Transactions on Fuzzy Sets and Systems
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 59-71
DOI: http://doi.org/10.30495/tfss.2022.1959658.1035

Article Type: Original Research Article (Invited Paper)

Forensic Dynamic Łukasiewicz Logic

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Abstract. A forensic dynamic n-valued Łukasiewicz logic $FD\text{Ł}_n$ is introduced on the base of n-valued Łukasiewicz logic Ł_n and corresponding to it forensic dynamic MV_n-algebra (FDL_n-algebra), $1 < n < \omega$, which are algebraic counterparts of the logic, that in turn represent two-sorted algebras $(\mathcal{M}, \mathcal{R}, \Diamond)$ that combine the varieties of MV_n-algebras $\mathcal{M} = (M, \oplus, \odot, \sim, 0, 1)$ and regular algebras $\mathcal{R} = (R, \cup, ;, ^*)$ into a single finitely axiomatized variety resemblig R-module with "scalar" multiplication $\Diamond$. Kripke semantics is developed for forensic dynamic Łukasiewicz logic $FD\text{Ł}_n$ with application to Digital Forensics.

AMS Subject Classification 2020: 03B421; 03B50; More
Keywords and Phrases: Lukasiewiz Logic, Dynamic Logic, Epistemic Logic, MV-algebra.

1 Introduction

Digital forensics involve securing and analyzing digital information stored on a computer for use as evidence in civil, criminal, or administrative cases. Forensics, network forensics, video forensics, and plenty of others are defined as the application of computer science and investigative procedures for a legal purpose involving the analysis of digital evidence (information of probative value that is stored or transmitted in binary form) after proper search authority, chain of keeping, validation with mathematics, use of validated tools, repeatability, reporting and possible expert presentation. The field of digital forensics can also encompass items such as research and incident response.

We introduce the notion of forensic dynamic n-valued Łukasiewicz logic $FD\text{Ł}_n$ $(1 < n < \omega)$ which permits compound investigation built up from given initial investigations and facts as well. Given investigations a and b, the compound investigations $a \cup b$, choice, is performed by performing one of a or b. The compound investigation $a;b$, sequence, is performed by performing first a and then b. The compound investigation a^*, iteration, is performed by performing a one or more times, sequentially. The constant investigation 0 does nothing and does not terminate, whereas the constant action 1, definable as 0^*, does nothing but does terminate.

Dynamic logic [14, 8] (see also [11] and cited their literature) is a classical formal system for reasoning about programs. Dynamic logic is a classical modal logic for reasoning about dynamic behavior taking into account a discrete time. Dynamic logic is an extension of modal logic originally intended for reasoning about computer programs.

Modal logic is characterized by the modal operators $\Box p$ asserting that p is necessarily the case, and $\Diamond p$ asserting that p is possibly the case. Dynamic logic extends this by associating every action (execution of the program) a the modal operators $[a]$ and $\langle a \rangle$, thereby making it a multimodal logic.

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Received: 30 May 2022; Revised: 12 June 2022; Accepted: 12 June 2022; Published Online: 7 November 2022.

How to cite: A. Di Nola, R. Grigolia, Forensic Dynamic Łukasiewicz Logic, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 59-71.

We adapt the dynamic logic, which is presented on the base of classical logic and R-module, to non-classical finitely valued Łukasiewicz logic $Ł_n$ and R-module, and the investigating group consisting of a set of investigators with communications between them represented as a Kripke frame, i.e. relational system - a non-empty set with binary relation on it. The meaning of $[a]p$ is that after performing fact-finding (investigation) a, i.e. to examine the validity of a hypothesis (proposition), it is necessarily the case that p holds, that is, a must bring about p. The meaning of $\langle a\rangle p$ is that after performing a it is possible that p, that is, a might bring about p. These operators are related by $[a]p \equiv \neg\langle a\rangle\neg p$ and $\langle a\rangle p \equiv \neg[a]\neg p$, analogously to the relationship between the universal $\forall$ and existential $\exists$ quantifiers.

Following D. Kozen [8] and V. Pratt [11], who have been introduced dynamic algebra, we propose the notion of a forensic dynamic MV_n-algebra[1] (FDL_n-algebra) ($1 < n < \omega$), which integrates an abstract notion of proposition with an equally abstract notion of investigation. Just as propositions tend to band together to form MV_n-algebras with operations $x \oplus y$, and $\sim x$, so do experiments organize themselves into regular algebras, with operations $a \cup b$, $a; b$, and a^*. Analogously to the proposition $p \underline{\vee} q$ being the strong disjunction (the algebraic counterpart of which is $x \oplus y$), $p \vee q$ being the disjunction of propositions p and q, and $\neg p$ the negation of p, the investigation $a \cup b$ is the choice of investigations a or b, $a; b$, or just ab, is the sequence a followed by b, and a^* is the iteration of a indefinitely often.

Just as $p \vee q$ has natural set theoretic interpretation, namely union, so do $a \cup b$, $a; b$ and a^* have natural interpretations on such concrete kinds of investigations as additive functions, binary relations, trajectory sets and languages over regular algebras, to name those regular algebras that are suited to foresinc dynamic MV_n-algebra.

It is natural to think of fact-finding as being able to bring about a proposition (hypothesis about the fact-findings). We write $\langle a\rangle p$ pronounced "$fact - finding\ a\ enables\ p$", as the proposition that fact-finding a can bring about proposition p. Then a forensic dynamic MV_n-algebra is a MV_n-algebra $(A, \oplus, \odot, \sim, 0, 1)$, a regular algebra $(R, \cup, ;, ^*)$, and the *enables operation* $\Diamond : R \times A \to A$.

Suppose now that either p holds, or a can bring about a situation from which a can eventually (by being iterated) bring about p. Then a can eventually bring about p. That is, $p \vee aa^*p \leq a^*p$. (We write $p \leq q$ to indicate that p implies q, defined as $p \vee q = q$). In turn, if a can eventually bring about p, then either p is already the case or a can eventually bring about a situation in which p is not the case but one further iteration of a will bring about p. That is, $a^*p \leq p \vee a^*(\neg p \wedge ap)$. $[a]$ is the dual of $\langle a\rangle$, and $[a]p$ asserts that whatever a does, p will hold.

We axiomatically define the Forensic Dynamic Lukasiewicz logic, its algebraic counterpart and the corresponding Kripke model which are suitable for digital forensics.

2 Forensic dynamic n-valued Łukasiewicz logic $FDŁ_n$

Forensic dynamic n-valued Łukasiewicz logic $FDŁ_n$ is designed for representing and reasoning about propositional Łukasiewicz logic expected results (hypothesis) of investigations. Its syntax is based upon two sets of symbols: a countable set **Var** $(= \{p, p_1, p_2, \ldots, q, q_1, q_2, \ldots\})$ of propositional variables, that encompass hypotheses, and a countable set **Inv** $(= \{a, b, c, ...\})$ of atomic investigations, that encompass the initial facts and investigations. So the language $\mathcal{L}$ of $FDŁ_n$ is given by a countable set **Var** of propositional variables and a countable set **Inv** of atomic investigations. We suppose that investigations are performed by some computer programs. Formulas and investigations $FI(\mathcal{L})$, which we name formulas, over this base are defined as follows:

- Every propositional variable is a formula;

[1] MV_n-algebras, which are algebraic models of n-valued Łukasiewicz logic $Ł_n$, where introduced by Grigolia in [6]. The variety $\mathbf{MV_n}$ of MV_n-algebras is a subvariety of the variety $\mathbf{MV}$ [2].

- $\perp$ $(false)$ is a formula;
- If φ is a formula then $\neg\varphi$ ($not\varphi$) is a formula;
- If φ and ψ are formulas then $(\phi\underline{\vee}\psi)$ ($\underline{\vee}$ is a strong disjunction) is a formula;
- If φ and ψ are formulas then $(\varphi\&\psi)$ ($\&$ is a strong conjunction) is a formula;
- If φ and ψ are formulas then $(\varphi \vee \psi)$ (φ *or* ψ) is a formula;
- If a is an investigation and φ is a formula then $[a]\varphi$ (every made investigation a from the present state leads to a state where φ is *true*) is a formula;
- Every atomic investigation is an investigation;
- If a and b are investigations then $(a; b)$ (do a followed by b) is a investigation;
- If a and b are investigations then $(a \cup b)$ (do a or b, *non-deterministically*) is an investigation;
- If a is an investigation then a^* (repeat a a finite, but non-deterministically determined, number of times) is an investigation.

The other Łukasiewicz connectives $1, \rightarrow$ and $\leftrightarrow$ are used as abbreviations in the standard way ($1 \equiv \perp\underline{\vee}\neg\perp,\ p \rightarrow q \equiv \neg p\underline{\vee}q,\ p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$). In addition, we abbreviate $\neg[a]\neg\varphi$ to $\langle a\rangle\varphi$ (performing some investigation a from the present state leads to a state where φ is true) as in modal logic. We write a^n for $a; \ldots ; a$ with n occurrences of a. More formally:

- $a^1 =_{df} a$
- $a^{n+1} =_{df} a; a^n$

The axioms of $FD\text{Ł}_n$ are the axioms of Łukasiewicz logic Ł:

(L1) $\varphi \rightarrow (\psi \rightarrow \varphi)$,
(L2) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$,
(L3) $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$,
(L4) $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi) \rightarrow \varphi)$,

plus the axioms of the logic Ł_n, that was given by R. Grigolia [6]:

(L_n5) $\varphi^n \leftrightarrow \varphi^{n-1}$,
(L_n6) $n(\varphi^k) \leftrightarrow (k(\varphi^{k-1}))^n$,

for every integer $2 \leq k \leq n-2$ that does not divide $n-1$ and for any formulas $\varphi,\ \psi$ and any investigation:

Ax0 $[a](\varphi \rightarrow \psi) \rightarrow ([a]\varphi \rightarrow [a]\psi)$,
Ax1 $[a]1 \leftrightarrow 1$,
Ax2 $[a; b]\varphi \leftrightarrow [a][b]\varphi$,
Ax3 $[a \cup b]\varphi \leftrightarrow [a]\varphi \wedge [b]\varphi$,
Ax4 $[a](\varphi \wedge \psi) \leftrightarrow ([a]\varphi \wedge [a]\psi)$.
Ax5 $[a^*]\varphi \leftrightarrow \varphi \wedge [a][a^*]\varphi$,
Ax6 $\varphi \wedge [a^*](\varphi \rightarrow [a]\varphi) \rightarrow [a^*]\varphi$,
Ax7 $[a](\varphi\&\varphi) \leftrightarrow [a]\varphi\&[a]\varphi$,
Ax8 $[a](\varphi \veebar \varphi) \leftrightarrow [a]\varphi \veebar [a]\varphi$.

and closed under the following rules of inference:

(MP) from φ and $\varphi \rightarrow \psi$ infer ψ,
(N) from φ infer $[a]\varphi$,
(I) from $\varphi \rightarrow [a]\varphi$ infer $\varphi \rightarrow [a^*]\varphi$.

3 Forensic dynamic MV_n-algebras

An algebra $A = (A, 0, \neg, \oplus)$ with one binary and one unary and one nullary operations is a *MV-algebras* if it satisfies: MV1. $(A, 0, \oplus)$ is an abelian monoid

MV2. $\neg\neg x = x$
MV2. $x \oplus \neg 0 = \neg 0$
MV3. $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus y)$.

We set $1 = \neg 0$ and $x \odot y = \neg(\neg x \oplus \neg y)$. We shall write ab for $a \odot b$ and a^n for $\underbrace{a \odot \cdots \odot a}_{n \text{ times}}$, for given $a, b \in A$. Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \text{ iff } \neg x \oplus y = 1.$$

Then $(A; \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra:

$$xy \leq x \wedge y \leq x \vee y \leq x \oplus y.$$

An MV-algebra $A = (A, 0, \neg, \oplus)$ is MV_n-algebra if it satisfies the identities: $x^n = x^{n-1}$, $n(x^k) = (k(x^{k-1}))^n$ for every integer $2 \leq k \leq n-2$ that does not divide $n-1$ [6].

Recall that MV_n-algebras are algebraic models of n-valued Łukasiewicz logic $Ł_n$.

The unit interval of real numbers $[0, 1]$ endowed with the following operations:

$$xx \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), \sim x = 1 - x,$$

becomes an MV-algebra [2]. From these operations are defined the lattice operations

$$x \vee y = \max(x, y) = (x \odot \sim y) \oplus y \quad and \quad x \wedge y = \min(x, y) = (\sim x \oplus y) \odot x.$$

It is well known that the MV-algebra $S = ([0, 1], \oplus, \odot, \sim, 0, 1)$ generate the variety $\mathbf{MV}$ of all MV-algebras, i.e. $\mathcal{V}(S) = \mathbf{MV}$. The algebra $S_n = (\{0, 1/n-1, ..., n-2/n-1, 1\}, \oplus, \odot, \sim, 0, 1)$ generates the subvariety $\mathbf{MV}_n$ $(1 < n < \omega)$, the algebras of which is called MV_n-algebras [6], i.e. $\mathcal{V}(S_n) = \mathbf{MV}_n$. Notice that $\mathbf{MV} = \mathcal{V}(\bigcup_{i=1}^{\infty} \mathbf{MV}_n)$.

The algebra $\mathcal{S} = ([0, 1], \odot, \Rightarrow, 0)$ (which is functionally equivalent to the MV-algebra defined above), where a binary operation $\odot$ called Łukasiewicz t-norm and defined as $x \odot y = max\{0, x + y - 1\}$, for all $x, y \in [0, 1]$; a binary operation $\Rightarrow$ called the residuum (of the t-norm $\odot$) and defined as $x \Rightarrow y = min\{1, 1 - x + y\}$, and $\sim x = x \Rightarrow 0 = 1 - x$, $x \oplus y = \sim (\sim x \odot \sim y) = min(1, x + y)$, for all $x, y \in [0, 1]$.

Firstly define regular algebras that are also named Kleene algebras. There exist several definitions of regular algebras. We use J.H. Conway's definition of regular algebras [3] to whom Kozen follows [8]. A *Kleene algebra* is a structure $(K, +, \cdot, ^*, 0, 1)$ such that $(K, +, 0)$ is a commutative monoid, $(K, \cdot, 1)$ is a monoid, and the following laws hold:

$$\begin{array}{ll} a + a = a, & a \cdot (a + b) = a \cdot a + a \cdot b, \\ a \cdot 0 = 0 \cdot a = 0, & (a + b) \cdot c = a \cdot c + b \cdot c, \\ 1 + a \cdot a^* = a^*, & b + a \cdot c \leq c \Rightarrow a^* \cdot b \leq c, \\ 1 + a^* \cdot a = a^*, & b + c \cdot a \leq c \Rightarrow b \cdot a^* \leq c, \end{array}$$

where $\leq$ is the partial order induced by $+$, that is, $a \leq b \Leftrightarrow a + b = b$.

For us it is interesting regular algebras represented by algebras of binary relations. Algebras of relations over a set X: $(2^{X \times X}, \cup, ;, ^*, \emptyset, Id)$, where $\cup$ is set-theoretic union, ; is relational composition, * is reflexive-transitive closure and Id is the identity relation. Notice that this algebra is a complete lattice with respect to $\cup$. In the sequel, following Pratt [11], we represent regular algebras as $(R, \cup, ;, ^*)$.

Forensic dynamic MV_n-algebra, $n \in Z^+$, combine MV_n-algebra $\mathcal{M} = (M, \oplus, \odot, \sim, 0, 1)$ and regular algebra $\mathcal{R} = (R, \cup, ;, ^*)$ into a single finitely axiomatized class $(\mathcal{M}, \mathcal{R}, \Diamond)$ resembling an R-module with scalar multiplication $\Diamond : R \times M \to M$. A forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ satisfies the following axioms: for any $x, y \in M$ and $a, b \in R$

1. $\mathcal{M}$ is MV_n-algebra.
2. $a0 = 0$.
3. $a(x \vee y) = ax \vee ay$.
4. $(a \cup b)x = ax \vee bx$.
5. $(ab)x = a(bx)$.
6. $a(x \oplus x) = ax \oplus ax$.
7. $a(x \odot x) = ax \odot ax$.
8. $x \vee aa^*x \leq a^*x \leq x \vee a^*(\sim x \wedge ax)$.

If in addition a dynamic MV_n-algebra satisfies the following condition

9. $x?y = x \wedge y$,

then it is called *test algebra*.

Notice that we may think $\langle a \rangle x$ as a function on M. The alternative notation ax is to suggest that we may think of a itself as a function, in spite of the fact that we may have $ax = bx$ for all $x \in M$ yet not have $a = b$.

In the following instead of a variable x sometimes we will use a propositional variable p. If $ap = bp$ for all p we call a and b *inseparable* and write $a \equiv b$, an equivalence relation which we shall later show to be a congruence relation on (forensic) dynamic algebras. We call *separable* any forensic dynamic algebra in which inseparability is the identity relation [8]. More precisely, forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$, $n \in Z^+$, is called *separable* iff $(\forall a_1, a_2 \in R)(\exists x \in M)(a_1 \neq a_2 \Rightarrow a_1x \neq a_2x)$. We let $\mathbf{SFD_nA}$ denote the class of separable forensic dynamic MV_n-algebras.

On R we define a quasiorder $\leq$: $a \leq b$ means that $ap \leq bp$ for all p. It follows that $\leq$ on R is reflexive and transitive but not antisymmetric, and so is a quasiorder. In a separable forensic dynamic MV_n-algebra it becomes a partial order.

Using the axioms 2, 3, 4 and 8, Pratt have proven in [11] that if $a \equiv b$ then $a^* \equiv b^*$ and hence $\equiv$ is a congruence relation on R. Moreover (a) if $a \leq b$ then $a^* \leq b^*$, (b) $a \leq a^*$, (c) $a^* = a^{**}$ [11].

Let us consider M as a lattice, and write aS for $\{as : s \in S\}$ for any $S \subset M$ and $a \in R$. We call a finitely additive (completely additive) if $a(\bigvee S) = \bigvee a(S)$ for any finite subset $S \subset M$ (for any subset $S \subset M$ for which $\bigvee S$ exists). Notice that the regular algebra operations $\cup, ;, ^*$ preserve finitely additivity (completely additivity), i.e. if a and b are finite (completely) additive, so are $a \cup b$, $a; b$, a^* [11].

Example 3.1. Full forensic dynamic MV_n-algebras. Given a complete MV_n-algebra $\mathcal{M} = (M, \oplus, \odot, \sim, 0, 1)$, let R be the set of all finitely (resp. completely) additive functions on M, with conditions $f(0) = 0$, $f(x \oplus x) = f(x) \oplus f(x)$ and $f(x \odot x) = f(x) \odot f(x)$, and let $\Diamond : R \times M \to M$ be application of elements of R to elements of M. We call it the full (completely full) forensic dynamic MV_n-algebra on M.

Example 3.2. Functional MV_n-algebra. Let W be non-empty set (of states) and

$$M_W = \{f : f \text{ is a function from } W \text{ } to \text{ } S_n\},$$

$n \in Z^+$, - the set of all functions, which is complete MV_n-algebra. More precisely, we have MV_n-algebra $(M_W, \oplus, \otimes, \sim, 0, 1)$, where $(f \circ g)(x) = f(x) \circ g(x)$, $\sim f(x) = f(\sim x)$ with conditions $f(x \circ x) = f(x) \circ f(x)$ where $\circ \in \{\oplus, \otimes\}$, and

$$R = \{r | r : M_W \to M_W \text{ is additive (completelly additive) functions}\}$$

with $r(f) = r \circ f$. Then the full functional MV_n-algebra on W is the completely full forensic dynamic MV_n-algebra on M_W.

Remark 3.3. Notice, that the full functional MV_n-algebra on W is separable. Indeed, recall that forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$, $n \in Z^+$, is called *separable* iff

$$(\forall a_1, a_2 \in R)(\exists x \in M)(a_1 \neq a_2 \Rightarrow a_1 x \neq a_2 x).$$

Then, if we take as element $x \in M$ the constant function 1, then if $a_1 \neq a_2$, then

$$a_1 1 = a_1 \circ 1 = a_1 \neq a_2 = a_2 \circ 1 = a_2 1.$$

4 Completeness Theorem

Recall that dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ is called separable iff

$$(\forall a_1, a_2 \in R)(\exists x \in M)(a_1 \neq a_2 \Rightarrow \Diamond(a_1, x) \neq \Diamond(a_2, x)).$$

In this case x is called a separator for the actions a_1 and a_2. $\mathbf{SFD_nA}$ denotes the class of all separable dynamic MV_n-algebras, and $\mathbf{V}_n$ denotes the variety generated by $\mathbf{SFD_nA}$, i.e. $\mathbf{V_n} = \mathcal{V}(\mathbf{SFD_nA})$.

The notion of heterogeneous algebra and products, subalgebras and homomorphisms of heterogeneous algebras can be found in [1]. A *subalgebra* $\mathcal{D}' = (\mathcal{M}', \mathcal{R}', \Diamond)$ of an algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ is a set of subsets $M' \subset M, R' \subset R$ closed under the corresponding operations, and $\Diamond(a', x') \in M'$ for any $a' \in R'$ and $x' \in M'$. A *homomorphism* $h : \mathcal{D} \to \mathcal{D}'$ is a pair (h_1, h_2) homomorphisms $h_1 : M \to M'$, $h_2 : R \to R'$, and $h(\Diamond(a, x)) = \Diamond(h_1(a), h_2(x))$. A *congruence* E on an algebra $\mathcal{D}$ is a pair of congruences (E_1, E_2) on M and R respectively, and if aE_1b and xE_2y, then $\Diamond(a, x)E_1\Diamond(b, y)$.

Let $\mathbf{D_n}$ be the variety of all forensic dynamic MV_n-algebra.

Let $\mathcal{F}(\mathbf{Var}, \mathbf{Inv})$ denote the absolutely free algebra (or term-algebra) with similarity $(2, 2, 1, 0, 0; 2, 2, 1)$ and generate by the set of variables and set of investigations. We can restrict the cardinality of the set of variables (say finite set of variable) and the cardinality of the set of ivestigations (say finite set of investigations). Then we will have finitely generated absolutely free algebra. Denote by $\mathcal{F}(\mathbf{Var}_f, \mathbf{Inv}_f)$ finitely generated absolutely free algebra.

Let $x, y..., a, b, ..., \alpha, \beta, ...$ range over the set of generators in $M, R, M \cup R$ respectively, and write M_0, R_0, D_0 for the respective generator sets. Let $\mathcal{F}_{\mathbf{V}_n}(M_0, R_0)$ denotes the free V_n-algebra (free algebra over $\mathbf{V}_n$) freely generated by the sets R_0 and M_0 as free generators of sorts MV_n-algebra and actions respectively [5]. We can represent $\mathcal{F}_{\mathbf{V}_n}(M_0, R_0)$ as $(\mathcal{F}_{\mathbf{MV}_n}(M_0), \mathcal{F}_{\mathbf{R}}(R_0), \Diamond)$.
Notice that $(\mathcal{F}_{\mathbf{MV}_n}(M_0), \mathcal{F}_{\mathbf{R}}(R_0), \Diamond)$ is a homomorphic image of the absolutely free term forensic dynamic MV_n-algebra. In other words $(\mathcal{F}_{\mathbf{MV}_n}(M_0), \mathcal{F}_{\mathbf{R}}(R_0), \Diamond)$ is a Lindenbaum algebra of the forensic dynamic Lukasiewicz logic on a finitely many generating sets.

According to well known Birkhoff's theorem we have

Theorem 4.1. *D_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ is isomorphic to a subdirect product of subdirectly irreducible D_n-algebras.*

According to this theorem $(\mathcal{F}_{\mathbf{MV}_n}(M_0), \mathcal{F}_{\mathbf{R}}(R_0), \Diamond)$ is represented as a subdirect product of subdirectly irreducible D_n-algebras where $\mathcal{F}_{\mathbf{MV}_n}(M_0)$ is a subdirect product of finite chain MV_n-algebras and $\mathcal{F}_{\mathbf{R}}(R_0)$ is a separable regular algebras. Notice that when M_0 is finite then $\mathcal{F}_{\mathbf{MV}_n}(M_0)$ is finite.

Taking into account that the variety of MV_n-algebras is locally finite and adapting Segerberg's technique of filtration (for modal logic) [14] for dynamic MV_n-algebras it holds

Theorem 4.2. *For a free forensic dynamic MV_n-algebra $\mathcal{F}_{\boldsymbol{D}_n}(M_0, R_0)$ and a finite subset M_g of $\mathcal{F}_{\boldsymbol{MV}_n}(M_0)$, there exists a forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ and a homomorphism $f : \mathcal{F}_{\boldsymbol{D}_n}(M_0, R_0) \to \mathcal{D}$ injective on M_g, with $f(\mathcal{F}_{\boldsymbol{D}_n}(M_0, R_0))$ finite and separable.*

Theorem 4.3. *Every finite separable forensic dynamic MV_n-algebra $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ is isomorphic to a (finite) functional MV_n-algebra.*

Proof. Let $\mathcal{D} = (\mathcal{M}, \mathcal{R}, \Diamond)$ be a finite separable forensic dynamic MV_n-algebra. Let (W, R, V) be the Kripke model such that:

i) W is the set of all additive functions $f : M \to S_n$;
ii) the binary relation R is defined on W by

$(u, v) \in R$ if for every formula $\varphi \in \mathcal{F}_{\mathbf{MV}_n}(M_0)$ and $a \in \mathbf{Inv}$

$$u([a]\varphi) = 1 \Rightarrow v(\varphi) = 1;$$

iii) the valuation map $V : W \times \mathbf{Var} \to S_n$ is defined by

$$V(u, p) = u(p).$$

By the fact that every finite MV_n-algebra is isomorphic to ta direct product $\prod_{i \in I} S_i$, where i divides n, and by separability, $\mathcal{D}$ is isomorphic to a subalgebra of the full (hence completely full by the finiteness of M) forensic dynamic MV_n-algebra, which is a functional MV_n-algebra by definition. □

From the theorems 1 - 3 we can conclude that the variety $\mathbf{V_n}$ coincides with $\mathbf{D_n}$.

Let $\theta(n) = (\theta(n)_1, \theta(n)_2$ be an equivalence relation on $\mathcal{F}(\mathbf{Var}_f, \mathbf{Inv}_f)$ defined as follows: $\alpha\theta_1\beta$ iff $\alpha \to \beta$ and $\beta \to \alpha$ are theorems of $FD\text{Ł}_n$ and $a\theta_2 b$ iff $ax = bx$ for all $x \in M$.

It holds

Theorem 4.4. $(\mathcal{F}(Var_f, \Pi_f)/\theta(n)$ *is forensic dynamic MV_n-algebra.*

Theorem 4.5. *(Completeness theorem) A formula φ of forensic dynamic logic $FD\text{Ł}_n$ is a tautology iff it is a theorem of the logic.*

Proof. It is obvious that if φ is a theorem, then φ is a tautology. Let us suppose that φ is not a theorem. Then $\varphi/\theta(n) \neq 1$ in the Lindenbaum algebra $\mathcal{F}(\mathbf{Var}_f, \mathbf{Inv}_f)/\theta(n)$ $(n \in \omega)$. $\mathcal{F}(\mathbf{Var}_f, \mathbf{Inv}_f)/\theta(n)$ is isomorphic to $\mathcal{F}_{\mathbf{V}_n}(M_0, R_0)$ for some finite M_0 and R_0. Then there exists a homomorphism $h : \mathcal{F}_{\mathbf{V}_n}(M_0, R_0) \to \mathcal{D}$ with injection on M_0, R_0 where $\mathcal{D}$ is finite and separable with $h(\varphi/\theta(n)) \neq 1$. So, φ is not a tautology. □

5 Kripke semantics

Formulas can be used to describe the properties that hold after the successful investigation. For example, the formula $[a \cup b]\varphi$ means that whenever investigations a or b is successfully finalized, a state is reached where φ holds, whereas the formula $\langle(a; b)^*\rangle\varphi$ means that there is a sequence of alternating investigations of a and b such that a state is reached where φ holds. Semantically speaking, formulas are interpreted by sets of states and investigations are interpreted by binary relations over states in a Kripke model. More precisely, the meaning of $FD\text{Ł}_n$ formulas and investigations are interpreted over Kripke models (KM) $\mathcal{K} = (W, R, V)$ where W is a nonempty set of worlds or states, R is a mapping from the set $\mathbf{Inv}$ of atomic investigations into binary relations on W (i.e. $R : \mathbf{Inv} \to r : W^2 \to \{0, 1\}$) and V is a mapping from the set $\mathbf{Var}$ of atomic formulas into S_n. Informally, the mapping R assigns to each atomic investigation $a \in \mathbf{Inv}$ some binary relation $R(a)$ on W with intended meaning $xR(a)y$ iff there exists an execution of a from x that leads to

y, whereas the mapping V assigns to each pair $(p, x) \in Var \times W$, where $p \in Var$ is an atomic formula and $x \in W$, some element $V(p, x) \in S_n$ with intended meaning $V(p, x) = 1$ iff p is true in x. Given our readings of $0, \neg\varphi, \varphi \veebar \psi$, $[a]\varphi, a; b, a \cup b, a^*$ and $\varphi?$, it is clear that R and V must be extended inductively as follows to supply the intended meanings for the complex investigations and formulas:

- $xR(a; b)y$ iff there exists a world z such that $xR(a)z$ and $zR(b)y$,
- $xR(a \cup b)y$ iff $xR(a)y$ or $xR(b)y$,
- $xR(a^*)y$ iff there exists a non-negative integer n and there exist worlds $z_0, \dots, z_n$ such that $z_0 = x$, $z_n = y$ and for all $k = 1, \dots, n$, $z_{k-1}R(a)z_k$,
- $xR(\varphi?)y$ iff $x = y$ and $V(\varphi, y) = 1$,
- $V(\perp) = 0$.
- $V(\neg\varphi, x) = 1 - V(\varphi, x)$,
- $V(\varphi \veebar \psi, x) = V(\varphi, x) \oplus V(\psi, x)$,
- $V(\varphi \vee \psi, x) = V(\varphi, x) \vee V(\psi, x)$,
- $V([a]\varphi, x) = \bigwedge\{V(\varphi, y) : xR(a)y\}$,
- $V(\langle a\rangle\varphi, x) = \bigvee\{V(\varphi, y) : xR(a)y\}$.

If $V(\varphi, x) = 1$ then we say that φ is satisfied at state x in $\mathcal{K}$, or "$\mathcal{K}, x$ *sat* φ".

Now consider a formula φ. We say that φ is *valid* in $\mathcal{K}$ or that $\mathcal{K}$ is a model of φ, or "$\mathcal{K} \vDash \varphi$", iff for all worlds x, $V(\varphi, x) = 1$. φ is said to be *valid*, or "$\vDash \varphi$", iff for all models $\mathcal{K}$, $\mathcal{K} \vDash \varphi$. We say that φ is *satisfiable* in $\mathcal{K}$ or that $\mathcal{K}$ *satisfies* φ, or "$\mathcal{K}$ *sat* φ", iff there exists a world x such that $V(\varphi, x) = 1$. φ is said to be *satisfiable*, or " *sat* φ", iff there exists a model $\mathcal{K}$ such that $\mathcal{K}$ sat φ. Interestingly, sat φ iff not $\vDash \neg\varphi$, $\vDash \varphi$ iff not sat $\neg\varphi$.

Some remarkable formulas of $FD\L_n$ are valid.

$\vDash [a; b]\varphi \leftrightarrow [a][b]\varphi$
$\vDash [a \cup b]\varphi \leftrightarrow [a]\varphi \vee [b]\varphi$
$\vDash [a^*]\varphi \leftrightarrow \varphi \wedge [a][a^*]\varphi$
$\vDash [\varphi?]\psi \leftrightarrow (\varphi \to \psi)$

Equivalently, we can write them under their dual form.

$\vDash \langle a; b\rangle\varphi \leftrightarrow \langle a\rangle\langle b\rangle\varphi$
$\vDash \langle a \cup b\rangle\varphi \leftrightarrow \langle a\rangle \wedge \langle b\rangle\varphi$
$\vDash \langle a^*\rangle\varphi \leftrightarrow \varphi \vee \langle a\rangle\langle a^*\rangle\varphi$
$\vDash \langle\varphi?\rangle\psi \leftrightarrow (\varphi \wedge \psi)$.

We define propositional forensic dynamic Łukasiewicz logic $FD\L_n$ as the set of all formulas that are valid in all Kripke models, i.e.

$$FD\L_n = \{\varphi : \vDash_{FD\L_n} \varphi\}.$$

Completeness theorem for classical and non-classical case with respect to Kripke models was proven by many authors. Adapting the existing methods for $FD\L_n$ it is easy to prove the following

Theorem 5.1. *(Completeness theorem) The following assertions are equivalent: for any formula φ*
i) φ is a theorem of FDL_n ($n \in Z^+$),
ii) φ is valid.

Proof. We give a sketch of the proof.

i) ⇒ ii). It follows from the immediate inspection, i.e. showing that every axiom Ax0 - Ax8 are valid and inference rules preserve validity. It is routine to check every axiom and inference rules. But we show validity of one of them, namely the axiom Ax0. Firstly, notice that the identity

$$\bigwedge_{i\in I}(x_i \oplus y_i) = \bigwedge_{i\in I}(x_i) \oplus \bigwedge_{i\in I}(y_i) \qquad (\#)$$

holds in the MV-algebra S and, hence, in the MV_n-algebra S_n. Let $\mathcal{K}_n = (W, R, V)$ be any Kripke model. Then $V([a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi), x) =$
$V(\neg[a](\neg\varphi\underline{\vee}\psi)\underline{\vee}(\neg[a]\varphi\underline{\vee}[a]\psi), x) =$
$\sim V([a](\neg\varphi\underline{\vee}\psi), x) \oplus (\sim V([a]\varphi, x) \oplus V([a]\psi), x)) =$
$\sim (\bigwedge_{y\in W}\{\sim V(\varphi, y) \oplus V(\psi, y) : xR_a y\}) \oplus (\sim \bigwedge_{y\in W}\{V(\varphi, y) : xR_a y\} \oplus \bigwedge_{y\in W}\{V(\psi, y) : xR_a y\})$. Using (#) we have
$\sim (\bigwedge_{y\in W}\{\sim V(\varphi, y) \oplus V(\psi, y) : xR_a y\}) = (\sim \bigwedge_{y\in W}\{V(\varphi, y) : xR_a y\} \oplus \bigwedge_{y\in W}\{V(\psi, y) : xR_a y\})$. So, $V([a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi), x) = 1$.

ii) ⇒ i). This part is the completeness theorem concerning Kripke models. The completeness theorem for the classical case was given by Segerberg [13], Parikh [10], Kozen and Parikh [9]. In the proof of the theorem, they mainly use the fact that the set of the subformulas of the formula is finite and by the Boolean combination on the given subformulas, we also get finite set (because of locally finiteness of Boolean algebras), and then use filtration method. Since we have locally finiteness of MV_n-algebras (which is an algebraic counterpart of n-valued Łukasiewicz logic) G. Hansoul and B. Teheux in [7] adapted the Segerberg's proof for (mono)modal n-valued Łukasiewicz logic where they have proved Kripke completeness of (mono)modal n-valued Łukasiewicz logic.

Using an abstract version of the modal logic technique of filtration, which is a Kripke structure setting is the process of dividing a Kripke model of a given formula φ by an equivalence relation on its worlds to yield a finite Kripke model of φ. Fischer and Ladner [4] showed that filtration could be made to work for propositional dynamic logic just as well as for modal logic. Prat [12] has extended their result that filtration does not depend on any special properties of Kripke structures but works for all dynamic logic. Adapting G. Hansoul and B. Teheux technique of filtration (for Łukasiewicz modal logic) [7] for (multimodal) dynamic propositional Łukasiewicz logic $FDŁ_n$ we arrive to the assertion $(ii) \Rightarrow (i)$. □

6 Application

We study logical system and their Kripke semantics (Kripke frames) for an application to the forensic system. In turn, the forensic system consists of special kind of investigations interacting between themselves, depending on the state of an environment, which afterward is predetermined by investigators behavior. So, their behavior depends on so far as finding facts (evidence) possess full information about the environment and presented facts.

Our basic aim is to give to the investigators some useful tools for diagnosis about a state of a forensic system having some initial data. These data represent some properties, which may estimate, that possess some parts of a forensic system, in particular some evidence being fundamental elements of the forensic system.

6.1 A fragment of a forensic system as a Kripke Frame

In this section, we try to represent some simple fragment of a forensic system by n-valued Kripke frame with the following interpretation in forensic models that is different, but similar.

Now we give a naive definition of forensic system FS. A forensic system FS is a set of investigations with some actions between them. Identifying some investigation with a possible world and an action between investigations with the relation between corresponding words we can represent a forensic system FS as a n-valued descriptive Kripke frame.

$FragmFS = (S, Q)$, where $S = \{Fact_1, ..., Fact_n, Inv_1, ..., Inv_m\}$, forms a fragment of a forensic system with communication between its members which is expressed by some reflexive and transitive binary relation Q pointed out in Fig. 1. In the sequel, we assume that the binary relation Q is reflexive and transitive.

Now we will give some representation of a fragment of a forensic system by Kripke frame. Let $\mathfrak{J} = (W, R)$ be n-valued Kripke frame, where $R \subset W \times W$ is a binary reflexive and transitive relation on finite set W (called *the accessibility relation between possible words* from W). By the representation of a forensic system $FragmImS = (S, Q)$ by Kripke frame $\mathfrak{J} = (W, R)$ we mean a bijective function $\varphi : S \to W$ such that $(t_1, t_2) \in Q \Rightarrow (\varphi(t_1), \varphi(t_2)) \in R$.

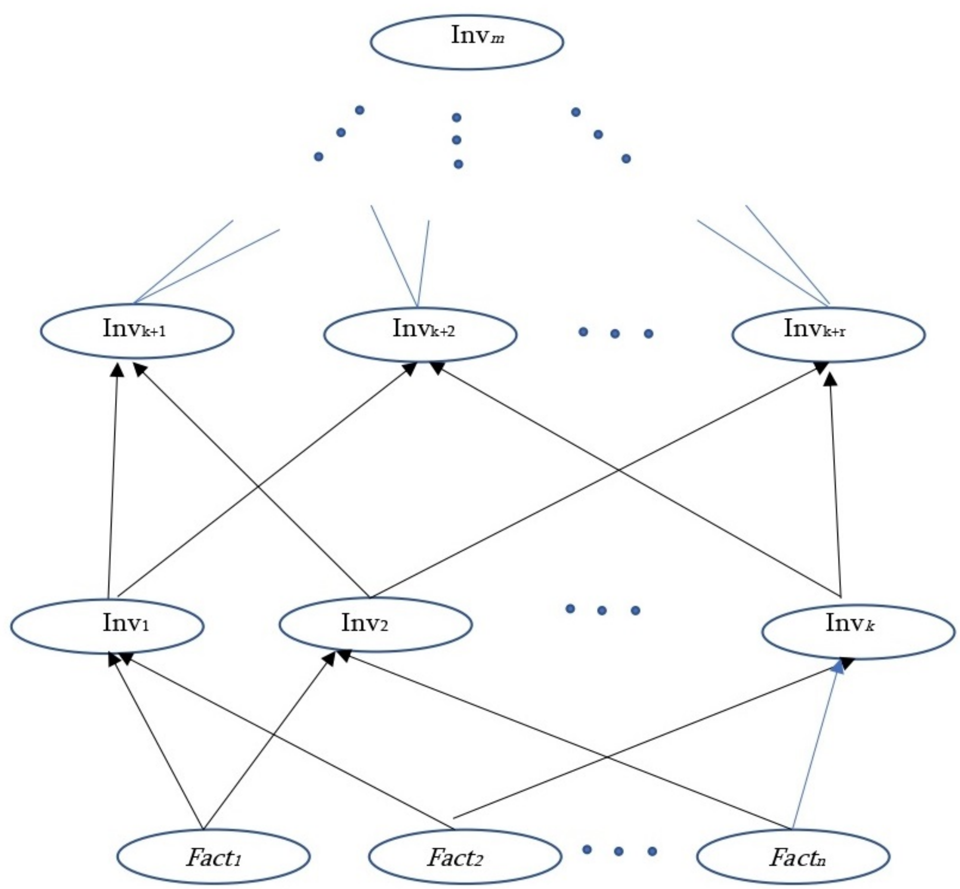

Fig. 1 Kripke frame

6.2 The modal aspects of forensic system

Given a n-valued Kripke frame $\mathfrak{J}$ which is a representation of a fragment of the forensic system, then we consider some a forensic system, represented by the Kripke frame $\mathfrak{J} = (W, R)$,where W is a finite set of forensic investigations, and let $\mathfrak{M} = (\mathfrak{J}, e)$ be n-valued Kripke model and $e : Var \times W \to S_n$. Representing a Kripke frame as a set of forensic investigations, in addition we can interpret a propositional variable $p \in Var$ as a sentence about the investigation $w \in W$. The value $e(p, w)$ expresses how much p fits a certain property of w.

We say that $w \in W$, where W is a finite set of investigations, is p-activated if $e(p, w) = 1$, it is not p-activated if $e(p, w) = 0$, it is p-activated in some degree $s \in S_n$ if $e(p, w) \in S_n - \{0, 1\}$. Note that for $w \in W$ there are finitely many ways to be p-activated for an investigation w. So, for evaluation e we have the set of points of W (i.e. the set of investigations) such that part of them are activated, part of them is not activated and part of them is activated to some degree.

A function $S : W \to S_n$ is named *a state function* (or simply *state*) if for every $w, w' \in W$ it holds

$$(w, w') \in R \Rightarrow (S(w) = 1 \Rightarrow S(w') = 1).$$

Let $e : Var \times W \to S_n$ be an evaluation. A formula φ defines a function $S^e_\varphi : W \to S_n$, such that $S^e_\varphi(w) = e(w, \varphi)$. We say that a formula φ is *labelled by the evaluation* e if S^e_φ is an state function and denote such kind of function by S^e_φ. The process of transformation of one state function S_1 $(= S^{e_1}_\varphi)$ to an another function S_2 $(= S^{e_2}_\varphi)$ will be named "$\varphi - activation$". So, for a formula φ a transferring of the state function $S^{e_1}_\varphi$ to the state function $S^{e_2}_\varphi$ is a φ-activation of points of W.

We described a forensic system as a Kripke Frame. It means that by Kripke frame we capture just the relational structure of a forensic system.

This representation of the forensic system neglects some information about the forensic system, that is some knowledge on the points w are not represented. So to recover such information we give the notion of *forensic system state function* (or simply *forensic state function* of a forensic system). This is done by a function S defined on all possible worlds to S_n. Of course S satisfies some suitable conditions, which are essentially compatibility conditions with respect to the relational structure of the forensic system. In this way we have a more faithful representation of the knowledge about the given forensic system. It is reasonable to think that to get the value $S(w)$ it is needed some intellectual work (maybe an experiment). We plan mathematically to study the set of all forensic states. Our aim is to help the investigators to have a formal and canonical way to explore the possible forensic state (function) of a forensic system. We have a variety of forensic state functions. Roughly speaking we have any allocation of the elements of S_n with any elements of W. But we need the allocations which are compatible with n-valued Kripke frame. So we single out such kinds of forensic state functions which are defined by some logical formulas, say φ, and an evaluation e is denoted as S^e_φ.

Since a forensic system, as defined in the paper, can be associated whit a logic which is complete with respect to certain Kripke frames, and since forensic system representation gives us a Kripke Frame, we use formulas of the logic of our Kripke Frame forensic system, to define some forensic states of the forensic system. Actually we use a formula φ and an evaluation e of φ, in the following way: $S^e_\varphi(w) = e(\varphi, w)$.

It is worth to note that a single formula φ essentially represents a set of forensic states (investigation), actually all such states are defined by S^e_φ when e varies in the set of all evaluations. In this way a given formula represents a collection of forensic states of the forensic system. It could be of interest to explore the possibility of checking whether given a collection of forensic states we can find a formula representing such a collection.

We defined the Activation function as a function defined on the set of all the forensic System States with value in the same set. This is a way to represent how changes the forensic information after, say an experiment, that produces new information about the forensic state values of all points w. To know facts about the function means to know facts about possible variations of the forensic state of the system, and to check whether these variations can be described by formulas.

Conflict of Interest: The authors declare that there are no conflict of interest.

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