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# Pure Ideals in Residuated Lattices

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**Abstract.** Ideals in MV algebras are, by definition, kernels of homomorphism. An ideal is the dual of a filter in some special logical algebras but not in non-regular residuated lattices. Ideals in residuated lattices are defined as natural generalizations of ideals in MV algebras. Spec(L), the spectrum of a residuated lattice L, is the set of all prime ideals of L, and it can be endowed with the spectral topology. The main scope of this paper is to characterize Spec(L), called the stable topology. In this paper, we introduce and investigate the notion of *pure* ideal in residuated lattices, and using these ideals we study the related spectral topologies.

Also, using the model of MV algebras, for a De Morgan residuated lattice L, we construct the *Belluce lattice* associated with L. This will provide information about the pure ideals and the prime ideals space of L. So, in this paper we generalize some results relative to MV algebras to the case of residuated lattices.

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## 1 Introduction

In fuzzy logic theory, residuated lattices play an important role because they provide an algebraic framework to fuzzy logic and fuzzy reasoning. From a logical point of view, various filters and ideals correspond to various sets of provable formulae. The notion of the ideal has been introduced in many algebraic structures such as lattices, rings of MV algebras. By definition, the ideals of MV algebras are kernels of homomorphisms. An ideal is the dual of a filter in some special logical algebras but not in non-regular residuated lattices. For terminology and theory of residuated Lattices we refer the reader to the papers (see [16], [18]).

For a residuated lattice, L, $\mathcal{P}(L)$, the set of all prime ideals of L, can be endowed with the spectral topology $\tau_L$ in the same manner as in the case of commutative rings of bounded distributive lattice.

For an ideal I of L, $V(I) = \{P \in Spec(L) : I \nsubseteq P\}$ is open in $(\mathcal{P}(L), \tau_L)$ and $\overline{V}(I) = \mathcal{P}(L) \backslash V(I) = \{P \in \mathcal{P}(L) : I \subseteq P\}$ is closed; Thus $V(I)$ is *stable under descent* and $\overline{V}(I)$ is stable under ascent. So, clopen sets are stable, that is, these are simultaneous stable under ascent and descent.

The characterization of open stable sets relies on the concept of *pure ideal* (see also, [7]) for commutative rings with the unit, (see [8]) for bounded distributive lattices, and (see [3], [6]) for MV algebras).

The scope of this paper is to introduce and investigate pure ideals in residuated lattices, using the model of MV algebras.

In Section 2 and Section 3 we recall basic results about residuated lattices and ideals in residuated lattices and we give new characterizations for prime and maximal ideals.

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In Section 4, we introduce the notion of *pure ideal*. Their properties and characterizations are obtained. We will use pure ideals in Section 6 to characterize the stable open sets relative to the spectral topology.

Using the model of MV algebras, (see [2]), in Section 5, for a De Morgan residuated lattice L, we construct the *Belluce lattice* [L] associated with L. The Belluce lattice will provide some insight about pure ideals and prime ideals space of L (see Theorem 5.8, Corollary 6.2, Corollary 6.5). The Belluce lattice [L] is a Boolean algebra iff L is a hyperarchimedean De Morgan residuated lattice, (see Theorem 5.4).

Section 6 contains topological results relative to the spectral topology $\tau_L$ and the stable topology $S_L$, coarser than the spectral one. For a De Morgan residuated lattice L, $\mathcal{P}(L)$, and Spec([L]) are homeomorphic, and (see Corollary 6.2) the stable topology $S_L$ coincides with the spectral topology $\tau_L$ iff L is a hyperarchimedean, (see Theorem 6.4, Corollary 6.5, Corollary 6.6, Corollary 6.7) study the connections between pure ideals of L and open stable subsets of $\mathcal{P}(L)$.

## 2 Preliminaries

A residuated lattice is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 1)$ of type $(2, 2, 2, 2, 0)$ satisfying the following axioms:

$(RL_1)$ $(L, \wedge, \vee)$ is a bounded lattice (the partial order is denoted by $\leq$);

$(RL_2)$ $(L, \odot, 1)$ is a commutative monoid;

$(RL_3)$ For every $x, y, z \in L$, $x \odot z \leq y$ iff $z \leq x \rightarrow y$ for any $x, y, z \in L$ (residuation).

A residuated lattice L is called an MTL algebra if $(x \rightarrow y) \vee (y \rightarrow x) = 1$ for every $x, y \in L$, (see [12], [13], [16]) and is called a *De Morgan residuated lattice* if $(x \wedge y)^* = x^* \vee y^*$, for every $x, y \in L$, (see [16], [18]). Examples of De Morgan residuated lattices are Boolean algebras, MV algebras, BL algebras, MTL algebras, Girard algebras.

MV algebras are particular cases of residuated lattices, (see [16]). A residuated lattice L is an MV algebras if it satisfies the additional condition:$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$, for every $x, y \in L$.

**Example 2.1.** (See [12]) Let $L = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$, and $a, b$ incomparable. L is a commutative residuated lattice with the following operations:

| $\rightarrow$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| a | b | 1 | b | 1 | 1 |
| b | a | a | 1 | 1 | 1 |
| c | 0 | a | b | 1 | 1 |
| 1 | 0 | a | b | c | 1 |

| $\odot$ | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | 0 | a | a |
| b | 0 | 0 | b | b | b |
| c | 0 | a | b | c | c |
| 1 | 0 | a | b | c | 1 |

**Example 2.2.** (See [12]) Let $L = \{0, b, c, d, 1\}$ with $0 < b, c < d < 1$ but $b, c$ are incomparable. L is a commutative residuated lattice with the following operations:

| $\rightarrow$ | 0 | b | c | d | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| b | d | 1 | d | 1 | 1 |
| c | d | d | 1 | 1 | 1 |
| d | d | d | d | 1 | 1 |
| 1 | 0 | b | c | d | 1 |

| $\odot$ | 0 | b | c | d | 1 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| b | 0 | 0 | 0 | 0 | b |
| c | 0 | 0 | 0 | 0 | c |
| d | 0 | 0 | 0 | 0 | d |
| 1 | 0 | b | c | d | 1 |

Let L be a residuated lattice. For $x \in L$ and $x \geq 0$ we denote $x^0 = 1, x^n = x^{n-1} \odot x$ for $n \geq 1, x^* = x \rightarrow 0$ and $x^{**} = (x^*)^*$.

Recall (see [1]) that an element $x \in L$ is called *complemented* if there is an element $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$; y is the complement of x.

If we denote by $B(L)$ the set of all complemented elements in the lattice $(L, \wedge, \vee, 0, 1)$, then $B(L)$ is a Boolean subalgebra of L, called *the Boolean center of* $L$ and $e \in B(L)$ iff $e \vee e^* = 1$, (see [16]).

For $x, y, z \in L$ we have the following rules of calculus, (see [14], [16], [18]):

$(c_1)$ $x \to 1 = 1$ and $1 \to x = x, x \to x = 1$;

$(c_2)$ $x \leq y$ iff $x \to y = 1$ and $x \leq y \to x, x \odot (x \to y) \leq y$;

$(c_3)$ If $x \leq y$ then $z \odot x \leq z \odot y, z \to x \leq z \to y, y \to z \leq x \to z, y^* \leq x^*$;

$(c_4)$ $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

$(c_5)$ $0^* = 1, 1^* = 0, x \odot x^* = 0, x \odot 0 = 0, x \leq (x^*)^*$;

$(c_6)$ $(x \vee y)^* = x^* \wedge y^*$ and $(x \wedge y)^* \geq x^* \vee y^*$;

$(c_7)$ $x \to y^* = y \to x^* = (x^*)^* \to y^* = (x \odot y)^*$;

$(c_8)$ $(x \to y)^{**} \leq x^{**} \to y^{**}, (x \odot y)^{**} = x^{**} \odot y^{**}$;

$(c_9)$ $x \vee y = 1$ implies $x \odot y = x \wedge y$ and $x^n \vee y^n = 1$, for every $n \geq 1$;

$(c_{10})$ for $x \geq 1, x^n \in B(L)$ iff $x \vee (x^n)^* = 1$.

In a residuated lattice L, for $x, y \in L$ we define $x \oplus y = x^* \to y$ and $x \boxplus y = (x^* \odot y^*)^* = x^* \to y^{**}$. We remark that $x \boxplus y = x \oplus y^{**}$ and for $x \in L$, we will use the notation $(n+1)x := nx \boxplus x$, for a natural number $n \geq 1$.

Let L be a commutative residuated lattice, for $x, y, z \in L$ and $m, n \geq 1$ we have the rules of calculus, (see [5] and [14]):

$(c_{11})$ $x, y \leq x \oplus y, (x \oplus y) \oplus z = x \oplus (y \oplus z)$;

$(c_{12})$ $x \boxplus y = y \boxplus x, (x \boxplus y) \boxplus z = x \boxplus (y \boxplus z)$;

$(c_{13})$ $x \wedge (y \boxplus z) \leq (x^{**} \wedge y^{**}) \boxplus (x^{**} \wedge z^{**})$ and $(mx) \wedge (ny) \leq (mn)(x^{**} \wedge y^{**})$.

**Lemma 2.3.** *If L is a De Morgan residuated lattices and $x, y, z \in L$, then*

$(c_{14})$ $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$*;*

**Proof.** To prove $(c_{14})$ we have to show that $(x \wedge y)^* \to z = (x^* \to z) \wedge (y^* \to z)$. To do this we prove that

$(i)$ $(x \wedge y)^* \to z \leq x^* \to z, y^* \to z$;

$(ii)$ If $t \leq x^* \to z, y^* \to z \Rightarrow t \leq (x \wedge y)^* \to z$.

We have $x \wedge y \leq x \Rightarrow x^* \leq (x \wedge y)^* \Rightarrow (x \wedge y)^* \to z \leq x^* \to z$ and similarly $(x \wedge y)^* \to z \leq x^* \to z$. Because L is a De Morgan residuated lattice, we have $x^* \leq t \to z, y^* \leq t \to z \Rightarrow (x \wedge y)^* = x^* \vee y^* \leq t \to z \Rightarrow (x \wedge y)^* \leq t \to z \Rightarrow t \leq (x \wedge y)^* \to z$. □

**Lemma 2.4.** *Let $x, y, z \in L$ and $n \geq 2$. Then:*

$(c_{15})$ $x \oplus (y \oplus z) = y \oplus (x \oplus z)$ *and* $1 \oplus x = x \oplus 1 = 1$ *and* $x \boxplus x^* = 1$*;*

($c_{16}$) *$x^* \odot y^* = (x \boxplus y)^*$ and $[(x^*)^n]^* = nx$;*

($c_{17}$) *If L is a De Morgan residuated lattice $x \wedge (y \oplus z) \leq (x \wedge z) \oplus (x \wedge z)$, $x \wedge y = x \wedge z = 0$ then $x \wedge (y \oplus z) = 0$.*

**Proof.** ($c_{15}$) $x \oplus (y \oplus z) = x^* \to (y^* \to z) = (y^* \to (x^* \to z)) = y \oplus (x \oplus z)$.

Also, $1 \oplus x = 1^* \to x = 0 \to x = 1$, $x \oplus 1 = x^* \to 1 = 1$ and $x \boxplus x^* = (x^* \odot x^{**})^* = 1$.

($c_{16}$) $x^* \odot y^* = (x^* \odot y^*)^{**} = (x^* \to y^{**})^* = (x \boxplus y)^*$. The proof that $[(x^*)^n]^* = nx$ for arbitrary n is a mathematical induction argument. $2x = x \boxplus x = x^* \to x^{**} = (x^* \odot x^*)^* = [(x^*)^2]^*$. If we suppose that $nx = [(x^*)^n]^*$, then $(n+1)x = x \boxplus (nx) = x^* \to (nx)^{**} = x^* \to [(x^*)^n]^* = [(x^*)^{n+1}]^*$.

($c_{17}$) From ($c_{14}$) we have $(x \wedge y) \oplus (x \wedge z) = [x \oplus (x \wedge z)] \wedge [y \oplus (x \wedge z)] = (x \oplus x) \wedge (x \oplus z) \wedge (y \oplus z) \wedge (y \oplus z) \geq x \wedge (y \oplus z)$ since by ($c_{11}$), $x \oplus x, x \oplus z, y \oplus x \geq x$. If $x \wedge y = x \wedge z = 0$, then $x \wedge (y \oplus z) \leq 0 \oplus 0 = 0^* \to 0 = 1 \to 0 = 0$, so $x \wedge (y \oplus z) = 0$. □

# 3 Ideals in residuated lattices

Let L be a residuated lattice. A nonempty subset I of a residuated lattice L will be called an ideal of L, (see [13], [14]) if it satisfies:

($I_1$) If $x \leq y$ and $y \in I$, then $x \in I$;

($I_2$) If $x, y \in I$, then $x \oplus y \in I$.

An ideal I called *proper* if $I \neq L$ (that is, $1 \notin I$). We denoted by $Id(L)$ the set of all ideals of L. If $I \in Id(L)$, then $0 \in I$ and $x \in I$ iff $x^{**} \in I$, (see [14]). Also, since $x, y \leq x \vee y \leq x \oplus y$, if $x, y \in I$ then $x \vee y \in I$, so I is a Lattice ideal.

**Remark 3.1.** *$I \in Id(L)$ iff it satisfies the conditions ($I_1$) and ($I_2'$): $x, y \in I$ implies $x \boxplus y \in I$. Indeed , if $I \in Id(L)$ then $x, y \in I$ implies $y^{**} \in I$, so, $x \oplus y^{**} = x \boxplus y \in I$. Conversely, if $I \subseteq L$ satisfies the conditions ($I_1$) and ($I_2'$), then $x \oplus y \leq x \boxplus y$, for every $x, y \in I$, so, $x \oplus y \in I$ and $I \in Id(L)$.*

Let L be a residuated lattice and $I \in Id(L)$. In (see [14]), on L is defined as a congruence relation $x \sim_I y$ iff $(x \to y)^*$, $(y \to x)^* \in I$. Moreover, $I = \{x \in L : x \sim_I 0\}$.

As an immediate consequence we have:

Let L be a residuated lattice. For $x \in L$ we denote by $x/I$ the congruence class of $x$ concerning to $\sim_I$ by $x/I$ and the quotient set $L/\sim_I$ by $L/I$. Since $\sim_I$ is a congruence on L, $L/I$ becomes a residuated lattice with the natural operations induced from those of L.

Clearly, in $L/I$, $\mathbf{0} = 0/I = \{x \in L : x \in I\}$, $\mathbf{1} = 1/I = \{x \in L : x^* \in I\}$ and for $x, y \in L$, $x/I \leq y/I$ iff $(x \to y)^* \in I$.

For a nonempty subset S of L, we denoted by $(S]$ the ideal of L generated by S and $x \in L$ we denoted by $(x] = (\{x\}]$.

Also, for $I \in Id(L)$ and $x \in L$ we denote by $I(x) = (I \cup \{x\}]$.

**Proposition 3.2.** *(See [5], [4]) Let L be a residuate lattice, $S \subseteq L$ a nonempty subset, $x, y \in L$ and $I \in Id(L)$. Then:*

*(i) $(S] = \{z \in L : z \leq s_1 \boxplus ... \boxplus s_n$, for some $n \geq 1$ and $s_1, ..., s_n \in S\}$ and $(x] = \{z \in L : z \leq nx$, for some $n \geq 1\}$;*

*(ii) $I(x) = \{z \in L : z \leq i \boxplus nx$ , for some $i \in L$ and $n \geq 0\}$and $I(x \wedge y) \subseteq I(x) \cap I(y) \subseteq I(x^{**} \wedge y^{**})$;*

*(iii) $(Id(L), \subseteq)$ is a complete Brouwerian lattice, where for $I_1, I_2 \in I_d(L), I_1 \wedge I_2 = I_1 \cap I_2$ and $I_1 \vee I_2 = (I_1 \cup I_2]$.*

**Remark 3.3.** *If $e \in B(L)$, then $(e] = \{z \in L : z \leq e\}$, since $e \boxplus e = e^* \to e^{**} = e^* \to e = e$, so $ne = e$, for every $n \geq 1$.*

In a residuated lattice L, the order of an element $x \in L$, denoted by $ord(x)$, is the smallest natural number n such that $x^n = 0$ and we write $ord(x) = n$. If no such n exists (that is, $x^n \neq 0$ for every $n \geq 1$) we say that the order of x is *infinite* and we write $ord(x) = \infty$.

A residuate lattice L is called *locally finite* if every non-unit element of L has finite order.

**Lemma 3.4.** *Let $L$ be a residuated lattice and $x \in L$. Then there is $I \in Id(L)$ proper such that $x \in I$ iff $ord(x^*) = \infty$.*

**Proof.** Let $I \in Id(L)$ proper ideal and $x \in I$ such that $ord(x^*) \neq \infty$. Then there is $n \geq 1$ such that $(x^*)^n = 0$ so, $[(x^*)^n]^* = 1$. From $(c_{16})$, $[(x^*)^n]^* = nx \in I$, thus, $1 \in I$, a contradiction so $ord(x^*) = \infty$.

Conversely, suppose that $ord(x^*) = \infty$. If $(x]$ is not proper then $1 \in (x]$, thus, $1 = nx$, so $0 = (nx)^*$, for some $n \geq 1$. Using $(c_{16})$, $(x^*)^n = 0$, so $ord(x^*) \neq \infty$, a contradiction. Thus, $(x]$ is proper. □

Using Lemma 3.4, we deduce that:

**Proposition 3.5.** *If $L$ is a residuated lattice and $x \in L$, then $(x]$ is proper iff $ord(x^*) = \infty$*

In a residuated lattice L, an ideal $P \in Id(L)$ is called *prime*, (see [15]) if $P \neq L$ and P is a prime element in $(Id(L), \subseteq)$, that is, if $I, J \in Id(L)$ and $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.

We denote by $\mathcal{P}(L)$ the set of prime of L. Since $(Id(L), \subseteq)$ is a distributive lattice, meet-irreductible and meet-prime elements coincide, so, $P \in \mathcal{P}(L)$ iff $[I, J \in Id(L)$ with $I \cap J = P$, implies $I = P$ or $J = P]$.

**Theorem 3.6.** *Let $L$ be a residuated lattice and $P \in Id(L)$. Then $P \in \mathcal{P}(L)$ iff $[x^{**} \wedge y^{**} \in P$ implies $x \in P$ or $y \in P]$.*

**Proof.** Let $P \in \mathcal{P}(L)$ and $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$. By Proposition 3.2, $P(x) \cap P(y) = P(x^{**} \wedge y^{**}) = P$. Since $P \in \mathcal{P}(L)$ we deduce that $P(x) = P$ or $P(y) = P$, that is, $x \in P$ or $y \in P$.

Conversely, let $I, J \in Id(L)$ such that $I \cap J \subseteq P$. If we suppose that $I \nsubseteq P$ and $J \nsubseteq P$, then there are $x \in I$ and $y \in J$ such that $x, y \notin P$. Then $x^{**} \in I, y^{**} \in J$ so $x^{**} \wedge y^{**} \in I \cap J \subseteq P$. By hypothesis, $x \in P$ or $y \in P$, a contradiction. □

**Theorem 3.7.** *Let $L$ be a residuated lattice and $P \in Id(L)$. We consider the following assertions:*

*(i)* $P \in \mathcal{P}(L)$;

*(ii)* *If $x \wedge y \in P$, then $x \in P$ or $y \in P$;*

*(iii)* *For every $x, y \in L, (x \to y)^* \in P$ or $(y \to x)^* \in P$;*

*(iv)* *$L/P$ is a chain.*

Then $(ii), (iii), (iv) \Rightarrow (i)$ but $(i) \nRightarrow (ii), (iii), (iv)$.

**Proof.** $(ii) \Rightarrow (iii)$. Let $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$. Since $x \wedge y \leq x^{**} \wedge y^{**}$ we deduce that $x \wedge y \in P$. From hypothesis, $x \in P$ or $y \in P$. Using Theorem 3.6, we conclude that $P \in \mathcal{P}(L)$.

$(iii) \Rightarrow (i)$. Let $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$ and we suppose that $(x \to y)^* \in P$.It follows that $(x \to y)^* \oplus (x^{**} \wedge y^{**}) = (x \to y)^{**} \to (x^{**} \wedge y^{**}) \in P$. From $(c_8)$, $(x \to y)^{**} \leq x^{**} \to y^{**}$, so $(x^{**} \to y^{**}) \to (x^{**} \wedge y^{**}) \leq (x \to y)^{**} \to (x^{**} \wedge y^{**})$. Since P is an ideal and $x^{**} \leq (x^{**} \to y^{**}) \to (x^{**} \wedge y^{**})$, we deduce that $x^{**} \in P$, thus $x \in P$. Similarly, if $(y \to x)^* \in P$ we obtain $y \in P$, so $P \in \mathcal{P}(L)$.

$(iv) \Rightarrow (i)$. Suppose that $L/P$ is a chain and let $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$. Then $x^{**}/P \wedge y^{**}/P = 0/P$, so $x^{**}/P = 0/P$ or $y^{**}/P = 0/P$. We deduce that, $x^{**} \in P$ or $y^{**} \in P$, so, $x \in P$ or $y \in P$. Hence $P \in \mathcal{P}(L)$.

$(i) \nRightarrow (ii), (iii), (iv)$. If we consider the residuated lattice $L = \{0, b, c, d, 1\}$ from Example 2.2, it is easy to see that $0^{**} = 0, b^{**} = c^{**} = d^{**} = d$ and $1^{**} = 1$. Obviously, $P = \{0\} \in \mathcal{P}(L)$ because if $x^{**} \wedge y^{**} = 0$ implies $x = 0$ or $y = 0$. But $b \wedge c = 0 \in P$ and $b, c \notin P$, thus $(i) \nRightarrow (ii)$.

Also, $(i) \nRightarrow (iii)$ since $(b \to c)^* = (c \to b)^* = d^* = d \notin P$

Also, for $b/P = \{x \in L : (b \to x)^* = (x \to b)^* = 0\} = \{x \in L : b \to x = x \to b = 1\} = \{b\}$ and $c/P = \{x \in L : (c \to x)^* = (x \to c)^* = 0\} = \{x \in L : c \to x = x \to c = 1\} = \{c\}$. But $\{b\} \nsubseteq \{c\}$ and $\{c\} \nsubseteq \{b\}$, so, $L/P$ is not a chain, thus, $(i) \nRightarrow (iv)$. □

If L is a De Morgan residuated lattice then $P \in \mathcal{P}(L)$ iff $[x \wedge y \in P$ implies $x \in P$ or $y \in P]$, (see [11]).

**Corollary 3.8.** *Let L be an MTL algebra and $P \in Id(L)$. Then the following conditions are equivalent:*

*(i) $P \in \mathcal{P}(L)$;*

*(ii) If $x \wedge y \in P$, then $x \in P$ or $y \in P$;*

*(iii) For every $x, y \in L, (x \to y)^* \in P$ or $(y \to x)^* \in P$;*

*(iv) $L/P$ is a chain;*

*(v) For $x, y \in L$, if $x \wedge y = 0$, then $x \in P$ or $y \in P$;*

*(vi) For every $x, y \in L, x \odot y^* \in P$ or $x^* \odot y \in P$.*

**Proof.** $(i) \Rightarrow (ii)$. (See [11]).

$(ii) \Rightarrow (iii)$. From $(x \to y) \vee (y \to x) = 1$, for every $x, y \in L$, we deduce that $(x \to y)^* \wedge (y \to x)^* = 0 \in P$. Thus, $(x \to y)^* \in P$ or $(y \to x)^* \in P$.

$(iii) \Rightarrow (i)$ From Theorem 3.7.

$(iv) \Rightarrow (ii)$. If $L/P$ is a chain and $x \wedge y \in P$ then $x/P \wedge y/P = 0/P$, so $x/P = 0/P$ or $y/P = 0/P$, that is, $x \in P$ or $y \in P$.

$(ii) \Rightarrow (v)$. Obviously, $x \wedge y = 0 \in P$, so $x \in P$ or $y \in P$.

$(v) \Rightarrow (iv)$. Let $x/P, y/P \in L/P$; since $(x \to y)^* \wedge (y \to x)^* = 0 \in P$, we deduce that $(x \to y)^* \in P$ or $(y \to x)^* \in P$, so $x/P \leq y/P$ or $y/P \leq x/P$.

$(i) \Rightarrow (iv)$. Since $(x \odot y^*)^{**} \wedge (x^* \odot y)^{**} = (y^* \to x^*)^* \wedge (x^* \to y^*) = [(y^* \to x^*) \vee (x^* \to y^*)]^* = 1^* = 0 \in P$, we deduced that $x \odot y^* \in P$ or $x^* \odot y \in P$.

$(vi) \Rightarrow (i)$. Suppose that $x^{**} \wedge y^{**} \in P$ and $x \odot y^* \in P$. It follows that $(x \odot y^*) \oplus (x^{**} \wedge y^{**}) \in P$. From $(c_{14})$,$(x \odot y^*) \oplus (x^{**} \wedge y^{**}) = [(x \odot y^*) \oplus x^{**}] \wedge [(x \odot y^*) \oplus y^{**}] \geq x \wedge x = x$, since $(x \odot y^*) \oplus x^{**} = (x \odot y^*)^* \to x^{**} \geq x^{**} \geq x$ and $(x \odot y^*) \oplus y^{**} = (x \odot y^*)^* \to y^{**} = (x \to y^{**}) \to y^{**} \geq x$. We conclude that $x \in P$, so, $P \in \mathcal{P}(L)$.

Similarly, if $x^* \odot y \in P$, we obtain that $y \in P$, so, P is a prime ideal of L. □

In general, in a residuated lattice L , if $P \in \mathcal{P}(L)$ and I is a proper ideal such that $P \subseteq I$, then I is not prime. Also, the set of proper ideals including a prime ideal is not a chain, (see [5]).

**Theorem 3.9.** *If L is an MTL algebra then:*

*(i) Every proper ideal of L that contains a prime ideal is prime;*

*(ii) For every prime ideal P of L, the set $\Im_P = \{I \in Id(L) : P \subseteq I$ and $I \neq L\}$ is totally ordered by inclusion.*

**Proof.** $(i)$. Let $P \in \mathcal{P}(L)$ and I a proper ideal of L such that $P \subseteq I$ and $x, y \in L$. From Corollary 3.8, $(vi)$, $x \odot y^* \in P$ or $y \odot x^* \in P$. Since $P \subseteq I$, we obtain $x \odot y^* \in I$ or $y \odot x^* \in I$, so $I \in \mathcal{P}(L)$.

$(ii)$. Let $I_1, I_2 \in \mathfrak{I}$ and suppose that $I_1 \nsubseteq I_2$ and $I_2 \nsubseteq I_1$. Then, there are $x_1, x_2 \in L$ such that $x_1 \in I_1 \backslash I_2$ and $x_2 \in I_2 \backslash I_1$. Since P is prime, $x_1 \odot x_2^* \in P \subseteq I_2$ or $x_2 \odot x_1^* \in P \subseteq I_1$. We deduce that $x_2 \oplus (x_1 \odot x_2^*) = x_2^* \to (x_1 \oplus x_2^*) \in I_2$ or $x_1 \oplus (x_1^* \odot x_2) = x_1^* \to (x_1^* \odot x_2^*) \in I_2$ or $x_1 \oplus (x_1^* \odot x_2) = x_1^* \to (x_1^* \odot x_2) \in I_1$. But $x_1 \leq x_2 \oplus (x_1 \odot x_2^*)$ and $x_2 \leq x_1 \oplus (x_1^* \odot x_2)$, so $x_1 \in I_2$ or $x_2 \in I_1$, a contradiction.
□

**Remark 3.10.** *(i) In a residuated lattice L, if $(P_i)_{i \in I} \subseteq \mathcal{P}(L)$ is a totally ordered family of prime ideals of L then $P = \cap_{i \in I} P_i \in Spec(L)$ and $Q = \vee_{i \in I} P_i \in Spec(L)$. Indeed, let $x, y \in L$ such that $x^{**} \wedge y^{**} \in P$, if by contrary $x \notin P$ and $y \notin P$ then there are $i_1, i_2 \in I$ such that $x \notin P_{i_1}$ and $y \notin P_{i_2}$. Since $P_{i_1}, P_{i_2}$ are prime ideals and $x^{**} \wedge y^{**} \in P_{i_1}, P_{i_2}$ then $x \in P_{i_2}$ and $y \in P_{i_1}$. Since the family $(P_i)_{i \in I}$ is totally ordered, then $P_{i_1} \subseteq P_{i_2}$ or $P_{i_2} \subseteq P_{i_1}$. If $P_{i_1} \subseteq P_{i_2}$ then $y \in P_{i_2}$, a contradiction. Similarly, if $P_{i_2} \subseteq P_{i_1}$. It follows that $x \in P$ or $y \in P$, that is, $P \in \mathcal{P}(L)$. Also, we remark that $Q = \cup_{i \in I} P_i$ and the proof for $Q \in \mathcal{P}(L)$ is obvious.*

*(ii) In general, an intersection of prime ideals in a residuated lattice is not necessary a prime ideal. For example, if we consider the residuated lattice L from Example 2.1, then $Id(L) = \{\{0\}, \{0, a\}, \{0, b\}, L\}$ and $\mathcal{P}(L) = \{\{0, a\}, \{0, b\}\}$. $\{0\} = \{0, a\} \cap \{0, b\} \notin \mathcal{P}(L), a^{**} \wedge b^{**} = 0$ but $a, b \neq 0$.*

**Theorem 3.11.** (Prime ideal theorem , see [5]) *Let L be a residuated lattice. If $I \in Id(L)$ and F is a filter of the lattice $(L, \wedge, \vee, 0, 1)$ such that $I \cap F = \emptyset$, then there is $P \in \mathcal{P}(L)$ such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Obviously, in a residuated lattice, any proper ideal of L can be extended to a prime ideal.

**Corollary 3.12.** *Let L be a residuated lattice and $x \in L$. Then $ord(x^*) < \infty$ iff $x \notin P$ for every $P \in \mathcal{P}(L)$.*

**Proof.** Suppose that $ord(x^*) < \infty$ and there exists $P \in \mathcal{P}(L)$ such that $x \in P$. Thus, there is $n \geq 1$ such that $(x^*)^n = 0$. Hence $1 = [(x^*)^n]^* = nx \in P$, so $P = L$, a contradiction. Conversely, we suppose that $x \notin P$ for every $P \in \mathcal{P}(L)$ and $ord(x^*) = \infty$. By Proposition 3.5 and Theorem 3.11, $(x]$ is proper so, there is $P \in \mathcal{P}(L)$ such that $(x] \subseteq P$, hence $x \in P$, is a contradiction. □

As immediate consequences of Theorem 3.11 we have:

**Corollary 3.13.** *If L is a residuated lattice then $\cap\{P \in \mathcal{P}(L)\} = \{0\}$ and for every $I \in Id(L), I = \cap\{P \in \mathcal{P}(L) : I \subseteq P\}$.*

**Proof.** If $x \neq 0$ there is a prime ideal $P \in \mathcal{P}(L)$ such that $x \notin P$, so $x \notin \cap\{P \in \mathcal{P}(L)\}$. □

**Proposition 3.14.** *Let L be a residuated Lattice, $L_1 \subseteq L$ a subalgebra of L and $P_1 \in \mathcal{P}(L_1)$. Then there exists $P \in \mathcal{P}(L)$ such that $P_1 = P \cap L_1$.*

**Proof.** Let I be the ideal generated by $P_1$ in L. Then $I = \{x \in L : x \leq x_1 \boxplus ... \boxplus x_n$, for some $x_1, ..., x_n \in P\}$. Then $I \cap (L_1 \backslash P_1) = \emptyset$. Indeed, if there is $i \in I \cap (L_1 \backslash P_1)$, then $i \in I, i \in L_1$ and $i \notin P_1$. From $i \in I$, there exists $p \in P_1$ such that $i \leq p$, hence $i \in P_1$, is a contradiction.

Clearly, $0 \notin L_1 \backslash P_1$ and $1 \in L_1 \backslash P_1$. Let $x, y \in L_1 \backslash P_1$. Then $x, y \notin P_1$ so $x \wedge y \notin P_1$ (since $P_1$ is prime in $L_1$). Thus, $x \wedge y \in L_1 \backslash P_1$, hence $L_1 \backslash P_1$ is a $\wedge -$ *closed* subset of L. By Theorem 3.11, there exists $P \in \mathcal{P}(L)$ such that $I \subseteq P$ and $P \cap (L_1 \backslash P_1) = \emptyset$, hence $P \cap L_1 \subseteq P_1$. Then $P_1 \subseteq I \cap L_1 \subseteq P \cap L_1 \subseteq P_1$, so $P_1 = P \cap L_1$.
□

We recall that an ideal M of a residuated lattice L is called *maximal*, (see [5], [14]), if it is proper and is not contained in any other proper ideal of L, i.e., for every ideal $I \neq L$, if $M \subseteq I$, then $M = I$.

We denote by $\mathcal{M}(L)$ the set of maximal ideals of L. Obviously, $\mathcal{M}(L) \subseteq \mathcal{P}(L)$.

Also, if M is a proper ideal of a residuated lattice L, then $M \in \mathcal{M}(L)$ iff for every $x \in L, x \notin M$ iff $(nx)^* \in M$, for some $n \geq 1$, (see [5], [15]).

**Theorem 3.15.** *Let $L$ be a residuated lattice and $M \in Id(L)$ be a proper ideal. Then $M \in \mathcal{M}(L)$ iff $L/M$ is locally finite.*

**Proof.** Suppose that $M \in \mathcal{M}(L)$ and let $x/M \neq 1/M$. Then $x^* \notin M$, so there is a natural number $n \geq 1$ such that $(nx^*)^* = [(x^{**})^n]^{**} \in M$. Since $M \in Id(L)$, $(x^{**})^n \in M$, so $x^n \in M$. We deduce that $x^n/M = (x/M)^n = 0/M$, so, $L/M$ is locally finite.

Conversely, let $I \in Id(L)$, $I \neq M$ be an ideal of L such that $M \subset I$. Then there is $x \in I \backslash M$, so, $x^*/M \neq 1/M$ (since if we suppose that $x^*/M = 1/M$, thus $x^{**} \in M$, so $x \in M$). But $L/M$ is locally finite, thus $(x^*/M)^n = 0/M$, for some $n \geq 1$. We conclude that $(x^*)^n \in M \subset I$. Since I is an ideal and $x \in I$, then $nx = [(x^*)^n]^* \in I$, so $(x^*)^n \oplus [(x^*)^n]^* = [(x^*)^n]^* \to [(x^*)^n]^* = 1 \in I$. Thus $I = L$ and $M \in \mathcal{M}(L)$.
□

As an immediate consequence of Zorn's lemma, every proper ideal of L can be extended to a maximal ideal.

**Theorem 3.16.** *Every prime ideal of an MTL algebra L is contained in a unique maximal ideal of L.*

**Proof.** For $P \in \mathcal{P}(L)$, the set $\mathfrak{I}_P = \{I \in Id(L) : P \subseteq I \text{ and } I \neq L\}$ is totally ordered by inclusion, from Theorem 3.9. Therefore, $\overline{P} = \cup_{I \in \mathfrak{I}_P}$ is proper, since $1 \notin \overline{P}$, so $\overline{P}$ is the only maximal ideal containing P. □

We recall that a residuated lattice L is called *local* if it has a unique maximal ideal (see [16]).

**Proposition 3.17.** *Let $L$ be a residuated lattice and $\mathfrak{I} = \{x \in L : ord(x^*) = \infty\}$. The following assertions are equivalent:*

*(i)* $\mathfrak{I} \in Id(L)$;

*(ii)* $(\mathfrak{I}]$ *is a proper ideal of L;*

*(iii)* $L$ *is local;*

*(iv)* $\mathcal{M}(L) = \{\mathfrak{I}\}$.

**Proof.** $(i) \Rightarrow (ii)$. Suppose $\mathfrak{I} \in Id(L)$ implies $(\mathfrak{I}] = \mathfrak{I} \neq L$ since $1 \notin \mathfrak{I}$.

$(ii) \Rightarrow (i)$. Obviously, $0 \in \mathfrak{I}$. Let $x, y \in L$ such that $x \leq y$ and $y \in \mathfrak{I}$. Then $ord(y^*) = \infty$. Since $y^* \leq x^*$ we deduce that $ord(x^*) = \infty$, thus, $x \in \mathfrak{I}$. Let now, $x, y \in \mathfrak{I}$. Since $\mathfrak{I} \subseteq (\mathfrak{I}]$ we have $x, y \in (\mathfrak{I}]$. If we suppose by contrary that $x \boxplus y \notin \mathfrak{I}$, then there is $n \geq 1$ such that $[(x \boxplus y)^*]^n = 0$. But $[(x \boxplus y)^*]^n = (x^* \odot y^*)^n = (x^*)^n \odot (y^*)^n = 0$. Thus, $1 = [(x^*)^n \odot (y^*)^n]^* = [(x^*)^n]^{**} \to [(y^*)^n]^* = (nx)^* \to (ny) = (nx) \oplus (ny)$, a contradiction since $(\mathfrak{I}]$ is proper.

We conclude that $\mathfrak{I} \in Id(L)$.

$(iv) \Rightarrow (iii)$. Clearly.

$(i) \Rightarrow (iv)$. To prove that $\mathfrak{I}$ is maximal, let $x \in L$ such that $x \notin \mathfrak{I}$. Then $(x^*)^n = 0$ for some $n \geq 1$. Thus, $(nx)^* = [(x^*)^n]^{**} = 0^{**} = 0 \in \mathfrak{I}$, so $\mathfrak{I} \in Max(L)$. To prove that $\mathfrak{I}$ is the unique maximal ideal of L, we consider $I_1 \in Id(L)$ such that $I_1 \neq L$. If by contrary, $I_1 \nsubseteq \mathfrak{I}$, then there is $x \in I_1$ such that $x \notin \mathfrak{I}$. Then $(x^*)^n = 0$ for some $n \geq 1$, hence $1 = [(x^*)^n]^* = nx \in I_1$ and $I_1 = L$, a contradiction. Therefore $\mathfrak{I}$ contains all the proper ideals of L, thus, $\mathfrak{I}$ is the unique maximal ideal of L.

$(iii) \Rightarrow (iv)$ and $(i)$. Let M be the unique maximal ideal of L. Since Proposition 3.5 every element $x \in \mathfrak{I}$ generates a proper ideal $(x]$ which can be extended to a maximal ideal $M_x$, we obtain $M = M_x$, so for every $x \in \mathfrak{I}, x \in M$ hence $\mathfrak{I} \subseteq M$. Since M is proper, from Lemma 3.4, $M \subseteq \mathfrak{I}$, hence $M = \mathfrak{I}$. □

**Theorem 3.18.** *In a local residuated lattice L, for every $x \in L, ord(x) < \infty$ or $ord(x^*) < \infty$.*

**Proof.** Suppose that there exists $x \in L$ such that $x^n > 0$ and $(x^*)^n > 0$ for every $n \geq 1$. Thus, $(x^{**})^n > 0$ for every $n \geq 1$. Then $x, x^* \in (\mathfrak{I}]$ so $x \boxplus x^* = 1 \in (\mathfrak{I}]$, so, $(\mathfrak{I}] = L$ in contradiction with Proposition 3.17.
□

## 4 Pure ideals in residuated lattices

Let L be a residuated lattice. For $x \in L$ we denote $x^{\perp} = \{y \in L : x \wedge y = 0\}$.

**Lemma 4.1.** *Let $L$ be a De Morgan residuated lattice and $x, y \in L$, $e \in B(L)$. Then:*

*(i)* $x^{\perp} \in Id(L)$ *and* $x \leq y$ *implies* $y^{\perp} \subseteq x^{\perp}$*;*

*(ii)* $x^{\perp} = L$ *iff* $x = 0$*;*

*(iii)* $x^{\perp} \cap y^{\perp} = (x \oplus y)^{\perp} = (x \vee y)^{\perp}$ *and* $e^{\perp} = (e^*]$.

*(iv)* $x^{\perp} \cap y^{\perp} = (x \boxplus y)^{\perp}$.

**Proof.**$(i)$ Let $t, z \in L$ such that $t \leq z$ and $z \in x^{\perp}$. Then $x \wedge z = 0$. Since $x \wedge t \leq x \wedge z = 0$, we deduce that $t \in x^{\perp}$. Also, if $t, z \in x^{\perp}$, then $x \wedge z = x \wedge y = 0$. Using $(c_{17})$, $x \wedge (t \oplus z) = 0$, so $t \oplus z \in x^{\perp}$ and $x^{\perp} \in Id(L)$. Now, suppose that $x \leq y$ and let $z \in y^{\perp}$. Then $z \wedge x \leq z \wedge y = 0$, so $z \wedge x = 0$, thus , $z \in x^{\perp}$.

$(ii)$ $x^{\perp} = L$ iff $1 \in x^{\perp}$ iff $1 \wedge x = 0$ iff $x = 0$.

$(iii)$. From $x, y \leq x \oplus y$, we deduce that $x, y \leq x \vee y \leq x \oplus y$. Using $(i)$, $(x \oplus y)^{\perp} \subseteq (x \vee y)^{\perp} \subseteq x^{\perp} \cap y^{\perp}$. Now $z \in (x \oplus y)^{\perp}$. Then $x \wedge z = y \wedge z = 0$. Using $(c_{17})$, $z \wedge (x \oplus y) = 0$, so, $z \in (x \oplus y)^{\perp}$ and $x^{\perp} \cap y^{\perp} \subseteq (x \oplus y)^{\perp}$ and we have obtained the equalities.

Finally, for $e \in B(L)$, since $e \wedge e^* = 0$ we deduce that $e^* \in e^{\perp}$ so, $(e^*] \subseteq e^{\perp}$. Let $x \in e^{\perp}$. Then $x \wedge e = 0$. Since $e^* \in B(L), x \wedge e^* = x \odot (x \rightarrow e^*) = x \odot (x \odot e)^* = x \odot 0^* = x \odot 1 = x$, so $x \leq e^*$, that is, $x \in (e^*]$, thus, $e^{\perp} = (e^*]$.

$(iv)$ From $x, y \leq x \boxplus y$ we deduce $(x \boxplus y)^{\perp} \subseteq x^{\perp} \cap y^{\perp}$. Now we consider $z \in x^{\perp} \cap y^{\perp}$. Then $x \wedge z = y \wedge z = 0$ From $(c_{13})$, $z \wedge (x \boxplus y) \leq (z^{**} \wedge x^{**}) \boxplus (z^{**} \wedge y^{**}) = (z \wedge x)^{**} \boxplus (z \wedge y)^{**} = 0 \boxplus 0 = 0^* \rightarrow 0^{**} = 1 \rightarrow 0 = 0$. We deduce that $z \in (x \boxplus y)^{\perp}$, thus, $x^{\perp} \cap y^{\perp} = (x \boxplus y)^{\perp}$. $\square$

For a residuated lattice L and $I \in Id(L)$ we denote by $\sigma(I) = \{x \in L$: there are $i \in I$ and $y \in x^{\perp}$ such that $i \oplus y = 1\}$. For MV-algebras, (see [6]).

Also, for a distributive lattice $(\mathcal{L}, \wedge, \vee, 0, 1)$ we denote by $Id(\mathcal{L})$ the set of ideals of L, $Spec(\mathcal{L})$ the set of prime ideals and by $Max(\mathcal{L})$ the set of maximal ideals of $\mathcal{L}$. About notations involving lattices and their spectral topologies, (see [8]).

We recall, (see [8], [9]), that if L is a distributive lattice $\mathcal{L}$, if $I \in Id(L)$, then $\sigma(I) = \{x \in L :$ there are $i \in I$ and $y \in x^{\perp}$ such that $i \vee y = 1\} \in Id(\mathcal{L})$ and $\sigma(I) \subseteq I$. Moreover, an ideal $I \in Id(\mathcal{L})$ is called *pure* if $\sigma(I) = I$, (see [8], [9]).

We denote by $Pure(\mathcal{L})$ the set of pure ideal of $\mathcal{L}$.

**Remark 4.2.** *In a residuated lattice L, if $I \in Id(L)$, then $\sigma(I) = I'$ where $I' = \{x \in L$: there are $i \in I$ and $y \in x^{\perp}$ such that $i \boxplus y = 1\}$. Obviously, $\sigma(I) \subseteq I'$ since $i \oplus y \leq i \boxplus y$. Conversely, let $x \in I'$. Then there are $i \in I$ and $y \in x^{\perp}$ such that $1 = i \boxplus y = i \oplus y^{**}$. Since $x^{\perp} \in Id(L)$ and $y \in x^{\perp}$ we deduce that $y^{**} \in x^{\perp}$, so $x \in \sigma(I)$ and $I' \subseteq \sigma(I)$.*

**Theorem 4.3.** *Let $L$ be a De Morgan residuated lattice and $I, J \in Id(L)$. Then*

*(i)* $\sigma(I) \in Id(L)$ *and* $\sigma(I) \subseteq I$*;*

*(ii)* $I \subseteq J$ *implies* $\sigma(I) \subseteq \sigma(J)$*;*

*(iii)* $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$ *and* $\sigma(I) \vee \sigma(J)) \subseteq \sigma(I \vee J)$.

*(iv)* $\sigma(I) \neq \{0\}$ *then there is* $i \in I$ *such that* $ord(i^{**}) = \infty$.

**Proof.** $(i)$. Let $x_1, x_2 \in L, x_1 \leq x_2$ and $x_2 \in \sigma(I)$, then there are $i \in I$ and $y \in x_2^{\perp}$ such that $i \oplus y = 1$.

Since $x_2^{\perp} \subseteq x_1^{\perp}$ so $y \in x_1^{\perp}$. We deduce that $x_1 \in \sigma(I)$.

For $x_1, x_2 \in \sigma(I)$, there are $i_1, i_2 \in I$ and $y_1 \in x_1^{\perp}, y_2 \in x_2^{\perp}$ such that $i_1 \oplus y_1 = i_2 \oplus y_2 = 1$. Denoting $i = i_1 \oplus i_2 \in I$ and $y = y_1 \wedge y_2$, we have $y \wedge (x_1 \oplus x_2) \leq (y \wedge x_1) \oplus (y \wedge x_2) = 0 \oplus 0 = 0$, so $y \in (x_1 \oplus x_2)^{\perp}$.

Also, $i \oplus y = (i_1 \oplus i_2) \oplus (y_1 \wedge y_2) = i_1 \oplus [(i_2 \oplus y_1) \wedge (i_2 \oplus y_2)] = i_1 \oplus [(i_2 \oplus y_1) \wedge 1] = i_1 \oplus (i_2 \oplus y_1) = i_2 \oplus (i_1 \oplus y_1) = i_2 \oplus 1 = 1$, so $x_1 \oplus x_2 \in \sigma(I)$, that is $\sigma(I) \in Id(L)$.

To prove that $\sigma(I) \subseteq I$, let $x \in \sigma(I)$. Then there are $i \in I$ and $y \in x^{\perp}$ such that $i \oplus y = 1$. We have $i^{**} = i \oplus 0 = i \oplus (x \wedge y) = (i \oplus x) \wedge (i \oplus y) = (i \oplus x) \wedge 1 = i \oplus x$. Hence $x \leq i^{**}$ , so $x \in I$and $\sigma(I) \subseteq I$.

$(ii)$ Obviously.

$(iii)$. By $(ii)$ $\sigma(I \cap J) \subseteq \sigma(I) \cap \sigma(J)$. Let $x \in \sigma(I) \cap \sigma(J)$. Then there are $i \in I, j \in J, y_1, y_2 \in x^{\perp}$ such that $i \oplus y_1 = j \oplus y_2 = 1$. Since $x^{\perp}, I, J$ are ideals we deduce that $y = y_1 \oplus y_2 \in x^{\perp}$ and $k = i \wedge j \in I \cap J$. Then $k \oplus y = (i \wedge j) \oplus y = (i \oplus y) \wedge (j \oplus y) = [i \oplus (y_1 \oplus y_2)] \wedge [j \oplus (y_1 \oplus y_2 \wedge)] = [(i \oplus y_1) \oplus y_2] \wedge [y_1 \oplus (j \oplus y_2)] = (1 \oplus y_2) \wedge (y_1 \oplus 1) = 1 \wedge 1 = 1$. We deduce that $x \in \sigma(I \cap J)$, so $\sigma(I) \cap \sigma(J) \subseteq \sigma(I \cap J)$. Hence $\sigma(I \cap J) = \sigma(I) \cap \sigma(J)$. From $(ii)$, we obtain $\sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$.

$(iv)$. For $x \in \sigma(I), x \neq 0$, there are $i \in I$ and $y \in x^{\perp}$ such that $i \oplus y = 1$. Then $i^* \rightarrow y = 1$, so $i^* \leq y$ and $(y^*)^n \leq (i^{**})$, for every $n \geq 1$. Obviously, if we prove that $ord(y^*) = \infty$, then $ord(i^{**}) = \infty$. From $x \wedge y = 0$ we deduce that $x^* \vee y^* = 1$, so, from $(c_9)$, $(x^*)^n \vee (y^*)^n = 1$, for every $n \geq 1$. If suppose by contrary that $(y^*)^n = 0$ for some $n \geq 1$, then $(x^*)^n = 1$, so, $x^* = 1$ and $x^{**} = 0$. Thus, $x = 0$, a contradiction. □

**Corollary 4.4.** *If L is a local De Morgan residuated lattices and $I \in Id(L)$ is proper,then $\sigma(I) = \{0\}$.*

**Proof.** Suppose $\sigma(I) \neq \{0\}$. From Theorem 4.3 (iv), there is $i \in I$ such that $ord(i^{**}) = \infty$. Since L is local, by Theorem 3.18, $ord(i^*) < \infty$, so, $(i^*)^n = 0$ for some $n \geq 1$. Thus, $1 = [(i^*)^n]^* = ni \in I$, so $I = L$, a contradiction. □

**Definition 4.5.** *An ideal $I$ of a residuated lattice $L$ is called pure in $L$ if $\sigma(I) = I$.*

For a residuated lattice L, we denote by $Pure(L)$ the set of pure ideals of L.

**Remark 4.6.** *For a residuated lattice $L$,*

**(i)** *$\{0\}, L \in Pure(L)$. Indeed, since $\{0\} \subseteq \sigma(\{0\}) \subseteq \{0\}$ we deduce that $\sigma(\{0\}) = \{0\}$. Also, since for every $x \in L$ there are $1 \in L$ and $0 \in x^{\perp}$ such that $1 \oplus 0 = 1$ we deduce that $x \in \sigma(L)$, so $\sigma(L) = L$.*

*(ii) If $I, J \in Pure(L)$, then $I \cap J$ and $I \vee J \in Pure(L)$ . Indeed, $\sigma(I) = I$ and $\sigma(J) = J$, so by Theorem 4.3, $\sigma(I \cap J) = \sigma(I) \cap \sigma(J) = I \cap J$, hence $I \cap J$ is a pure ideal in L. Also, we deduce that $I \vee J = \sigma(I) \vee \sigma(J) \subseteq \sigma(I \vee J)$, so, $\sigma(I \vee J) = I \vee J$, hence $I \vee J$ is pure in L.*

By Corollary 4.4 we deduce that:

**Corollary 4.7.** *If L is a local MTL algebra, then the unique pure ideals in L are $\{0\}$ and L.*

**Example 4.8.** If we consider the residuated lattice $L = \{0, a, b, c, 1\}$ from Example 2.1 then $0^{\perp} = L, a^{\perp} = \{0, b\}, b^{\perp} = \{0, a\}$ and $1^{\perp} = c^{\perp} = \{0\}$. It is easy to prove that every ideal of L is a pure ideal, so $Pure = Id(L)$.

# 5 The Belluce lattice associated with a De Morgan residuated lattice

In this section, we consider L a De Morgan residuated lattice L.

On L we define the relation $\equiv (mod\mathcal{P}(L))$ on L by $x \equiv y(mod\mathcal{P}(L))$ iff for every $P \in \mathcal{P}(L), x \in P$ iff $y \in P$. Thus, $x \equiv y(mod\mathcal{P}(L))$ iff no prime $P \in \mathcal{P}(L)$ can separate x and y.

**Lemma 5.1.** *$\equiv (mod\mathcal{P}(L))$ is an equivalence relation compatible with $\wedge$ and $\vee$.*

**Proof.** Obviously, $\equiv (mod\mathcal{P}(L))$ is an equivalence relation on L. Let $x, y, z, t \in L$ such that $x \equiv y(mod\mathcal{S})$ and $z \equiv t(mod\mathcal{S})$. Also, let $P \in \mathcal{S}$ such that $x \vee z \in P$. Since $x, z \leq x \vee z$ then $x, z \in P$ , $y, t \in P$ and $y \vee t \in P$. Then $y \oplus t \in P$. But P is an ideal and $y \vee t \leq y \oplus t$, so $y \vee t \in P$. Suppose now $x \wedge z \in P$, since P is prime then $x \in P$ or $z \in P$. Thus $y \in P$ or $t \in P$. In either case $y \wedge t \in P$. So, $\equiv (mod\mathcal{P}(L))$ is compatible with $\vee$ and $\wedge$. □

For every $x \in L$ we denote by $[x]$ the equivalence class of x and by $[L]_{\mathcal{S}}$ the set of these equivalence classes.

In this case, we denote $[L]_{\mathcal{S}}$ by $[L]$. On $[L]$ we define $[x] \wedge [y] = [x \wedge y], [x \vee y] = [x] \vee [y], \mathbf{0} = [0] = \cap\{P : P \in \mathcal{P}(L)\} = \{0\}$ and $\mathbf{1} = [1] = \{x \in L : x \notin P$, for every $P \in \mathcal{P}(L)\}$. Also, we define $[x] \leq [y]$ iff $[x] \wedge [y] = [x]$ iff $[x] \vee [y] = [y]$. Obviously, the relation $\leq$ is well defined and $([L], \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice.

Using the model of MV algebra, (see [2], [3]), $[L]$ will called *Belluce lattice associated with L.*

**Lemma 5.2.** *Let $x, y \in L$ then:*

*(i)* $x \leq y$ *implies* $[x] \leq [y]$*;*

*(ii)* $[x] = \mathbf{0}$ *iff* $x = 0$ *and* $[x] = \mathbf{1}$ *iff* $ord(x^*) < \infty$*;*

*(iii)* $[x \vee y] = [x \oplus y] = [x \boxplus y]$*, so* $[nx] = [x]$*, for every* $n \geq 1$*.*

**Proof.** $(i)$. $x \leq y$ implies $x \wedge y = x$, so, $[x] = [x \wedge y] = [x] \wedge [y]$. Thus, $[x] \leq [y]$. $(ii)$ $x = 0$, implies $[x] = \mathbf{0}$. Conversely, let $x \in L$ such that $[x] = \mathbf{0}$, then $x \in \cap\{P : P \in \mathcal{P}(L)\} = \{0\}$, since $0 \in P$ for every $P \in \mathcal{P}(L)$. Thus, $[x] = \mathbf{0}$ iff $x = 0$.

Now, let $x \in L$ such that $ord(x^*) < \infty$. Then there exists $n \geq 1$ such that $(x^*)^n = 0$, so, $(x^*)^n \in P$ for every $P \in \mathcal{P}(L)$. Hence $x \notin P$ for every $P \in \mathcal{P}(L)$, since if we suppose that there is $P \in \mathcal{P}(L)$ such that $x \in P$, then $(nx) \boxplus (x^*)^n \in P$. But $(nx) \boxplus (x^*)^n = [(x^*)^n]^* \boxplus (x^*)^n = [(x^*)^n]^{**} \rightarrow [(x^*)^n]^{**} = 1 \in P$, a contradiction. Hence $[x] = \mathbf{1}$. Conversely, suppose that $[x] = \mathbf{1}$ but $ord(x^*) = \infty$. Then, using Proposition 3.5 $(x]$ is proper there is $P \in \mathcal{P}(L)$ such that $(x] \subseteq P$. Thus $x \in P$, a contradiction. We conclude that $[x] = \mathbf{1}$ iff $ord(x^*) < \infty$.

$(iii)$. Let $P \in \mathcal{P}(L)$ if $x \vee y \in P$, then $x, y \in P$, so $x \boxplus y \in P$. Conversely, since $x \vee y \leq x \oplus y \leq x \boxplus y \in P$ if $x \boxplus y \in P$ then $x \vee y \in P$. Using $(i)$, $[x \vee y] = [x \oplus y] = [x \boxplus y]$. Obviously, $[nx] = [x]$, for every $n \geq 1$ since $[x \boxplus y] = [x \vee y]$. □

**Theorem 5.3.** $([L], \wedge, \vee, \mathbf{0}, \mathbf{1})$ *is a distributive lattice.*

**Proof.** For $x, y, z \in L$, we have $[x] \vee ([y] \vee [z]) = ([x] \vee [y]) \wedge ([x] \vee [z])$ iff $[x \vee (y \wedge z)] = [(x \vee y) \wedge (x \vee z)]$. To prove this equality, let $P \in \mathcal{P}(L)$ such that $x \vee (y \wedge z) \in P$. Since $P \in \mathcal{P}(L)$, we have $x, y \in P$ or $x, z \in P$. If $x, y \in P$ then $(x \vee y) \wedge (x \vee z) \in P$ and similarly, if $x, z \in P$. Conversely, if $(x \vee y) \wedge (x \vee z) \in P$, then $x \vee y \in P$ or $x \vee z \in P$. We deduce that $x \in P, y \in P$ or $x \in P, z \in P$. Hence $x, y \wedge z \in P$, so $x \vee (y \wedge z) \in P$. We deduce that $[L]$ is distributive bounded lattice. □

As in case of MV algebra, (see [3]), for residuated lattice L, an element $x \in L$ is called *archimedean* if there is $n \geq 1$ such that $nx \in B(L)$. The residuated lattice L is called *hyperarchimedean* if all its elements are archimedean.

Remembering that a De Morgan residuated lattice L is hyperarchimedean iff $\mathcal{P}(L) = \mathcal{M}(L)$, (see [9]), we have:

**Theorem 5.4.** *Let L be a De Morgan residuated lattice. Then $[L]$ is a Boolean algebra iff L is hyperarchimedean.*

**Proof.** If $[L]$ is a Boolean algebra, then for every $x \in L$, there is $y \in L$ such that $[x] \vee [y] = \mathbf{1}$ and $[x] \wedge [y] = \mathbf{0}$. From $[x] \vee [y] = \mathbf{1}$ we deduce that $[x \vee y] = \mathbf{1}$, so by Theorem 5.3, $ord(x \vee y)^* = ord(x^* \wedge y^*) < \infty$, hence there is $n \geq 1$ such that $(x^* \wedge y^*)^n = 0$. Since $[x] \wedge [y] = [0]$ then $x \wedge y = 0$, hence $x^* \vee y^* = 1$. From $(c_9)$, $(x^*)^n \vee (y^*)^n = 1$. Also, $(x^*)^n \wedge (y^*)^n = (x^*)^n \odot (y^*)^n = (x^* \odot y^*)^n \leq (x^* \wedge y^*)^n = 0$, hence $(x^*)^n \in B(L)$, so $[(x^*)^n]^* = nx \in B(L)$ and L is hyperarchimedean.

Conversely, suppose that L is hyperarchimedean. From Theorem 5.3, $[L]$ is a bounded distributive lattice and for every $x \in L$ there is $n \geq 1$ such that $nx \in B(L)$ i.e., $(nx) \vee (nx)^* = 1$ and $(nx) \wedge (nx)^* = 0$. Then $[x] \vee [(nx)^*] = [1] = \mathbf{1}$ and $[x] \wedge [(nx)^*] = [0] = \mathbf{0}$, so, $[L]$ is a Boolean algebra.
□

For $I \in Id(L)$ and $J \in Id([L])$, we denote $I^* = \{[x] : x \in I\}$ and $J_* = \cup\{[x] : [x] \in J\}$.

**Proposition 5.5.** *(i) If $I \in Id(L)$, then $I^* \in Id(L)$; Moreover, if $P \in \mathcal{P}(L)$, then $P^* \in Spec([L])$;*

*(ii) If $J \in Id([L])$, then $J_* \in Id(L)$; Moreover, if $Q \in Spec([L])$, then $Q_* \in \mathcal{P}(L)$;*

*(iii) If $I_1, I_2 \in Id(L)$ and $I_2 \in \mathcal{P}(L)$, then $I_1 \subseteq I_2$ iff $I_1^* \subseteq I_2^*$;*

*(iv) If $J_1, J_2 \in Id([L])$, then $J_1 \subseteq J_2$ iff $(J_1)_* \subseteq (J_2)_*$.*

**Proof.** $(i)$ Let $x, y \in L$ such that $[x] \leq [y]$ and $[y] \in I^*$. Thus, there is $y_1 \in I$ such that $[y] = [y_1]$. Then $[x] = [x] \wedge [y] = [x] \wedge [y_1] \in I^*$, since $y_1 \in I$ and $x \wedge y_1 \leq y_1$. If $[x], [y] \in I^*$, then there are $x_1, y_1 \in I$ such that $[x] = [x_1]$ and $[y] = [y_1]$. Hence $x_1 \vee y_1 \in I$ and $[x] \vee [y] = [x_1] \vee [y_1] = [x_1 \vee y_1] \in I^*$, so $I^* \in Id([L])$.

Also, if $P \in \mathcal{P}(L)$, then $P \neq L$, we deduce $P^* \neq [L]$. If by contrary, $P^* = [L]$ then $\mathbf{1} \in P^*$, so $1 \in P$ and $P = L$, a contradiction. Let $x, y \in L$ such that $[x] \wedge [y] \in P^*$. Then $[x \wedge y] \in P^*$, so $x \wedge y \in P$. Since $P \in \mathcal{P}(L), x \in P$ or $y \in P$. We deduce that $[x] \in P^*$ or $[y] \in P^*$, that is $P^* \in Spec([L])$.

$(ii)$. Let $x, y \in L$ such that $x \leq y$ and $y \in J_*$(hence $[y] \in J$). Then by Lemma 5.2, $(i), [x] \leq [y]$ and since $[y] \in J$ then $[x] \in J$, so $x \in J_*$. If $x, y \in J_*$ then $[x], [y] \in J$ so $[x] \vee [y] = [x \vee y] \in J$. Since $[x \vee y] = [x \oplus y]$, we obtain that $[x \oplus y] \in J$, so $x \oplus y \in J_*$ and $J_* \in Id(L)$. Also, for $Q \in Spec([L])$, if $Q_* = L$, then $1 \in Q_*$, so, $1 \in [x]$. Thus, $[1] = [x] \in Q$, so $Q = [L]$, a contradiction. Let $x, y \in L$ such that $x \wedge y \in Q_*$. Then $[x \wedge y] = [x] \wedge [y] \in Q$. Since $Q \in Spec([L]), [x] \in Q$ or $[y] \in Q$, so $x \in Q_*$ or $y \in Q_*$. Thus, $Q_* \in \mathcal{P}(L)$.

$(iii)$ Suppose that $I_1 \subseteq I_2$ and we consider $x \in I_1$ such that $[x] \in I_1^*$; then $x \in I_2$, so $[x] \in I_2^*$ that is , $I_1^* \subseteq I_2^*$. Suppose now that $I_1^* \subseteq I_2^*$ and let $x \in I_1$. Then $[x] \in I_1^* \subseteq I_2^*$ so $[x] \in I_2^*$. Then there is $y \in I_2$ such that $[x] = [y]$. Since $I_2 \in \mathcal{P}(L)$ and $y \in I_2$ we deduce that $x \in I_2$, so $I_1 \subseteq I_2$.

$(iv)$ Suppose $J_1 \subseteq J_2$ and let $x \in (J_1)_*$. Thus, $[x] \in J_1$. Then $[x] \in J_2$ so $x \in (J_2)_*$. We deduce $(J_1)_* \subseteq (J_2)_*$. Conversely, suppose $(J_1)_* \subseteq (J_2)_*$ and let $[x] \in J_1$. Then $x \in (J_1)_* \subseteq (J_2)_*$, thus $x \in (J_2)_*$. Hence $[x] \in J_2$, so $J_1 \subseteq J_2$. □

The following results hold:

**Proposition 5.6.** *Let $I \in Id(L), J \in Id([L])$ and $x \in L$. Then*

*(i) $x \in \sigma(I)$ implies $[x] \in \sigma(I^*)$;*

*(ii) If $[x] \in \sigma(I^*)$, then there exists $z \in [x]$ such that $z \in \sigma(I)$;*

*(iii) $[x] \in \sigma(J)$ iff $x \in \sigma(J_*)$;*

*(iv) $(\sigma(I))^* = \sigma(I^*)$ and $(\sigma(J))_* = \sigma(J_*)$.*

**Proof.** $(i)$. $x \in \sigma(I) \subseteq I$ implies $x \in I$, so $[x] \in I^*$. From $x \in \sigma(I)$ there are $i \in I$ and $y \in x^{\perp}$ such that $i \boxplus y = 1$. Hence $[1] = [i \boxplus y] = [i \vee y] = [i] \vee [y]$ and $[x] \wedge [y] = [x \wedge y] = [0]$. Since $[i] \in I^*$ and $[y] \in [x]^{\perp}$ we deduce that $[x] \in \sigma(I^*)$.

$(ii)$. For $[x] \in \sigma(I^*) \subseteq I^*$ there is $z \in [x] \cap I$ such that $[x] = [z]$.

Since $[L]$ is a distributive lattice and $[x] \in \sigma(I^*)$ there are $[i] \in I^*, [y] \in [x]^\perp$ such that $[i] \vee [y] = [1]$ and $[x] \wedge [y] = [0]$. Thus, $\mathbf{0} = [z] \wedge [y] = [z \wedge y]$ so, $z \wedge y = 0$. We conclude that $y \in z^\perp$.

Since $[1] = [i] \vee [y] = [i \vee y] = [i \boxplus y]$, we deduce that $i \boxplus y \notin P$ for every $P \in \mathcal{P}(L)$. Using Corollary 3.12, $ord((i \boxplus y)^*) < \infty$ so there is $n \geq 1$ such that $[(i \boxplus y)^*]^n = 0$. Since $n, [i] = [ni] \in I^*$ we deduce that there is $t \in [ni] \cap I$ such that $[t] = [ni]$.

To prove that $ord([(ny) \boxplus t]^*) < \infty$, we show that $(ny) \boxplus t \notin P$ for every $P \in \mathcal{P}(L)$. If $(ny) \boxplus t \in Q$ for some $Q \in \mathcal{P}(L)$ then $ny, t \in Q$. Since $t \in Q$ we deduce that $ni \in Q$, so $(ni) \boxplus (ny) = n(i \boxplus y) = n \cdot 1 = 1 \in Q$, a contradiction.

Then there is a natural number $m$ such that $ord([(ny) \boxplus t]^*) = m$, so, $1 = \{[(ny \boxplus t)^*]^m\}^* = m[(ny) \boxplus t] = (mny) \boxplus (mt)$, with $mt \in I$. Since $y \in z^\perp$ and $z^\perp \in Id(L)$, we deduce that $mny \in z^\perp$. Hence $z \in \sigma(I)$.

$(iii)$ First, suppose $[x] \in \sigma(J) \subseteq J$. Then $[x] \in J$ and $x \in J_*$. From $[x] \in \sigma(J)$ there are $[j] \in J$ and $[y] \in [x]^\perp$ such that $[j] \vee [y] = [1]$. Thus $[1] = [j \vee y] = [j \boxplus y]$, so $j \boxplus y \notin P$ for every $P \in \mathcal{P}(L)$, that is, $ord((j \boxplus y)^*) < \infty$. Then $[(j \boxplus y)^*]^n = 0$ for some $n \geq 1$, so $1 = \{[(j \boxplus y)^*]^n\}^* = n(j \boxplus y) = (nj) \boxplus (ny)$. Also, from $[y] \in [x]^\perp$ we deduce that $[0] = [x] \wedge [y] = [x \wedge y]$, so $x \wedge y = 0$. Since $j \in J_*, y \in x^\perp$ and $J_*, x^\perp \in Id(L)$. We obtain that $nj \in J_*, ny \in x^\perp$, so $x \in \sigma(J_*)$. Conversely, let $x \in L$ such that $x \in \sigma(J_*) \subseteq J_*$. Then $x \in J_*$ and $[x] \in J$. Moreover there are $j \in J_*, y \in x^\perp$ such that $j \boxplus y = 1$. We have that $[j] \vee [y] = [j \vee y] = [j \boxplus y] = [1]$ and $[y] \in [x]^\perp$, since $x \wedge y = 0$ implies $[x] \wedge [y] = [0]$. Hence, $[x] \in \sigma(J)$.

$(iv)$ Let $[x] \in (\sigma(I))^*$. Then $[x] = [x_1]$ with $x_1 \in \sigma(I)$. From Proposition 5.6, $(i)$, $[x_1] \in \sigma(I^*)$, so $(\sigma(I))^* \subseteq \sigma(I^*)$. Conversely, let $x \in L$ such that $[x] \in \sigma(I^*)$. By Proposition 5.6, $(ii)$, there exists $z \in [x]$ such that $z \in \sigma(I)$. We deduce that $[z] \in (\sigma(I))^*$. But $z \in [x]$ so $[z] = [x]$. Then $[x] \in (\sigma(I))^*$, so $\sigma(I^*) \subseteq (\sigma(I))^*$. Thus, $(\sigma(I))^* = \sigma(I^*)$.

Finally, $x \in (\sigma(J))_*$, then $[x] \in \sigma(J)$, so $x \in \sigma(J_*)$ and $(\sigma(J))_* \subseteq \sigma(J_*)$. Conversely, if $x \in \sigma(J_*)$ then $[x] \in \sigma(J)$. Implies $x \in (\sigma(J))_*$ so $\sigma(J_*) \subseteq (\sigma(J))_*$. We conclude that $(\sigma(J))_* = \sigma(J_*)$. □

**Theorem 5.7.** *$(i)$ If $I \in Id(L)$, then $(I^*)_* = I$;*

*$(ii)$ If $J \in Id([L])$, then $(J_*)^* = J$;*

**(***$(iii)$ If $M \in Max(L)$, then $M^* \in Max([L])$.*

**Proof.** $(i)$. Clearly, $I \subseteq (I^*)_*$. Let $x \in (I^*)_*$. Then $x \in \cup\{[y] : [y] \in I^*\}$, so there exists $y_0 \in I$ such that $x \in [y_0]$. Since $I = \cap\{P \in Spec(L) : I \subseteq P\}$ so for every $P \in \mathcal{P}(L)$ such that $I \subseteq P$ we deduce $y_0 \in P$ so $x \in P$. Thus, $(I^*)_* \subseteq \cap\{P \in \mathcal{P}(L) : I \subseteq P\} = I$, so $(I^*)_* \subseteq I$. Hence $(I^*)_* = I$.

$(ii)$. For $x \in L, [x] \in (J_*)^*$ iff $[x] \in J$, so, $(J_*)^* = J$.

$(iii)$. Obviously, $M^*$ is a proper ideal in $[L]$. Let, $J \in Id([L])$ such that $M^* \subseteq J$. Then $(M^*)_* \subseteq J_*$ so, $M \subseteq J_*$. Thus, $J_* = L$ or $J_* = M$. If $J_* = L$, then $J = [L]$. If $J_* = M$, then $J = (J_*)^* = M^*$. Thus $M^* \in Max([L])$. □

**Theorem 5.8.** *The assignment $P \rightsquigarrow P^*$ is an one-one map from $\mathcal{P}(L)$ to $Spec([L])$. This mapping carries $\mathcal{M}(L)$ onto in $Max([L])$.*

**Proof.** Let $P, Q \in \mathcal{P}(L)$ such that $P^* = Q^*$. Using Proposition 5.5 and Theorem 5.7 , $P^*, Q^* \in Spec([L])$ and $P = (P^*)_* = (Q^*)_* = Q$. If $R \in Spec([L])$, then $R_* \in Spec(L)$ and $(R_*)^* = R$. Let $M \in \mathcal{M}(L)$. From Theorem 5.7, $M^* \in Max([L])$. Let $I \in Max([L])$ and J a proper ideal of L such that $I_* \subseteq J$. Then $I = (I_*)^* \subseteq J^* \neq [L]$. Hence $I = J^*$. If $x \in J$, then $[x] \in I$ so $x \in I_*$. Thus $J = I_*$, so $I_* \in \mathcal{M}(L)$ and this map carries $\mathcal{M}(L)$ onto in $Max([L])$. □

**Theorem 5.9.** *Let $I \in Id(L)$ and $J \in Id([L])$. Then*

*$(i)$ $\sigma(I) \in Pure(L)$;*

*(ii)* *If* $I \in Pure(L)$ *then* $I^* \in Pure([L])$*;*

*(iii)* *If* $\sigma(I) \in \mathcal{P}(L)$ *then* $I \in Pure(L)$ *iff* $I^* \in Pure([L])$*;*

*(iv)* $J \in Pure([L])$ *iff* $J_* \in Pure(L)$*.*

**Proof.** $(i)$ Dualizing Lemma 3.3 from ([9]) we obtain that $\sigma(I^*)$ is pure, that is, $\sigma(I^*) = \sigma(\sigma(I^*))$. Now, from Proposition 5.6, Theorem 5.7 we obtain $\sigma(I) = \sigma(\sigma(I))$, that is, $\sigma(I)$ is a pure ideal.

$(ii)$. $I \in Pure(L)$ implies $\sigma(I) = I$. By Proposition 5.6, $I^* = (\sigma(I))^* = \sigma(I^*)$.

$(iii)$. From Proposition 5.6, $(\sigma(I))^* = \sigma(I^*) = I^*$ and using Proposition 5.5 we obtain $I \in Pure(L)$.

$(iv)$ $J \in Pure([L])$ implies $\sigma(J) = J$, so, by Proposition 5.6, $J_* = (\sigma(J))_* = \sigma(J_*)$. Thus, $J_* \in Pure(L)$. Conversely, $J_* \in Pure(L)$ implies, using Proposition 5.6, $J_* = \sigma(J_*) = (\sigma(J))_*$. From Proposition 5.5, $J \in Pure([L])$. □

# 6 The spectral topology on a residuated lattice

In ([15]), for a residuated lattice L, $\mathcal{P}(L)$ was endowed with the spectral topology as in case of bounded distributive lattices. For $I \in Id(L)$ we denote $V(I) = \{P \in Spec(L) : I \nsubseteq P\}$. Then $\tau_L = \{V(I) : I \in Id(L)\}$ is a topology on $\mathcal{P}(L)$, called the spectral topology. Moreover, the mapping $V : Id(L) \to \tau_L$ defined above is a bijection. Also, for every $x \in L$, we denote $V(x) = \{P \in Spec(L) : x \notin P\}$. We recall that the family $\{V(x) : x \in L\}$ is a basis for the topology $\tau_L$ on $\mathcal{P}(L)$ and the compact open subsets of $\mathcal{P}(L)$ are exactly the sets of the form $V(x)$.

Now, let L be a De Morgan residuated lattice. We compare the spectral topologies on $\mathcal{P}(L)$ and $Spec([L])$. Since $\{V(x)\}_{x\in L}$ generate the spectral topology $\tau_L$ on $\mathcal{P}(L)$, we consider the family of sets $V([x]) = \{Q \in Spec([L]) : [x] \notin Q\}$ which determines a topology on $[L]$.

For a subsets $\mathcal{S} \subseteq \mathcal{P}(L)$ we denote $\mathcal{S}^* = \{P^* \in \mathcal{S}\}$.

**Theorem 6.1.** *Let* $L$ *be a De Morgan residuated lattice and* $x, y \in L$*. Then*

*(i)* $(V(x))^* = V([x])$ *and* $(V(x))^* = (V(y))^*$ *implies* $V(x) = V(y)$*;*

*(ii)* $(V(x) \cap V(y))^* = (V(x))^* \cap (V(y))^*$*;*

*(iii)* $(\cup_{x\in I} V(x))^* = \cup_{x\in I}(V(x))^*$*, for* $I \subseteq L$*.*

**Proof.**$(i)$ Let $R^* \in (V(x))^* = \{P^* : P \in V(x)\}$. Then $x \notin R$, so $[x] \notin R^*$. Thus, $R^* \in V([x])$. Conversely, let $I \in V([x])$. Then by Proposition 5.5 and Theorem 5.7, $I = P^*$ for some $P \in \mathcal{P}(L)$. So $[x] \notin P^*$, hence $x \notin (P^*)_* = P$. So $P \in V(x)$ and $P^* = I \in (V(x))^*$. Finally, $(V(x))^* = (V(y))^*$ implies $V([x]) = V([y])$. So for every $P \in Spec(L)$ we have $[x] \notin P^*$ iff $[y] \notin P^*$. This implies $x \notin P$ iff $y \notin P$ since $P = P^*_*$. Therefore $V(x) = V(y)$.

$(ii)$ From ([15]) $V(x) \cap V(y) = V(x^{**} \wedge y^{**})$. Thus, by, (i), $(V(x) \cap V(y))^* = (V(x^{**} \wedge y^{**}))^* = V([x^{**} \wedge y^{**}]) = V([(x \wedge y)^{**}]) = V([x \wedge y]) = V([x]) \cap V([y]) = (V(x))^* \cap (V(y))^*$.

$(iii)$Let $P^* \in (\cup_{x\in I} V(x))^*$. Then $P \in \cup_{x\in I} V(x)$, so, for some $x \in I, P \in V(x)$ Thus $P^* \in (V(x))^*$. So $(\cup_{x\in I} V(x))^* \subseteq \cup_{x\in I}(V(x))^*$. Conversely, if $P^* \in \cup_{x\in I}(V(x))^*$ then $P^* \in (V(x))^*$ for some $x \in I$ so $P \in V(x) \subseteq \cup_{x\in I} V(x)$. Hence $P^* \in (\cup_{x\in I} V(x))^*$ and $(\cup_{x\in I} V(x))^* \supseteq \cup_{x\in I}(V(x))^*$.

□

To summarize, we have:

**Corollary 6.2.** *If* $L$ *is a De Morgan residuated lattice, then*

*(i)* *the map* $V(x) \rightsquigarrow (V(x))^*$ *is one-one, onto and preserves arbitrary unions and finite intersections;*

*(ii) the prime ideal spaces $\mathcal{P}(L)$ and Spec([L]) are homeomorphic.*

Since in a residuated lattice L, for $I \in Id(L), V(I) = \{P \in \mathcal{P}(L) : I \nsubseteq P\}$ is open in $\mathcal{P}(L)$ and $\overline{V}(I) = \mathcal{P}(L) \backslash V(I) = \{P \in \mathcal{P}(L) : I \subseteq P\}$ is closed, then obviously, $V(I)$ is stable under descent (that is, if $P \in V(I), Q \in \mathcal{P}(L)$ and $Q \subseteq \mathcal{P}(L)$ and $P \subseteq Q$, then $Q \in \overline{V}(I)$ and $\overline{V}(I)$ is stable under ascent (that is, if $P \in \overline{V}(I), Q \in \mathcal{P}(L)$ and $P \subseteq Q$ then $Q \in \overline{V}(I)$).

So, the sets simultaneous open and closed (*clopen sets* in $\mathcal{P}(L)$, are *stable*, that is, are stable under ascent and descent.

As in the case of MV algebras, by *stable topology* for L, we mean a collection $S_L$ of stable open subsets $V(I)$ of $\mathcal{P}(L)$, that is $S_L = \{V(I) : I \in Id(L)\}$ and $V(I)$ is stable under ascent.

**Proposition 6.3.** *Let L be a residuated lattice and $I \in Id(L)$. Then $V(I)$ is stable in $\mathcal{P}(L)$ iff $V(I^*)$ is stable in Spec([L]).*

**Proof.** Suppose that $V(I)$ is stable in $\mathcal{P}(L)$ and let $P, Q \in Spec([L])$ such that $P \subseteq Q$ and $P \in V(I^*)$. Then $I^* \nsubseteq P$ and by Theorem 5.7 we deduce that $I = (I^*)_* \nsubseteq P_*$, so $P_* \in V(I)$. Since $P_* \nsubseteq Q_*$ and $V(I)$ is stable, then $Q_* \in V(I)$. But $Q_* \in V(I)$ iff $I \nsubseteq Q_*$. Then $I^* \nsubseteq Q_* = Q$ so $Q \in V(I^*)$. Thus, $V(I^*)$ is stable in $\mathcal{P}(L)$. Conversely, suppose that $V(I^*)$ is stable in Spec([L]) then for $P, Q \in \mathcal{P}(L)$ such that $P \subseteq Q$ and $P \in V(I)$. We have $I \nsubseteq P$. Thus $I^* \nsubseteq P^*$, so $P^* \in V(I^*)$. Since $P^* \subseteq Q^*$ and $V(I^*)$ is stable in Spec([L]) then $Q^* \in V(I^*)$. But $Q^* \in V(I^*)$ iff $I^* \nsubseteq Q^*$ iff $I \nsubseteq Q$. Thus, $Q \in V(I)$, that is ,$V(I)$ is stable in $\mathcal{P}(L)$. □

**Theorem 6.4.** *Let L be a De Morgan residuated lattice and $I \in Id(L)$. Then $I \in Pure(L)$ iff $V(I)$ is stable in $\mathcal{P}(L)$.*

**Proof.** Suppose that $I \in \mathcal{P}(L)$ and let $P, Q \in \mathcal{P}(L)$ such that $P \subseteq Q$ and $P \in V(I)$. Then $I \nsubseteq P$, so there exists $i_0 \in I$ such that $i_0 \notin P$. Since $I = \sigma(I)$, then $i_0 \in \sigma(I)$, so $i_0^{**} \in \sigma(I)$. Then there are $i \in I$ and $y \in (i_0^{**})^{\perp}$ such that $i \oplus y = 1$. Since $y^{**} \in (i_0^{**})^{\perp}$ we deduce that $i_0^{**} \wedge y^{**} = 0 \in P$. But $i_0 \notin P$ so, $y \in P \subseteq Q$, thus $y \in Q$. If by the contrary, $Q \notin V(I)$ then $I \subseteq Q$ so $i \in Q$. From $y, i \in Q$ we deduce that $i \oplus y = 1 \in Q$. Hence $Q = L$, a contradiction.

Conversely, we suppose that $V(I)$ is stable in $\mathcal{P}(L)$. If by contrary I is not pure in L, then there is $x_0 \in I$ such that $x_0 \notin \sigma(I)$, so $x_0 \neq 0$. From (see [14], Corollary 23), there is a minimal prime ideal P such that $\sigma(I) \subseteq P$ and $x_0 \notin P$. Thus $I \nsubseteq P$, hence $P \in V(I)$. Since $x_0 \notin \sigma(I)$, then for every $i \in I$ and $x_0^{\perp}$ we have $i \boxplus y \neq 1$. This implies that $i \notin x_0^{\perp} \vee I$, that is $x_0^{\perp} \vee I$ is proper in L. From Theorem 3.11, there is $Q \in Spec(L)$ such that $x_0^{\perp} \vee I \subseteq Q$. But $\sigma(I) \subseteq I \subseteq Q$ and by minimally of P, $P \subseteq Q$. Since $V(I)$ is stable, we deduce $Q \in V(I)$. But $I \subseteq Q$, hence $Q \notin V(I)$, a contradiction. Thus, $\sigma(I) = I$ and I is pure L. □

From Proposition 6.3 and Theorem 6.4 we obtain:

**Corollary 6.5.** *Let L be a De Morgan residuated lattice and $I \in Id(L)$. Then the following are equivalent:*

*(i) $I \in Pure(L)$;*

*(ii) $V(I)$ is stable in $\mathcal{P}(L)$;*

*(iii) $V(I^*)$ is stable in Spec([L]).*

**Corollary 6.6.** *For a residuated lattice L, the assignment $I \rightsquigarrow V(I)$ is a bijection between from Pure(L) and the set of stable open subsets of $\mathcal{P}(L)$.*

**Corollary 6.7.** *Let L be a De Morgan residuated lattice. Then the spectral topology coincides with a stable topology on $\mathcal{P}(L)$ iff L is hyperarchimedean.*

**Proof.** By Theorem 5.4, L is hyperarchimedean iff $[L]$ is a Boolean algebra. Using Corollary 6.2 and Theorem 4, (see [8]) we deduce the conclusion. □

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