of demand resources [3]. Another application of double Boolean algebra is in multi-layer neural networks, in fact considering multilayer neural network design. Different blocks made between layers represent ordered structures of dBas. So with a specific dBa, we can easily design a multilayer neural network based on connection between layers. This task can therefore be added to artificial intelligence purpose on designing circuits that are used in digital computers.

So far, fuzzification of ideals has been studied on bounded lattices [1, 7, 10]. Mezzomo et *al*, based on Chon's approach [4], has defined the notion of fuzzy ideals and fuzzy filters on the product operators of bounded lattices. They have also proved some properties that are analogous to the classical theory of fuzzy ideals and fuzzy filters such as, the class of fuzzy ideals being closed under fuzzy union and fuzzy intersection. Thus this leads to the study of fuzzy topology on bounded lattices. Attalah has studied complete fuzzy prime ideals on distributive lattices. Fuzzification of ideals has been tackled in other algebraic structures such as, IL

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algebras [6], in that structure the concept of fuzzy ideal generalizes the notion of fuzzy ideals in BL-algebra and MTL-algebra. Kuanyum et *al* [12] have tackled the question of fuzzy ideals in residuated lattices. They first defined the generalized fuzzy ideals and they showed that the set of generalized fuzzy ideals is endowed with a lattice structure. As it is known that a double Boolean algebra is a more general structure than a residuated lattice and does not necessarily have a subjacent structure of lattice and Tatuéné [5] has shown that transfer of structure from that algebra to the fuzzy structure does not holds. The question that captures our interest is whether for the case of a double Boolean algebra, we still have a lattice structure by endowing the set of fuzzy ideals with some operators. From the best of our knowledge, this direction has not yet been tackled. So our goal in this paper is mainly focused on the study of fuzzy ideals of a double Boolean algebra. We fuzzify the notion of ideals on double Boolean algebra. Moreover we prove that the collection of fuzzy ideals of a double Boolean algebra $\underline{D}$ is endowed with a lattice structure which is an extension of the work done by Kuanyun et *al* [12].

The paper is organized as follows: in section 2, we present a background which contains definitions and related properties of ideals and filters in the double Boolean algebra for a better understanding of the structure. In section 3, we introduce the concept of fuzzy ideals and fuzzy filters on double Boolean algebras and then we characterize them. In section 4, we study the lattice structure of the set of all fuzzy ideals of a double Boolean algebra $\underline{D}$. In section 5, we draw a generalization of the concept of fuzzy ideal and then we study the bounded lattice structure of the set of generalized fuzzy ideals of a double Boolean algebra.

2 Background

In this section, we present double Boolean algebras, ideals of double Boolean algebras and related properties. We then give some results obtained by Tenkeu et al [11] for this structure. These notions will be useful for the rest of the paper.

2.1 Double Boolean algebras and related properties

Definition 2.1. [9] A double Boolean algebra is an algebra $\underline{D} = (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ of type $(2, 2, 1, 1, 0, 0)$ that satisfies (1a) to (11a) and (1b) to (11b).

(1a) $(x \sqcap x) \sqcap y = x \sqcap y$

(2a) $x \sqcap y = y \sqcap x$

(3a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$

(4a) $\neg(x \sqcap x) = \neg x$

(5a) $x \sqcap (x \sqcup y) = x \sqcap x$

(6a) $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$

(7a) $x \sqcap (x \vee y) = x \sqcap x$

(8a) $\neg\neg(x \sqcap y) = x \sqcap y$

(9a) $x \sqcap \neg x = \bot$

(10a) $\neg\bot = \top \sqcap \top$

(11a) $\neg\top = \bot$

(1b) $(x \sqcup x) \sqcup y = x \sqcup y$

(2b) $x \sqcup y = y \sqcup x$

(3b) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$

(4b) $\lrcorner(x \sqcup x) = \lrcorner x$

(5b) $x \sqcup (x \sqcap y) = x \sqcup x$

(6b) $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$

(7b) $x \sqcup (x \wedge y) = x \sqcup x$

(8b) $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$

(9b) $x \sqcup \lrcorner x = \top$

(10b) $\lrcorner\top = \bot \sqcup \bot$

(11b) $\lrcorner\bot = \top$

$$(12)\ (x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$$

Where the supremum (join) is defined by $x \vee y := \neg(\neg x \sqcap \neg y)$, and the infimum (meet) is defined by: $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$, $1 := \neg\bot$ and $0 := \lrcorner\top$. The relation defined by $x \sqsubseteq y \Longleftrightarrow x \sqcap y = x \sqcap x \ \ and \ \ x \sqcup y = y \sqcup y$ is a quasi-order.

A double Boolean algebra is called pure if it satisfies: $x \sqcap x = x \ \ or \ \ x \sqcup x = x$. This relation also holds in algebra of semiconcepts.

The following notations are adopted $x_\sqcap := x \sqcap x$, $D_\sqcap = \{x_\sqcap : x \in D\}$ and $x_\sqcup = x \sqcup x$, $D_\sqcup = \{x_\sqcup : x \in D\}$. The algebras $(D_\sqcap, \sqcap, \vee, \neg, \bot, 1)$ and $(D_\sqcup, \wedge, \sqcup, \lrcorner, 0, \top)$ are Boolean algebras.

$$\begin{aligned} x_\sqcap \leq y_\sqcap &\Longleftrightarrow x \sqsubseteq y \\ &\Longleftrightarrow x_\sqcap \sqcap y_\sqcap = (x \sqcap x) \sqcap (y \sqcap y) = x \sqcap x = x_\sqcap \end{aligned}$$

and in the same way, we have: $x_\sqcup \sqcup y_\sqcup = y_\sqcup$. As $\sqcap$ is the meet and $\sqcup$ is the join operator in the Boolean algebra $(D_\sqcap, \sqcap, \vee, \neg, \bot, 1)$ and $(D_\sqcup, \wedge, \sqcup, \lrcorner, 0, \top)$ respectively. We get $x \sqsubseteq y \Longleftrightarrow x_\sqcap \leq y_\sqcap \ \ and \ \ x_\sqcup \leq y_\sqcup$ where $\leq$ is the induced order in the corresponding Boolean algebra.

A double Boolean algebra is called regular if the relation $\sqsubseteq$ is an order relation.

Example 2.2. Let $D = \{a, b, c, d, e = a \sqcap b = c \sqcup d, \bot, \top\}$ with the diagram given by the following:

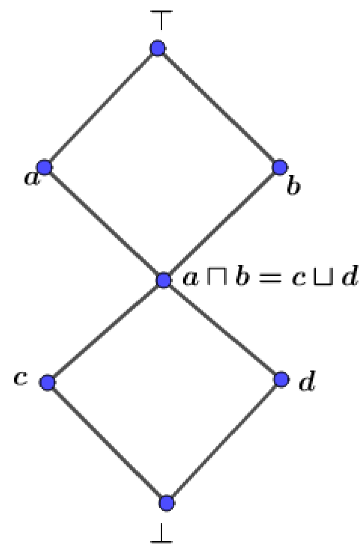


Figure 1: dBa pure

With the following table:

$\sqcup$	a	b	c	d	$\bot$	$\top$	e
a	a	$\top$	a	a	a	$\top$	a
b	$\top$	b	b	b	b	$\top$	b
c	a	b	e	e	e	$\top$	e
d	a	b	e	e	e	$\top$	e
e	a	b	e	e	e	$\top$	e
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	e	b	e	e	$\bot$	$\top$	e

x	a	b	c	d	$\bot$	$\top$	e
$\neg x$	$\bot$	$\bot$	d	c	e	$\bot$	$\bot$

x	a	b	c	d	$\bot$	$\top$	e
$\lrcorner x$	b	a	$\top$	$\top$	$\top$	e	$\top$

$\sqcap$	a	b	c	d	$\bot$	$\top$	e
a	e	e	c	d	$\bot$	e	e
b	e	b	c	d	$\bot$	b	b
c	c	c	c	$\bot$	$\bot$	c	e
d	d	d	$\bot$	d	$\bot$	d	e
$\top$	a	b	c	d	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	e
e	a	b	e	e	e	$\top$	e

$D = D_{\sqcup} \cup D_{\sqcap}$, *with* $D_{\sqcup} = \{a, b, a \sqcap b, \top\}$and $D_{\sqcap} = \{c, d, c \sqcup d, \bot\}$ $a \sqcup a = a$, $b \sqcup b = b$, $, \top \sqcup \top = \top$. Hence D is a pure double Boolean algebra.

Example 2.3. Let $D = \{\alpha, \beta, \gamma, \lambda, \bot, \top\}$ with the diagram and tables given by the following:

$\sqcap$	$\bot$	γ	λ	β	α	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
γ	$\bot$	γ	$\bot$	γ	γ	γ
λ	$\bot$	$\bot$	λ	λ	$\bot$	λ
β	$\bot$	λ	β	γ	γ	β
α	$\bot$	γ	$\bot$	γ	γ	γ
$\top$	$\bot$	γ	λ	β	γ	β

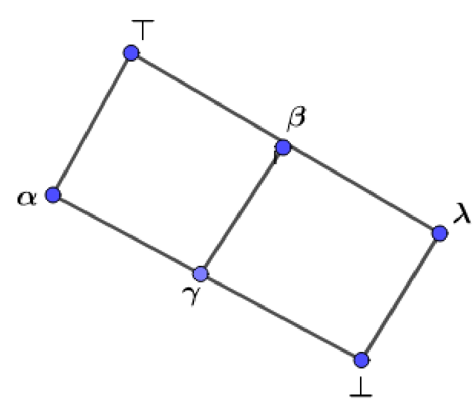


Figure 2: Hasse diagram of D

$\sqcup$	$\bot$	γ	λ	β	α	$\top$
$\bot$	γ	γ	β	β	α	$\top$
γ	γ	γ	β	β	α	$\top$
λ	β	β	β	β	$\top$	$\top$
β	β	β	β	β	$\top$	$\top$
α	α	α	$\top$	$\top$	α	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

x	$\bot$	γ	λ	β	α	$\top$
$\neg x$	β	λ	γ	$\bot$	λ	$\bot$
$\lrcorner x$	$\top$	$\top$	α	α	β	γ

The dBa D is pure and we have $D_\sqcap = \{\bot, \gamma, \lambda, \beta\}$ and $D_\sqcup = \{\alpha, \beta, \gamma, \top\}$

2.2 Ideals on double Boolean algebras and their properties

Definition 2.4. (see [8]) Let $\underline{D}$ be a double Boolean algebra. A non empty subset F of $\underline{D}$ is called a filter if it satisfies:

(i) $x, y \in F \Longrightarrow x \sqcap y \in F$;
(ii) $x \in F, y \in D, x \sqsubseteq y \Longrightarrow y \in F$.

Dually, ideals of double Boolean algebras is defined.

Definition 2.5. (see [8]) Let $\underline{D}$ be a double Boolean algebra. A filter F is called proper if $F \neq D$, and primary if it is proper and satisfies $x \in F$ or $\neg x \in F$, for all $x \in D$.
Dually are defined primary ideals.

Tenkeu et *al.* [11] showed that primary ideals are exactly maximal ideals in the framework of double Boolean algebras.

Proposition 2.6. *(see [8]) Let I an ideal of D, then $I_\sqcup = \{x_\sqcup : x \in I\}$ is an ideal of $D_\sqcup$.*

We call double Boolean algebra **trivial** iff $\top \sqcap \top = \bot \sqcup \bot$.

Proposition 2.7. *(see [11]) Let D be a dBa and $X \subseteq D$ a non empty subset of D. F_1, F_2 two filters of $\underline{D}$ and I_1, I_2 two ideals of $\underline{D}$.*

1. *$I(a) = \{x \in D : x \sqsubseteq a \sqcup a\}$, where $I(a)$ stand for the ideal generated by a and $F(a) = \{x \in D : a \sqcap a \sqsubseteq x\}$, where $F(a)$ stand for the filter generated by a.*
2. *$Ideal(\emptyset) = I(\bot) = \{x \in D : x \subseteq \bot \sqcup \bot\}$ and $Filter(\emptyset) = F(\top) = \{x \in D : \top \sqcap \top \sqsubseteq x\}$*
3. *$Ideal(X) = \{x \in D : x \sqsubseteq b_1 \sqcup b_2 \sqcup \cdots \sqcup b_n,$ for some $b_1, b_2, \cdots, b_n \in X, n \geq 1\}$*
4. *$Filter(X) = \{x \in D : x \sqsupseteq b_1 \sqcap b_2 \sqcap \cdots \sqcap b_n,$ for some $b_1, b_2, \cdots, b_n \in X, n \geq 1\}$*
5. *$Ideal\,(I_1 \cup I_2) = \{x \in D : x \sqsubseteq i_1 \sqcup i_2, i_1 \in I_1, i_2 \in I_2\} = I_1 \vee I_2$*
6. *$Filter\,(F_1 \cup F_2) = \{x \in D : x \sqsubseteq f_1 \sqcap f_2, f_1 \in F_1, f_2 \in F_2\} = F_1 \vee F_2$*

The following proposition gives the distributivity-like property of dBa.

Proposition 2.8. *(see [11]) Let $\underline{D} = (D, \sqcap, \sqcup, \bot, \top)$ be a dBa and $a, b, c \in D$. We have:*

(i) $a \vee (b \sqcap c) = (a \vee b) \sqcap (a \vee c)$ (iv) $a \wedge (a \sqcup b) = a \sqcup a$
(ii) $a \wedge (b \sqcup c) = (a \wedge b) \sqcup (a \wedge c)$ (v) $(a \sqcap a) \vee (b \sqcap b) = a \vee b$
(iii) $a \vee (a \sqcap b) = a \sqcap a$ (vi) $(a \sqcup a) \wedge (b \sqcup b) = a \wedge b.$
(vii) $a \sqcup b,\ c \sqcup d \Longrightarrow a \sqcap c \sqsubseteq b \sqcap d,\ a \sqcup c \sqsubseteq b \sqcup d,\ a \vee c \sqsubseteq b \vee d$ *and* $a \wedge c \sqsubseteq b \wedge d$.

In the next section we are going to introduce fuzzy ideals and fuzzy filters on double Boolean algebras and give some related properties.

3 Fuzzy ideals and fuzzy filters on double Boolean algebras

In this section we introduce the notion of fuzzy ideals and fuzzy filters in the context of double Boolean algebras. Namely, we characterized these concepts. A fuzzy set on $\underline{D}$ is a function $\mu : D \longrightarrow [0,1]$. Let $\alpha \in [0,1]$, the α-cut of μ is defined by $\mu_\alpha = \{x \in D : \mu(x) \geq \alpha\}$.

Definition 3.1. Let $\underline{D} = (D, \sqcap, \sqcup, \neg, \lrcorner, \bot, \top)$ be a double Boolean algebra and μ, ν two fuzzy subsets of $\underline{D}$. The fuzzy subset μ of $\underline{D}$ is a fuzzy filter if for all $x, y \in D$

(i) $\mu(x \sqcap y) \geq \mu(x) \wedge \mu(y)$;

(ii) $x \sqsubseteq y \Longrightarrow \mu(x) \leq \mu(y)$.

Dually the fuzzy subset ν of $\underline{D}$ is a fuzzy ideal if for all $x, y \in D$, the following inequalities hold:

(i) $\nu(x \sqcup y) \geq \nu(x) \wedge \nu(y)$;

(ii) $x \sqsubseteq y \Longrightarrow \nu(x) \geq \nu(y)$.

The above two relations on Definition 3.1 are equivalent to say that : μ is fuzzy ideal of $\underline{D}$ iff $\mu(x \sqcup y) = \mu(x) \wedge \mu(y)$, for all $x, y \in D$. And μ is fuzzy filter of $\underline{D}$ iff $\mu(x \sqcap y) = \mu(x) \wedge \mu(y)$, for all $x, y \in D$.

Proposition 3.2. *Let μ be a fuzzy ideal of $\underline{D}$, and ν the fuzzy filter of $\underline{D}$, then we have the following: $\mu(x \sqcup x) = \mu(x)$ and $\nu(x \sqcap x) = \nu(x)$, for all $x \in D$.*

Proof. Since μ is a fuzzy ideal then we have: $\mu(x \sqcup x) \geq \mu(x) \wedge \mu(x) = \mu(x)$ this implies that $\mu(x \sqcup x) \geq \mu(x)$. In addition, we have $x \sqsubseteq x \sqcup x$, then $\mu(x) \geq \mu(x \sqcup x)$. Hence $\mu(x \sqcup x) = \mu(x)$. Similarly, the cases of the fuzzy filter can be shown. □

The previous proposition shows that the fuzzy ideals of the Boolean algebra $D_\sqcup$ can be extended as fuzzy ideals of the double Boolean $\underline{D}$.

Let denote by $FI(D_\sqcup)$ and $FI(D)$ the collection of fuzzy ideals of $D_\sqcup$ and D respectively. We have the following proposition which is a characterization of fuzzy ideals and fuzzy filters with their α-cuts.

Proposition 3.3. *Let μ, ν be two fuzzy subsets of $\underline{D}$.*

(i) μ is a fuzzy ideal iff for all $\alpha \in [0,1]$, $\mu_\alpha = \emptyset$ or μ_α is an ideal of $\underline{D}$.

(ii) ν is a fuzzy filter iff for all $\alpha \in [0,1]$, $\nu_\alpha = \emptyset$ or ν_α is an filter of $\underline{D}$.

Proof.

Case of (i)

Let μ be a fuzzy subset of $\underline{D}$.

$\Longrightarrow$) Let us assume that μ is a fuzzy ideal of $\underline{D}$. Let $\alpha \in [0,1]$, such that $\mu_\alpha \neq \emptyset$. we need to show that μ_α is an ideal of $\underline{D}$.

Suppose $x, y \in \mu_\alpha$ then we have $\mu(x) \geq \alpha$ and $\mu(y) \geq \alpha$, hence $\mu(x) \wedge \mu(y) \geq \alpha$. But by hypothesis, μ is a fuzzy ideal of $\underline{D}$. So we have $\mu(x \sqcup y) \geq \mu(x) \wedge \mu(y)$. Thus $\mu(x \sqcup y) \geq \alpha$. Hence $x \sqcup y \in \mu_\alpha$.

Now suppose that $x \sqsubseteq y$ then $\mu(y) \leq \mu(x)$ since μ is a fuzzy ideal of $\underline{D}$. And since $y \in \mu_\alpha$, this means that $\mu(y) \geq \alpha$, so $\mu(x) \geq \alpha$, by transitivity of $\leq$, therefore $x \in \mu_\alpha$. Hence μ_α is an ideal of $\underline{D}$.

$\Longleftarrow$) Let us assume that for any $\alpha \in [0,1]$, μ_α is an ideal of $\underline{D}$.

Let $x, y \in D$, for $\alpha = \mu(x) \wedge \mu(y)$, $x, y \in \mu_\alpha$, since μ_α is an ideal of $\underline{D}$, we have $x \sqcup y \in \mu_\alpha$. Thus $\mu(x \sqcup y) \geq \mu(x) \wedge \mu(y)$. Also $x \sqsubseteq y \Longrightarrow x \sqcup y = y \sqcup y$ (Prop.3). For $\alpha = \mu(y)$, $y \in \mu_\alpha$, but $x \sqsubseteq y$ and μ_α ideal implies $x \in \mu_\alpha$. Thus $\mu(x) \geq \alpha = \mu(y)$.

The proof of (ii) is similar.
□

Example 3.4. Let us consider the double Boolean algebra defined in the example 2.2 with it diagram in Figure 1. Then we have the following:

$$\mu(x) = \begin{cases} \frac{1}{2} & if \quad x \in \{a, b, c, d, e, \top\} \\ 1 & if \quad x = \bot \end{cases}$$

is a fuzzy ideal of the double Boolean algebra D, defined in example 1 of the paper. In fact, let $\alpha \in [0, 1]$, if $\alpha \leq \frac{1}{2}$, $\mu_\alpha = \{a, b, c, d, e, \top\}$ which is an ideal of D. If $\alpha > \frac{1}{2}$, then $\mu_\alpha = \{\bot\}$, which is an ideal of D.

$$\nu(x) = \begin{cases} \frac{3}{4} & if \quad x \in \{a, b, e\} \\ 1 & if \quad x \in \{c, d, \top\} \\ 0 & if \quad x = \bot \end{cases}$$

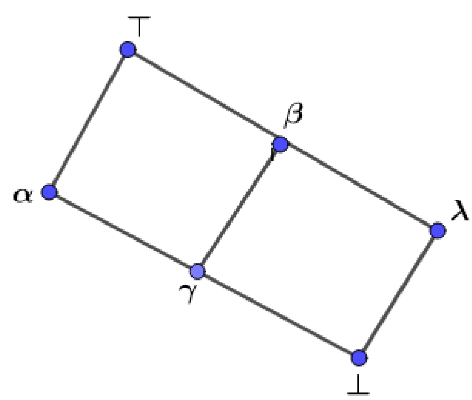
is a fuzzy filter of the double Boolean algebra D, defined in example 2.2

Definition 3.5. Let $\underline{D}$ a double Boolean algebra. A fuzzy filter μ is called proper if μ is a non constant function.

Definition 3.6. Let μ be a fuzzy subset of $\underline{D}$, μ is a fuzzy primary filter of $\underline{D}$ if it is proper and satisfies $\mu(x) \vee \mu(\neg x) = 1$, for all $x \in D$. Dually, a fuzzy subset ν is a fuzzy primary ideal of $\underline{D}$ if $\nu(x) \vee \nu(\lrcorner x) = 1$, for all $x \in D$.

Example 3.7. Let us consider the double Boolean algebra of the example 2.3, where $D = \{\bot, \alpha, \beta, \gamma, \lambda, \top\}$ with it diagram in Figure 2.

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \{\bot, \alpha, \beta, \gamma\} \\ \frac{1}{3} & \text{if } x = \lambda \\ \frac{1}{10} & \text{if } x = \top \end{cases} \qquad (1)$$



Then μ is a fuzzy primary ideal of D. In fact, μ is a fuzzy ideal of D, since it is a decreasing function. And it is primary: let $x \in D = \{\bot, \alpha, \beta, \gamma, \lambda, \top\}$

If $x = \bot$, then since $\mu(\bot) = 1$, we directly have $\mu(x) \vee \mu(\lrcorner x) = 1$;

If $x = \alpha$, then since $\mu(\alpha) = 1$, we directly have $\mu(\alpha) \vee \mu(\lrcorner \alpha) = 1$;

If $x = \beta$, then since $\mu(\beta) = 1$, we directly have $\mu(\beta) \vee \mu(\lrcorner \beta) = 1$;

If $x = \gamma$, then since $\mu(\gamma) = 1$, we directly have $\mu(\gamma) \vee \mu(\lrcorner \gamma) = 1$;

If $x = \lambda$, then since $\mu(\lrcorner \lambda) = 1$, we directly have $\mu(\lambda) \vee \mu(\lrcorner \lambda) = 1$;

If $x = \top$, then since $\mu(\lrcorner \top) = 1$, we directly have $\mu(\top) \vee \mu(\lrcorner \top) = 1$.

Proposition 3.8. *Let ν and μ be two fuzzy subsets of D,*

(i) ν is a fuzzy primary ideal of D if and only if for all $\alpha \in [0,1]$, such that $\emptyset \neq \nu_\alpha \neq D$, ν_α is a primary ideal of D;

(ii) μ is a fuzzy primary filter of D if and only if for all $\alpha \in [0,1]$, $\emptyset \neq \mu_\alpha \neq D$, μ_α is a primary filter of D.

Proof. Case of (i)
$\Longrightarrow$) Suppose ν is a fuzzy primary ideal of D, let $\alpha \in [0,1]$ and $x \in D$. We have $\nu(x) \vee \nu(\lrcorner x) = 1$ this implies $\nu(x) \vee \nu(\lrcorner x) \geq \alpha$. Thus we have: $\nu(x) \geq \alpha$ or $\nu(\lrcorner x) \geq \alpha$. Hence $x \in \nu_\alpha$ or $\lrcorner x \in \nu_\alpha$.
$\Longleftarrow$) Conversely, let's assume that for any $\alpha \in [0,1]$, ν_α is a primary ideal of D. let us show that ν is a fuzzy primary ideal of D. Let $x \in D$, since ν_α is a primary ideal of D, either $x \in \nu_\alpha$ or $\lrcorner x \in \nu_\alpha$ this implies $\nu(x) \geq \alpha$ or $\nu(\lrcorner x) \geq \alpha$, thus $\nu(x) \vee \nu(\lrcorner x) \geq \alpha$ by taking on both side the sup on α, we get $\nu(x) \vee \nu(\lrcorner x) \geq 1$. Hence $\nu(x) \vee \nu(\lrcorner x) = 1$. The proof of (ii) is similar to the one of (i). □

Definition 3.9. Let μ be a fuzzy subset of $\underline{D}$. Then μ is a fuzzy maximal ideal if $\mu \neq \underline{1}$ and for any fuzzy ideal ν, $\mu \leq \nu \Longrightarrow \nu = \underline{1}$ or $\mu = \nu$.

Dually, fuzzy maximal filter of $\underline{D}$ is defined.

Example 3.10. Let us consider the double Boolean algebra illustrated in example 2.2, with it diagram in Figure 1. Then we have the following:

$$\mu(x) = \begin{cases} \frac{1}{2} & if \quad x \in \{a, b, c, d, e, \top\} \\ 1 & if \quad x = \bot \end{cases}$$

is a maximal fuzzy ideal. In fact by taking $\alpha \in [0,1]$, we have $\mu_\alpha = D$, if $\alpha > \frac{3}{5}$ and $\mu_\alpha = \{a, b, c, d, e, \top\}$ if $\alpha \leq \frac{3}{5}$. So μ_α is a maximal ideal of D.

Proposition 3.11. *Let μ, ν be two fuzzy subsets of $\underline{D}$.*

(i) If μ is fuzzy primary ideal iff μ is fuzzy maximal ideal.

(ii) If ν is fuzzy primary filter iff ν is fuzzy maximal filter.

Proof. Case of (i)
Let μ a fuzzy primary ideal of $\underline{D}$, let us show that μ is a fuzzy maximal ideal of $\underline{D}$. Since μ is a fuzzy primary ideal, μ is proper, then $\mu \neq \underline{1}$. Let ν be a fuzzy ideal of D such that $\mu \leq \nu$ and let us assume that $\nu \neq \underline{1}$ and show that necessarily we have $\mu = \nu$. If $\mu \neq \nu$ then there exists $a \in D$, such that $\mu(a) < \nu(a)$. According to Proposition 3.8, for $\alpha = 1$, $\mu_1 = \{x \in D : \ \mu(x) = 1\}$, since $\mu \neq 1$, and $\bot \in \mu_1$ then $\emptyset \neq \mu_1 \neq D$ is a primary ideal of D. But $\mu \leq \nu$ implies that $\mu_1 \subseteq \nu_1$ and using the fact that any primary ideal of D is maximal, we therefore have $\mu_1 = \nu_1$ or $\nu_1 = D$. Having $\mu < \nu$ implies that $\mu_1 \neq \nu_1$. Hence by maximality of μ_1, we have $\nu_1 = D$ and this implies that $\nu = \underline{1}$ which is absurd ($\nu \neq \underline{1}$ by hypothesis). Conversely, let assume that μ is a maximal fuzzy ideal and let us show that μ is a fuzzy primary ideal.

Suppose that there exists $x \in D$ such that $\mu(x) \vee \mu(\lrcorner x) < 1$, then we have $\mu(x) < 1$ and $\mu(\lrcorner x) < 1$. Then there exists $x \in D$, $x \notin \mu_1$ and $\lrcorner x \notin \mu_1$. Since μ is maximal fuzzy ideal, then μ_1 is a maximal ideal. But $x \notin \mu_1$ implies that $Ideal < x > \cup \mu_1$ strictly contains μ_1 and by maximality of μ_1, we have $Ideal < x > \cup \mu_1 = D$. Thus $\lrcorner x \in \mu_1$, which is absurd.

The case of (ii) is the dual version. □

Proposition 3.12. *Let μ be a fuzzy subset of $\underline{D}$.*

(i) μ is a fuzzy ideal of D iff $\mu(\bot) \geq \mu(x)$ and $\mu(y) \geq \mu(x) \wedge \mu(\lrcorner x \wedge y), \forall x, y \in D$

(ii) μ *is a fuzzy filter of* D *iff* $\mu(\top) \geq \mu(x)$ *and* $\mu(x) \geq \mu(y) \wedge \mu(x \vee \neg y)$.

Proof. Case (i)
$\Longrightarrow$) Let us assume that μ is a fuzzy ideal of $\underline{D}$. Then it is obvious that $\mu(\bot) \geq \mu(x)$. Since μ is a fuzzy ideal of D, we have: $\mu(x \sqcup (\lrcorner x \wedge y)) = \mu(x) \wedge \mu(\lrcorner x \wedge y)$ but $y \sqsubseteq x \sqcup (\lrcorner x \wedge y)$, since $x \sqcup (\lrcorner x \wedge y) = (x \sqcup \lrcorner x) \wedge (x \sqcup y) = \top \wedge (x \sqcup y) = x \sqcup y$. Thus $\mu(y) \geq \mu(x) \wedge \mu(\lrcorner x \wedge y)$.

$\Longleftarrow$) Now let us assume that $\mu(\bot) \geq \mu(x)$ and $\mu(y) \geq \mu(x) \wedge \mu(\lrcorner x \wedge y), \forall x, y \in D$. Then we have $\mu(x \sqcup y) \leq \mu(x)$ and $\mu(x \sqcup y) \leq \mu(y)$. Thus $\mu(x \sqcup y) \leq \mu(x) \wedge \mu(y)$ and $\mu(x \sqcup y) \geq \mu(x) \wedge \mu(\lrcorner x \sqcup (x \sqcup y)) \geq \mu(y) \wedge \mu(y)$. Thus $\mu(x \sqcup y) = \mu(x) \wedge \mu(y)$.

Let x, y such that $x \sqsubseteq y$, then we have

$$\begin{aligned} \mu(x) &\geq \mu(y) \wedge \mu(\lrcorner y \wedge x) \\ &= \mu(y \sqcup (\lrcorner y \wedge x)) \quad \textit{By definition} \\ &\geq \mu(y \sqcup y) = \mu(y) \quad \textit{Proposition } 3.3 \end{aligned}$$

Finally, we have the equivalence. The case (ii) is similar. □

4 Lattices of fuzzy ideals in double Boolean algebras

Kaunyun et *al.* in [12] have introduced the concept of tip-extended. In the context of double Boolean algebra, having μ and ν to be two fuzzy ideals, it is not always true that $\mu \vee \nu$ is a fuzzy ideal too. So to solve this problem we need to introduce the concept of tip-extended in double Boolean algebra.

Definition 4.1. Let μ and ν be two fuzzy sets of $\underline{D}$. Then the tip-extended pair of μ and ν of $\underline{D}$ can be defined as follows:

$$\mu^{\nu}(x) = \begin{cases} \mu(x) & \text{if } x \neq \bot \\ \mu(\bot) \vee \nu(\bot) & \text{if } x = \bot \end{cases}$$

and

$$\nu^{\mu}(x) = \begin{cases} \nu(x) & \text{if } x \neq \bot \\ \nu(\bot) \vee \mu(\bot) & \text{if } x = \bot \end{cases} .$$

Lemma 4.2. *Let* μ *be a fuzzy ideal of* $\underline{D}$ *and* $t \in [0, 1]$. *Then*

$$\mu^{t}(x) = \begin{cases} \mu(x) & \textit{if } x \neq \bot \\ \mu(\bot) \vee t & \textit{if } x = \bot \end{cases}$$

is a fuzzy ideal of $\underline{D}$.

Proof. Let $x, y \in D$, and $t \in [0, 1]$

$$\mu^{t}(x \sqcup y) = \begin{cases} \mu(x \sqcup y) & \text{if } x \sqcup y \neq \bot \\ \mu(\bot) \vee t & \text{if } x \sqcup y = \bot \end{cases}$$

- If $x \sqcup y \neq \bot$, then we have $\mu^{t}(x \sqcup y) = \mu(x \sqcup y) = \mu(x) \wedge \mu(y)$

- If $x = \bot$ and $y \neq \bot$

$$\begin{aligned}\mu^t(x \sqcup y) &= \mu(x \sqcup y) \\ &= \mu(x) \wedge \mu(y) \\ &= (\mu(\bot) \vee t) \wedge \mu^t(y) \\ &= \mu^t(x) \wedge \mu^t(y)\end{aligned}$$

- If $x \neq \bot$ and $y \neq \bot$, then

$$\begin{aligned}\mu^t(x \sqcup y) &= \mu(x \sqcup y) \\ &= \mu(x) \wedge \mu(y) \\ &= \mu^t(x) \wedge \mu^t(y)\end{aligned}$$

Let $x, y \in D$ such that $x \sqsubseteq y$ we need to show that $\mu^t(x) \geq \mu^t(y)$.
If $y = \bot$, then $\mu^t(x) = \mu^t(y)$.
If $y \neq \bot$ and $x = \bot$ then we have $\mu^t(y) = \mu(y) \leq \mu(\bot) = \mu(x) \leq \mu(\bot) \vee t = \mu^t(x)$. If $y \neq \bot$ and $x \neq \bot$ we have $\mu^t(y) = \mu(y) \geq \mu(x) = \mu^t(x)$. Thus in any case, $\mu^t(y) \geq \mu^t(x)$. Thus for all $x, y \in D$, $\mu^t(x \sqcup y) = \mu^t(x) \wedge \mu^t(y)$.

□ In general, when ν is a non constant function, the tip-extended pair μ^ν is a fuzzy ideal. Here we defined the join of two fuzzy ideals.

Definition 4.3. Let μ and ν be two fuzzy sets of $\underline{D}$. Then the operation $\sqcup^*$ is defined as follows:

$$(\mu \sqcup^* \nu)(x) = \bigvee_{x \sqsubseteq y \sqcup z} (\mu(y) \vee \nu(z)), \quad \forall x \in D. \tag{2}$$

The following theorem characterizes the fuzzy ideal of $\underline{D}$ generated by a fuzzy subset.

Lemma 4.4. *Let μ be a fuzzy set of D. Define a fuzzy set ν of D as follows:*

$$\nu(x) = \bigvee_{x \sqsubseteq x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n} (\mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_n)) \tag{3}$$

for some $x_1, x_2 \cdots, x_n \in D$. Then ν is the smallest fuzzy ideal of D that contains μ.

Proof. Let us first show that ν is a fuzzy ideal of D. Let $x, y \in D$. By definition of $\nu(x \sqcup y)$, we have

$$\nu(x \sqcup y) = \bigvee_{x \sqcup y \sqsubseteq x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n} (\mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_n)).$$

By definition of $\nu(x)$ and $\nu(y)$, we have

$$\nu(x) = \bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} (\mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n))$$

$$\nu(y) = \bigvee_{y \sqsubseteq b_1 \sqcup b_2 \sqcup \cdots \sqcup b_m} (\mu(b_1) \wedge \mu(b_2) \wedge \cdots \wedge \mu(b_m)).$$

We know from [11] that if $x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n$ and $y \sqsubseteq b_1 \sqcup b_2 \sqcup \cdots \sqcup b_m$, then $x \sqcup y \sqsubseteq (a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n) \sqcup (b_1 \sqcup b_2 \sqcup \cdots \sqcup b_m)$ thus we have the following:

$$\begin{aligned}\nu(x) \wedge \nu(y) &= \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n) \right) \wedge \\ &\quad \left(\bigvee_{y \sqsubseteq b_1 \sqcup b_2 \sqcup \cdots \sqcup b_m} \mu(b_1) \wedge \mu(b_2) \wedge \cdots \wedge \mu(b_m) \right) \\ &= \bigvee \left(\mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n) \wedge \mu(b_1) \wedge \mu(b_2) \wedge \cdots \wedge \mu(b_m) \right) \\ &= \nu(x \sqcup y).\end{aligned}$$

Thus ν is a fuzzy ideal of $\underline{D}$.

Let $x \in D$, then we have that $\nu(x) \geq \mu(x)$, this shows that ν contains μ.

Let η be a fuzzy ideal of L that contains μ ($\eta(x) \geq \mu(x)$) and let $x \in D$,

$$\begin{aligned}\nu(x) &= \bigvee_{x \sqsubseteq x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n} \mu(x_1) \wedge \mu(x_2) \wedge \cdots \wedge \mu(x_n) \\ &\leq \bigvee_{x \sqcup y \sqsubseteq x_1 \sqcup x_2 \sqcup \cdots \sqcup x_n} \eta(x_1) \wedge \eta(x_2) \wedge \cdots \wedge \eta(x_n) \\ &\leq \eta(x).\end{aligned}$$

□

Notation:

Let μ be a fuzzy set of $\underline{D}$. We denote by $< \mu >$, the fuzzy ideal generated by μ. That is the smallest fuzzy ideal containing μ.

Lemma 4.5. *Let $\underline{D}$ be a double Boolean algebra, μ and ν two fuzzy ideals of $\underline{D}$. Then $\mu^{\nu} \sqcup^{*} \nu^{\mu} = < \mu \vee \nu >$. That is: the fuzzy ideal generated by μ and ν is exactly the supremum between the tip-extended pair of μ and ν.*

Proof. We need first to show that $\mu^{\nu} \sqcup^{*} \nu^{\mu}$ is a fuzzy ideal.

Let $x, y \in D$,

$$\begin{aligned}\left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x \sqcup y) &= \bigvee_{x \sqcup y \sqsubseteq a \sqcup b} \mu^{\nu}(a) \wedge \nu^{\mu}(b) \\ &\geq \bigvee_{x \sqsubseteq q \sqcup p, y \sqsubseteq r \sqcup s} \mu^{\nu}(p \sqcup r) \wedge \nu^{\mu}(q \sqcup s) \\ &\geq \bigvee_{x \sqsubseteq q \sqcup p, y \sqsubseteq r \sqcup s} \mu^{\nu}(p) \wedge \mu^{\nu}(r) \wedge \nu^{\mu}(q) \wedge \nu^{\mu}(s) \\ &= \left(\bigvee_{x \sqsubseteq p \sqcup q} \mu^{\nu}(p) \wedge \nu^{\mu}(q) \right) \wedge \left(\bigvee_{y \sqsubseteq r \sqcup s} \mu^{\nu}(r) \wedge \nu^{\mu}(s) \right) \\ &= \left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x) \wedge \left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(y).\end{aligned}$$

Let $x, y \in D$ such that $x \sqsubseteq y$, we need to show that $\left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x) \geq \left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(y)$

If $y = \perp$ then it is obvious that $x = \perp$ and the result holds.
If $y \neq \perp$

$$\left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x) = \bigvee_{x \sqsubseteq t \sqcup z} \mu^{\nu}(t) \wedge \nu^{\mu}(z) \tag{4}$$

$$\left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(y) = \bigvee_{y \sqsubseteq r \sqcup s} \mu^{\nu}(r) \wedge \nu^{\mu}(s) \tag{5}$$

since $x \sqsubseteq y$ so by transitivity of $\sqsubseteq$, $x \sqsubseteq r \sqcup s$.
Thus $\bigvee_{y \sqsubseteq r \sqcup s} \mu^{\nu}(r) \wedge \nu^{\mu}(s) \leq \bigvee_{x \sqsubseteq t \sqcup z} \mu^{\nu}(t) \wedge \nu^{\mu}(z)$. That is $\left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x) \leq \left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x)$
Thus $\mu^{\nu} \sqcup^{*} \nu^{\mu}$ is a fuzzy ideal of D. Now let us show that we have $\mu^{\nu} \sqcup^{*} \nu^{\mu} \geq \mu \vee \nu$ to conclude with Lemma 4.2 and Lemma 4.4.
Let $x \in D$,

$$\begin{aligned}
\left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x) &= \bigvee_{x \sqsubseteq y \sqcup z} \mu^{\nu}(y) \wedge \nu^{\mu}(z) \\
&\geq \mu^{\nu}(x) \wedge \nu^{\mu}(\perp) \\
&\geq \mu(x) \wedge \mu(\perp) \\
&= \mu(x).
\end{aligned}$$

Thus $\mu^{\nu} \sqcup^{*} \nu^{\mu}$ contains μ. Similarly,

$$\begin{aligned}
\left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x) &= \bigvee_{x \sqsubseteq y \sqcup z} \mu^{\nu}(y) \wedge \nu^{\mu}(z) \\
&\geq \nu^{\mu}(x) \wedge \mu^{\nu}(\perp) \\
&\geq \nu(x) \wedge \nu(\perp) \\
&= \nu(x).
\end{aligned}$$

Let η be a fuzzy ideal of D containing $\mu \vee \nu$. We need to show that $\eta(x) \geq \left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x)$
Let $x \in D$, if $x = \perp$, then $\left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(\perp) = \mu(\perp) \vee \nu(\perp) \leq \eta(\perp)$.
If $x \neq \perp$

$$\begin{aligned}
\left(\mu^{\nu} \sqcup^{*} \nu^{\mu}\right)(x) &= \bigvee_{x \sqsubseteq y \sqcup z} \mu^{\nu}(y) \wedge \nu^{\mu}(z) \\
&= \left(\bigvee_{x \sqsubseteq y \sqcup z, y \neq \perp, z \neq \perp} \mu(y) \wedge \nu(z)\right) \vee \left(\bigvee_{x \sqsubseteq y} \mu(y)\right) \vee \left(\bigvee_{x \sqsubseteq z} \nu(z)\right) \\
&= \left(\bigvee_{x \sqsubseteq y \sqcup z, y \neq \perp, z \neq \perp} \mu(y) \wedge \nu(z)\right) \vee \left(\bigvee_{x \sqsubseteq y} \mu(y)\right) \vee \left(\bigvee_{x \sqsubseteq z} \nu(z)\right) \\
&\leq \left(\bigvee_{x \sqsubseteq y \sqcup z, y \neq \perp, z \neq \perp} \eta(y) \wedge \eta(z)\right) \vee \left(\bigvee_{x \sqsubseteq y} \eta(y)\right) \vee \left(\bigvee_{x \sqsubseteq z} \eta(z)\right) \\
&= \bigvee_{x \sqsubseteq y \sqcup z} \eta(y) \wedge \eta(z) \\
&\leq \eta(x).
\end{aligned}$$

Thus $\mu^{\nu} \sqcup^{*} \nu^{\mu} = < \mu \vee \nu >$. □

Theorem 4.6. *If we consider $FI(D)$ to be the collection of all fuzzy ideals of $\underline{D}$, then $(FI(D), \sqcup^*, \sqcap^*, \underline{0}, \underline{1})$ is a bounded lattice, where $\sqcap^*$ is defined as follows: $\mu \sqcap^* \nu = \mu \wedge \nu$, $\underline{0} : D \mapsto [0,1]$, $\underline{0}(x) = 0$, for all $x \in D$. $\underline{1} : D \mapsto [0,1]$, and $\underline{1}(x) = 1$, for all $x \in D$.*

Proof. From the previous proposition, any pair of elements of $FI(D)$ has a supremum (Lemma 4.5) and the upper bound is $\underline{1}$, the lower bound is $\underline{0}$. □

Theorem 4.7. *The map*

$$\begin{aligned} \overline{\varphi} \colon FI(D_\sqcup) &\longrightarrow FI(D) \\ \mu &\longmapsto \tilde{\mu} \end{aligned}$$

Where

$$\begin{aligned} \tilde{\mu} \colon D &\longrightarrow [0,1] \\ x &\longmapsto \mu(x \sqcup x) \end{aligned}$$

is an isomorphism of lattices.

Proof. The map $\overline{\varphi}$ is well defined, in fact for $\mu \in FI(D_\sqcup)$, $\tilde{\mu} \in FI(D)$ according to Proposition 3.2. $\overline{\varphi}$ preserve the order $\leq$. Suppose $\mu_1 \leq \mu_2$ then let $x \in D$, $\tilde{\mu_1}(x) = \mu_1(x \sqcup x) \leq \mu_2(x \sqcup x)$. Thus $\tilde{\mu_1}(x) \leq \tilde{\mu_2}(x)$. Hence $\tilde{\mu_1} \leq \tilde{\mu_2}$. Conversely, if $\tilde{\mu_1} \leq \tilde{\mu_2}$, then $\tilde{\mu_1}_{/D_\sqcup} = \tilde{\mu_2}_{/D_\sqcup}$ i.e $\mu_1 \leq \mu_2$. Thus $\mu_1 \leq \mu_2 \iff \overline{\varphi}(\mu_1) \leq \overline{\varphi}(\mu_2)$. $\overline{\varphi}$ is surjective since for any $\mu \in FI(D)$, $\mu_{/D_\sqcup} \in FI(D_\sqcup)$ and $\overline{\varphi}\left(\mu_{/D_\sqcup}\right) = \mu$. $\overline{\varphi}$ is injective, in fact let $\mu_1, \mu_2 \in FI(D_\sqcup)$ such that $\tilde{\mu_1} = \tilde{\mu_2}$ Let us show that $\mu_1 = \mu_2$. Let $x \in D_\sqcup$,

$$\begin{aligned} \mu_1(x) &= \mu_1(x \sqcup x) \quad \textit{by (i) of Proposition 3.2} \\ &= \tilde{\mu_1}(x) \quad \textit{by definition of } \overline{\varphi} \\ &= \tilde{\mu_2}(x) \quad \textit{by hypothesis } (\tilde{\mu_1} = \tilde{\mu_2}) \\ &= \mu_2(x \sqcup x) \\ &= \mu_2(x). \end{aligned}$$

Thus $\mu_1 = \mu_2$. □

5 Lattices of generalized fuzzy ideals in dbas

In this section, we introduce the notion of generalized fuzzy ideal which is a more general definition of fuzzy ideal on dBas. Namely we show that the collection of generalized fuzzy ideals of a dBa is endowed with a lattice structure.

Let us first define the concept of generalized fuzzy ideal.

Let $m, n \in [0,1]$ and $m < n$, then a fuzzy set μ of D is called a generalized fuzzy ideal of D if:

(I_1) $\mu(x \sqcup y) \vee m \geq \mu(x) \wedge \mu(y) \wedge n$

(I_2) $x \sqsubseteq y \Longrightarrow \mu(x) \vee m \geq \mu(y) \wedge n$.

Here generalized fuzzy ideals are defined with respect to a fixed pair (m, n). We denote by $GFI(D)$ the set of all generalized fuzzy ideals.

Let $m, n \in [0,1]$ and $m < n$. Then the fuzzy set μ of $\underline{D}$ is called generalized fuzzy ideal if :

(I_3) $\mu(\perp) \vee m \geq \mu(x) \wedge n$;

(I_4) $\mu(y) \vee m \geq \mu(x) \wedge \mu(\lrcorner x \sqcup y) \wedge n$.

Proof. Let μ be a generalized fuzzy ideal of $\underline{D}$. Since $\perp \sqsubseteq x$, for any $x \in D$, it follows that $\mu(\perp) \vee m \geq \mu(x) \wedge n$. On other hand, since $y \sqsubseteq x \sqcup (\lrcorner x \sqcup y)$, we have $\mu(y) \vee m \geq \mu\left(x \sqcup (\neg x \sqcup y)\right) \wedge n$.

Thus one can write:

$$\begin{aligned}\mu(y) \vee m \vee m &= \mu(y) \vee m \\ &\geq \left(\mu\left(x \sqcup (\lrcorner x \sqcup y)\right) \wedge n\right) \vee m \\ &= (\mu(x \sqcup (\lrcorner x \sqcup y)) \vee m) \wedge (n \vee m) \\ &\geq \mu(x) \wedge \mu(\lrcorner x \sqcup y) \wedge n \wedge n \\ &= \mu(x) \wedge \mu(\lrcorner x \sqcup y) \wedge n.\end{aligned}$$

Conversely, let us now assume that I_3 and I_4 hold. Let $x, y \in L$ and $x \sqsubseteq y$. Then $\lrcorner y \sqsubseteq \lrcorner x$ and $x \wedge \lrcorner y \sqsubseteq x \wedge \lrcorner y$ by compatibility. So $\mu(\perp) = \mu(\neg y \wedge x)$ and from (I_2) we have that

$$\begin{aligned}\mu(x) \vee m &\geq \mu(y) \wedge \mu(\neg y \wedge x) \wedge n \\ &= \mu(y) \wedge \mu(\perp) \wedge n.\end{aligned}$$

Thus

$$\begin{aligned}\mu(x) \vee m \vee m &\geq ((\mu(y) \wedge n) \vee m) \wedge (\mu(\perp) \vee m) \\ &\geq (\mu(y) \wedge n) \vee m) \wedge (\mu(y) \wedge n) \\ &= \mu(y) \wedge n.\end{aligned}$$

This implies that I_2 holds. On the other hand, since $\lrcorner x \wedge (x \sqcup y) \sqsubseteq y$,
we have $\mu\left(\lrcorner x \wedge (x \sqcup y)\right) \vee m \geq \mu(y) \wedge n$ and from I_4 we have that $\mu(x \sqcup y) \vee m \geq \mu(x) \wedge \mu(\lrcorner x \wedge (x \sqcup y)) \wedge n$.
Thus

$$\begin{aligned}\mu(x \sqcup y) \vee m \vee m &\geq ((\mu(x) \wedge n) \vee m) \wedge (\mu(y) \wedge n) \vee m \\ &= ((\mu(x) \wedge n) \vee m) \wedge \left(\mu\left(\lrcorner x \wedge (x \sqcup y)\right) \vee m\right) \\ &= ((\mu(x) \wedge n) \vee m) \wedge (\mu(y) \wedge n) \\ &= (\mu(x) \wedge n) \wedge (\mu(y) \wedge n) \\ &= \mu(x) \wedge \mu(y) \wedge n.\end{aligned}$$

Therefore I_1 is satisfied and this shows that μ is a generalized fuzzy ideal of D. □

Theorem 5.1. *Let $m, n \in [0, 1]$ and $m < n$, Then a fuzzy set μ of D is a generalized fuzzy ideal if and only if for all $x, y, z \in D$,*

$$x \sqsubseteq y \sqcup z \Longrightarrow \mu(x) \vee m \geq \mu(y) \wedge \mu(z) \wedge n.$$

Proof. Let μ be a generalized fuzzy ideal of $\underline{D}$ and $x \sqsubseteq y \sqcup z$, then $\mu(x) \vee m \geq \mu(y \sqcup z) \wedge n$.

Hence one can write

$$\begin{aligned}\mu(x) \vee m &\geq (\mu(y \sqcup z) \vee m) \wedge (n \vee m) \\ &\geq \mu(y) \wedge \mu(z) \wedge n \wedge n \\ &= \mu(y) \wedge \mu(z) \wedge n.\end{aligned}$$

Thus $\mu(x) \vee m \geq \mu(y) \wedge \mu(z) \wedge n$, conversely, since we know that we have $\mu(\bot) \vee m \geq \mu(x) \wedge \mu(x) \wedge n$. Thus $\mu(\bot) \vee m \geq \mu(x) \wedge n$.

On the other side, since we know that $y \sqsubseteq x \sqcup (\lrcorner x \wedge y)$ we have:

$$\mu(y) \vee m \geq \mu(x) \wedge \mu(\lrcorner x \wedge y) \wedge n$$

Hence μ is a generalized fuzzy ideal of D.

□

Example 5.2. By considering the dBa of example 2.2, we can easily show that the following is a generalized fuzzy dBa:

$$\mu(x) = \begin{cases} \frac{1}{3} & if \quad x \in \{a, b, c, d, e, \top\} \\ 1 & if \quad x = \bot \end{cases}$$

In fact, let $m, n \in [0, 1]$ such that $m < n$. Then the following property is verify:

$$x \sqsubseteq y \sqcup z \Longrightarrow \mu(x) \vee m \geq \mu(z) \wedge n.$$

Corollary 5.3. *Let $m, n \in [0, 1]$ and $m < n$. Then a fuzzy set μ of $\underline{D}$ is called a generalized fuzzy ideal if and only if for all $x, y_1, \cdots, y_n \in D$, $x \sqsubseteq y_1 \sqcup y_2 \sqcup \cdots \sqcup y_n$ implies that $\mu(x) \vee m \geq \mu(y_1) \wedge \cdots \wedge \mu(y_n) \wedge n$.*

Let μ be a fuzzy set of $\underline{D}$, $m, n \in [0, 1]$ and $m < n$. Then the intersection of all generalized fuzzy ideals containing μ is called the generated generalized fuzzy ideal by μ, denoted $< \mu >^{(m,n)}$.

Theorem 5.4. *Let $\underline{D}$ be a double Boolean algebra μ be a fuzzy set of L, $m, n \in [0, 1]$ and $m < n$ then*

$$\begin{aligned} \langle \mu \rangle^{(m,n)}(x) &= m \vee \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \mu(a_1) \wedge \cdots \wedge \mu(a_n) \wedge n \right) \\ &= \bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} (\mu(a_1) \vee m) \wedge \cdots \wedge (\mu(a_n) \vee m) \wedge n, \quad \textit{for all } x \in D. \end{aligned}$$

Proof. Let

$$\theta(x) = m \vee \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \mu(a_1) \wedge \cdots \wedge \mu(a_n) \wedge n \right).$$

Let us show that θ is a generalized fuzzy ideal which contains μ.

For $n = 1$ and $a_1 = x$, $m \vee (n \wedge \mu(x)) \leq \theta(x)$. Hence $n \wedge \mu(x) \leq \theta(x) \vee m$. We can easy check that θ is a generalized fuzzy ideal. Now let us focus on how to show that θ is the smallest generalized fuzzy ideal which contains μ.

Let us assume that there is a generalized fuzzy ideal η such that $\forall x \in D$, $\eta(x) \vee m \geq n \wedge \mu(x)$.

Then we need to show that η contains θ too.

$$\begin{aligned} n \wedge \theta(x) &= n \wedge \left(m \vee \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \mu(a_1) \wedge \mu(a_2) \wedge \cdots \wedge \mu(a_n) \wedge n \right) \right) \\ &\leq m \vee \left(\bigvee_{x \sqsubseteq a_1 \sqcup a_2 \sqcup \cdots \sqcup a_n} \eta(x) \wedge \cdots \wedge \eta(x) \wedge n \right) \\ &\leq m \vee \eta(x). \end{aligned}$$

Thus θ is the smallest generalized fuzzy ideal which contains μ. □

Definition 5.5. Let μ and ν be two fuzzy sets of L, $m, n \in [0,1]$ and $m < n$ the operation $\tilde{\sqcup}^{(m,n)}$ is defined by:

$$\mu\tilde{\sqcup}^{(m,n)}\nu = \bigvee_{x\sqsubseteq y\sqcup z} \left((\mu(y) \vee m) \wedge (\nu(z) \vee m) \wedge n\right).$$

Let $GFI(D)$ the set of generalized fuzzy ideals of $\underline{D}$.

Remark 5.6. Let μ be a generalized fuzzy ideal of D and $t \in [0,1]$. Then μ^t is also a generalized fuzzy ideal of $\underline{D}$.

Theorem 5.7. *Let $\underline{D}$ be a double Boolean algebra, $m, n \in [0,1]$ such that $m < n$. Then $\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu = \langle\mu \vee \nu\rangle^{(m,n)}$. More over, $(GFI(D), \tilde{\sqcap}, \tilde{\sqcup}, \underline{0}, \underline{1})$ is a bounded lattice, where $GFI(D)$ the set of generalized fuzzy ideals of $\underline{D}$.*

Proof.

$$\begin{aligned}
m \vee \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x \wedge y) &= \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x \wedge y) \\
&= \bigvee_{x\sqcup y\sqsubseteq u\sqcup v} \left((\mu^\nu(u) \vee m) \wedge (\nu^\mu(v) \vee m) \wedge n\right) \\
&\geq \bigvee_{x\sqsubseteq p\sqcup q, y\sqsubseteq r\sqcup s} (\mu^\nu(p \sqcup r) \vee m) \wedge (\nu^\mu(q) \wedge \nu^\mu(p) \wedge n) \\
&= \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x) \wedge \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(y) \\
&= \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x) \wedge \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(y) \wedge n.
\end{aligned}$$

Let $x, y \in D$ and $x \sqsubseteq y$. Then it is easy to see that $m \vee \left(\mu^\nu\tilde{\sqcup}\nu^\mu\right)(x) \geq n \wedge \left(\mu^\nu\tilde{\sqcup}\nu^\mu\right)(y)$.
So $\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu$ is a generalized fuzzy ideal of $\underline{D}$.
Let $x \in D$,

$$\begin{aligned}
m \vee \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x) &= \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(x) \\
&= \bigvee_{x\sqsubseteq y\sqcup z} \left((\mu^\nu(y) \vee m) \wedge (\nu^\mu(z) \vee m) \wedge n\right) \\
&\geq \left((\mu^\nu(x) \vee m) \wedge (\nu^\mu(\perp) \vee m) \wedge n\right) \\
&\geq \left((\mu^\nu(x) \vee m) \wedge (\mu^\nu(\perp) \vee m) \wedge n\right) \\
&= \left((\mu^\nu(x) \wedge n) \vee m\right) \\
&\geq \mu^\nu(x) \wedge n \\
&\geq \mu(x) \wedge n.
\end{aligned}$$

Hence $m \vee \mu^\nu\tilde{\sqcup}\nu^\mu \geq \mu \wedge n$. In a similar way, we have $m \vee \mu^\nu\tilde{\sqcup}\nu^\mu \geq \nu \wedge n$.
Hence $\mu^\nu\tilde{\sqcup}\nu^\mu \geq \mu \vee \nu^{(m,n)}$.
Last, let us verify that is the smallest. Let $\lambda \in GFI(D)$ such that $n \wedge (\mu(x) \vee \nu(x)) \leq \lambda(x) \wedge n$.
Consider the following cases:
If $x = 0$ then $n \wedge \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(\perp) = \left(\mu^\nu\tilde{\sqcup}^{(m,n)}\nu^\mu\right)(0) = (\mu(\perp) \vee \nu(\perp) \vee m) \wedge n \leq \lambda(\perp) \vee m$.

Else:

$$\begin{aligned} n \wedge \left(\mu^{\nu} \tilde{\sqcup}^{(m,n)} \nu^{\mu}\right)(x) &= \left(\mu^{\nu} \tilde{\sqcup}^{(m,n)} \nu^{\mu}\right)(x) \\ &= \bigvee_{x \sqsubseteq y \sqcup z} (((\mu^{\nu}(y) \vee m)) \wedge (\nu^{\mu}(y) \vee m) \wedge n) \\ &\leq m \vee \bigvee_{x \sqsubseteq y \sqcup z} (\lambda(y) \wedge \lambda(z) \wedge n) \\ &\leq m \vee \lambda(x). \end{aligned}$$

Thus

$$n \wedge \left(\mu^{\nu} \tilde{\sqcup}^{(m,n)} \nu^{\mu}\right)(x) \leq m \vee \lambda(x)$$

We can conclude that $\mu^{\nu} \tilde{\sqcup}^{(m,n)} \nu^{\mu} = \langle \mu \vee \nu \rangle^{(m,n)}$. Thus we have just proved that any pair of elements of $FI(D)$ has a supremum. The upper bound is $\underline{1}$, the lower bound is $\underline{0}$.

□

6 Conclusion

In this paper fuzzy ideals on double Boolean algebras have been studied. Various properties and characterizations of fuzzy ideals have been proved. Based on the notion of tip-extended pair inspired by Kuanyun [12], it was proved that the set of fuzzy ideals on double Boolean algebra is endowed with a structure of the bounded lattice. This is a new type of lattice structure constructed from fuzzy ideals. This lattice structure has the particularity that it is isomorphic to the collection of fuzzy ideals of the Boolean algebra $D_{\sqcup}$. Thus we can conclude that the collection of fuzzy ideals of the dBa D can fully be determined by knowing just fuzzy ideals of the Boolean algebra $D_{\sqcup}$. We have also introduced the concept of fuzzy primary ideals and fuzzy primary filters and have established that the set of generalized fuzzy ideals of a double Boolean algebra has the structure of a bounded lattice. Generalized fuzzy ideals of D can also be entirely determined by generalized fuzzy ideals of $D_{\sqcup}$. In the future, it will be interesting to investigate L-fuzzy ideals (where L is a bounded lattice) and generalized L-fuzzy ideals in the framework of double Boolean algebras.

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References

[1] Attallah M. Complete Fuzzy Prime Ideals of Distributives Lattices. *The journal of fuzzy Mathematics*, 2000; 8(1): 151-156.

[2] Awoke T, Rout M, Mohandi L, and Satapathy S. Bitcoin Price Prediction and Analysis Using Deep Learning Models. *Networks and Systems,*. 2020; 1-5. DOI: https://doi.org/10.1007/978-981-15-5397-4-63

[3] Azari A, Papapetrou P, Denic S and Peters G. Cellular Traffic Prediction and Classification: a comparative evaluation of LSTM and ARIMA. *arXiv*. 2019; 1-8. DOI: https://doi.org/10.48550/arXiv.1906.00939

[4] Chon I. Fuzzy Partial Order Relations and Fuzzy Lattices. *Korean Journal Mathematics*. 2009; 17(4): 361-374.

[5] Fotso Tatuéné A, Tonga M. *Transfer of structure between a universal algebra and its fuzzy algebra*. 2016.

[6] Islam S, Sanyal A and Sen J. Fuzzy filters of IL-algebras. *Foundation, algebraic, and analytical methods in soft computing, Springer*. 2022; 26: 7017-7027. DOI: https://doi.org/10.1007/s00500-022-06985-1

[7] Koguep BBN, Nkuimi C and Lele C. On Fuzzy Primes Ideals of Lattices. *SAMSA Journal of Pure and Applied Mathematics, (SJPAM)*. 2008; (3): 1-11.

[8] Kwuida L. Prime ideal theorem for double Boolean algebra. *Discussiones Mathematicae. General algebra and Applications*. 2007; 27: 263-275. DOI: 10.7151/dmgaa.1130.

[9] Howlader P and Banerjee M. Remarks on Prime Ideal and Representation Theorems for Double Boolean Algebras. *Proceedings of the 15th International conference on Concept Lattices and their Applications, CLA*. 2020; 83-94.

[10] Mezzomo I, Bredegal BC, Santiago RHN and Reiser RHS. Fuzzy ideals of product operators on bounded fuzzy lattices. *Second workshop-school of theoretical computer sciences*. 2014; 1-7.

[11] Tenkeu Jeufack Y, Temgoua Alomo E and Kwuida L. Filters Ideals and Congruences on Double Boolean Algebras. *International conference on Formal Concept analysis*. 2021; 270-280. DOI: https://doi.org/10.1007/978-3-030-77867-5-18

[12] Zhu K, Wang J and Yang Y. Lattices of (Generalized) Fuzzy Ideals in Residuated Lattices. *International Journal of Applied Mathematics*. 2020; 50(3).

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