Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://tfss.journals.iau.ir/
Online ISSN: 2821-0131
Vol.1, No.2, (2022), 46-58
DOI: http://10.30495/tfss.2022.1960147.1036

Article Type: Original Research Article (Invited Paper)

Managing the Uncertainty: From Probability to Fuzziness, Neutrosophy and Soft Sets

Michael Gr. Voskoglou

Abstract. The present paper reviews and compares the main theories reported in the literature for managing the existing real life uncertainty by listing their advantages and disadvantages. Starting with a comparison of the bivalent logic (including probability) and fuzzy logic, proceeds to a brief description of the primary generalizations of fuzzy sets (FSs) including interval valued FSs, type-2 FSs, intuitionistic FSs, neutrosophic sets, etc. Alternative theories related to fuzziness are also examined including grey system theory, rough sets and soft sets. The conclusion obtained at the end of this discussion is that there is no ideal model for managing the uncertainty; it all depends upon the form, the available data and the existing knowledge about the problem under solution. The combination of all the existing models, however, provides a sufficient framework for efficiently tackling several types of uncertainty appearing in real life.

AMS Subject Classification 2020: Primary 03E72; Secondary; 03B52; 03B53
Keywords and Phrases: Uncertainty, Fuzzy set (FS), Interval valued FS (IVFS), Type-2 FS, Intuitionistic FS (IFS), Neutrosophic set (NS), Rough set, Soft set, Grey system (GS).

1 Introduction

The frequently appearing in the real world, in science and in everyday life *uncertainty* is due to a shortage of knowledge regarding some situations. Roughly speaking, the amount of the existing uncertainty is equal to the difference in the amount of the necessary knowledge needed for interpreting or determining the evolution of a situation, minus the already existing knowledge about this situation. In other words, uncertainty represents the total amount of potential information in the situation, which implies that a reduction of uncertainty due to new evidence (e.g. receipt of a message) indicates a gain of an equal amount of information. This is the reason for which the classical measures of uncertainty under crisp or fuzzy conditions (Hartley's formula, Shannon's entropy, etc. [15, Chapter 5]) have also been adopted as measures of information, comprising a powerful tool for dealing with problems such as systems modeling analysis and design, decision making, etc. Different kinds of uncertain environments exist in real life [15]. A typical taxonomy of the uncertainties that can arise includes *vagueness, imprecision, ambiguity* and *inconsistency.* The uncertainty due to vagueness is created when one is unable to clearly differentiate between two classes, such as "a person of average height" and "a tall person". In case of imprecision the available information has not an exact value; e.g. "the temperature tomorrow will be between 27° and 32° C". In ambiguity then existing information leads to several interpretations by different observers. For example, the statement "Boy no girl" written as "Boy, no

Corresponding author: M. Gr. Voskoglou, Email: voskoglou@teiwest.gr, ORCID: 0000-0002-4727-0089
Received: 1 June 2022; Revised: 11 June 2022; Accepted: 11 June 2022; Published Online: 7 November 2022.

How to cite: M. Gr. Voskoglou, Managing the uncertainty: from probability to fuzziness, neutrosophy and soft sets, *Trans. Fuzzy Sets Syst.*, 1(2) (2022), 46-58.

girl" means boy, but written as "Boy no, girl" means girl. Finally, inconsistency appears when two or more pieces of information cannot be true at the same time. As a result the obtainable in this case information is conflicted or undetermined. For example, "the probability for raining tomorrow is 80%, but this does not mean that the probability of not raining is 20%, because they might be hidden weather factors".

Note that several other taxonomies of the uncertainty exist. One such taxonomy, for example, includes the *epistemic (or subjective)* uncertainty and the *linguistic* uncertainty. The former is due to a lack of knowledge, whether the latter is produced by statements expressed in natural language. Another taxonomy includes the uncertainty due to *randomness* and the uncertainty due to *imprecision.* The uncertainty due to randomness is related to well-defined events whose outcomes cannot be predicted in advance, like the turning of a coin, the throwing of a die, etc. On the other hand, uncertainty due to imprecision occurs when the events are well defined, but the possible outcomes cannot be expressed in a crisp form.

The uncertain problems need imprecise methods that could deal with different types of uncertainties to increase the understanding of the outcomes. Several theories have been proposed for tackling such kinds of problems. The target of the present work is to review and compare the primary among those theories and list their advantages and disadvantages. The rest of the paper is formulated as follows: Section 2 compares the bivalent logic (including probability) with the fuzzy logic. Section 2 describes the headlines of the primary generalizations of fuzzy sets (FSs), such as interval valued FSs, type-2 FSs, intuitionistic FSs, neutrosophic sets, etc. Alternative theories related to fuzziness are examined in section 3, including grey system theory, rough sets and soft sets. The paper closes with a discussion including some hints for future research and the final conclusion, which are contained in section 5.

2 Fuzzy Vs Bivalent Logic

Logic is the study of correct reasoning, involving the drawing of inferences. There is no doubt that the enormous progress of science and technology owes a lot to Aristotle's (384-322 BC) *bivalent logic,* which dominated for centuries the human way of thinking.

Bivalent logic is based on Aristotle's law of the *excluded middle,* according to which, for all propositions p, either p or not p must be true and there is no middle (third) true proposition between them; all its other principles are mere elaborations of this law [16].

From the time of Buddha Siddhartha Gautama, however, who lived in India around 500 BC, Heraclitus (535-475 BC) and Plato (427-377 BC) views have appeared to discuss the existence of a third area between "true" and "false", where those two opposites can exist together. More recent philosophers like Hegel, Marx, Engels, Russel and others supported and cultivated further those ideas, but the first integrated propositions of multivalued logics appeared only during the 20^{th} century by Jan Lukasiewicz (1858-1956) and Alfred Tarski (1901-1983) [37, Section 2]. Max Black [2] introduced in 1937 the concept of the *vague set* being a premonition of the Zadeh's *fuzzy* set (FS) introduced in 1965 [44].

Let U be the universal set of the discourse. It is recalled that a fuzzy set F on U is defined with the help of its *membership function* $m : U \to [0, 1]$ as the set of the ordered pairs

$$F = \{(x, m(x)) \ : \ x \in U\}. \tag{1}$$

The real number $m(x)$ is called the *membership degree* of x in F. The greater is $m(x)$, the more x satisfies the characteristic property of F.

A crisp subset A of U is a fuzzy set on U with a membership function taking the values $m(x) = 1$ if x belongs to A and 0 otherwise. Most notions and operations concerning the crisp sets, e.g. subset, complement, union, intersection, Cartesian product, binary and other relations, etc., can be extended to FS. For general facts about FSs and the connected to them uncertainty we refer to the chapters 4-7 of the book [36].

The infinite-valued on the interval $[0, 1]$ *fuzzy logic* (FL) is defined with the help of the concept of FS [45].

Through FL, the fuzzy terminology is translated by algorithmic procedures into numerical values, operations are performed upon those values and the outcomes are returned into natural language statements in a reliable manner [17]. FL is useful for handling real-life situations that are inherently fuzzy, calculating the existing in such situations fuzzy data and describing the operation of the corresponding fuzzy systems. An important advantage of FL is that its rules are set in natural language with the help of linguistic, and therefore fuzzy, variables [46].

The process of reasoning with fuzzy rules involves:

• *Fuzzification* of the problem's data by utilizing the suitable membership functions to define the required FSs.

• Application of FL operators on the defined FSs and combination of them to obtain the final result in the form of a unique FS.

• *Defuzzification* of the final FS to return to a crisp output value, in order to apply it to the real world situation for resolving the corresponding problem.

Among the more than 30 defuzzification methods in use, the most popular is probably the *Centre of Gravity* (*COG*) technique. According to it, a problem's fuzzy solution is represented by the coordinates of the COG of the level's section contained between the graph of the MF involved and the OX axis [35].

But, while Zadeh was trying to spread out the message of fuzziness, he received many tough critiques for his radical ideas from three different directions [10].

The first direction of critique came from a great number of scientists who asked for some practical applications. In fact, such applications started to appear in the industry during the 1970's, the first one being in the area of cement kiln control [42]. This is an operation demanding the control of a highly complex set of chemical interactions by dynamically managing 40-50 "rules of thumb". This was followed by E. H. Mamdani's [21] work in the Queen Mary College of London, who designed the first fuzzy system for controlling a steam engine and later the operation of traffic lights. Another type of fuzzy inference system was developed later in Japan by Takagi-Sugeno-Kang [32]. Nowadays FSs and FL have found many important applications in almost all sectors of human activity. It must be mentioned that fuzzy mathematics has also been significantly developed on a theoretical level, providing important contributions even in branches of classical mathematics, such as algebra, analysis, geometry, etc. (e.g. see [3]).

The second direction was related to a great part of the *probability* theorists, who claimed that FL could not do any more than probability does. Membership degrees, taking values in the same with probabilities interval $[0, 1]$, are actually hidden probabilities, fuzziness is a kind of disguised randomness, and the multi-valued logic is not a new idea. It took a long time to become universally understood that fuzziness does not oppose probability, but actually supports and completes it by successfully treating the cases of the existing the real world uncertainty which is caused by reasons different from randomness [9].

The expressions "John's membership degree in the FS of clever people is 0.7" and "the probability of John to be clever is 0.7", although they look similar, they actually have essentially different meanings. The former means that John is a rather clever person, whereas the latter means that John, according to the principle of the excluded middle, is either clever or not, but his outlines (heredity, academic studies, etc.) suggest that the probability to be clever is high (70%).

There are also other differences between the two theories mainly arising from the way of defining the corresponding notions and operations. For instance, whereas the sum of the probabilities of all the single events (singleton subsets) of the universal set is always equal to 1 (probability of the certain event), this is not necessarily true for the membership degrees. Consequently a probability distribution could be used to define membership degrees, but the converse does not hold in general.

Note that Edwin T. Jaynes, Professor of Physics at the University of Washington, argued that Probability theory could be considered as a generalization of the bivalent logic reducing to it in the special case where our hypothesis is either absolutely true or absolutely false [12]. Many eminent scientists have been inspired

by the ideas of Janes, like the expert in Algebraic Geometry David Mumford, who believes that Probability and Statistics are emerging as a better way of building scientific models [26].
Probability and Statistics are related mathematical topics having, however, fundamental differences. In fact, Probability is a branch of theoretical mathematics dealing with the estimation of the likelihood of future events, whereas Statistics is an applied branch , which tries to make sense by analyzing the frequencies of past events. Nevertheless, both Probability and Statistics have been developed on the basis of the principles of the bivalent logic. As a result, they are tackling effectively only the cases of the existing in the real world uncertainty which are due to randomness [18]. In other words, Janes' probabilistic logic "covers" only the cases of uncertainty wich are due to randomness.
One could argue, however, that *Bayesian Reasoning* constitutes an interface between bivalent and FL [38]. In fact, the *Bayes' rule* (see equation 2 below) calculates the *conditional probability* $P(A/B)$ with the help of the *inverse in time* conditional probability $P(B/A)$, the *prior probability* $P(A)$ and the *posterior probability* $P(B)$:

$$P(A/B) = \frac{P(B/A)P(A)}{P(B)}. \tag{2}$$

In other words, the Bayes' rule calculates the probability of an event based on prior knowledge of conditions related to that event. The value of the prior probability $P(A)$ is fixed before the experiment, whereas the value of the posterior probability is derived from the experiment's data. Usually, however, there exists an uncertainty (not necessarily due to randomness) about the value of $P(A)$. In such cases, considering all the possible values of $P(A)$, we obtain different values for the conditional probability $P(A/B)$. Therefore, the Bayes' rule introduces a kind of multi-valued logic tackling the existing, due to the different values of the prior probability, uncertainty in a way analogous to FL.
The third direction of the critiques against FL comes from bivalent logic. Many of its traditional supporters, based on a culture of centuries, argue that, since this logic works effectively in science, functions the computers and explains satisfactorily the phenomena of the real world, except perhaps those that happen in the boundaries, there is no reason to make things more complicated by introducing the unstable principles of a multi-valued logic.
FL, however, aims exactly at smoothing the situation in the boundaries! Look, for example, at the graph in Figure 1 corresponding to the FS T of "tall people". People with heights less than 1.50 m are considered to have a membership degree 0 in T. The membership degree is continuously increasing for heights greater than 1.50 m, taking its maximal value 1 for heights equal or greater than 1.80 m. Therefore, the "fuzzy part" of the graph - which is conventionally represented in Figure 1 by the straight line segment AC, but its exact form depends upon the way in which the membership function has been defined - lies in the area of the rectangle ABCD defined by the OX axis, its parallel through the point E and the two perpendiculars to it lines at the points A and B.

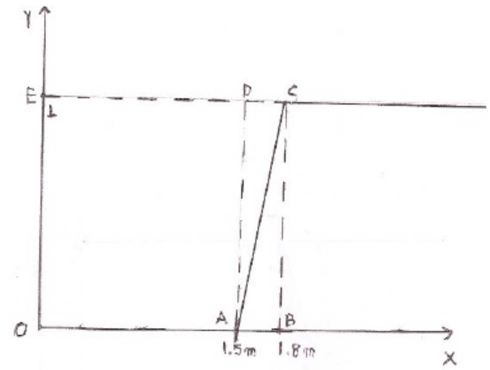

Figure 1: The fuzzy set of "tall people"

In fact, the way of perceiving a concept (e.g. "tall") is different from person to person, depending on the "signals" that each one receives from the real world about it. Mathematically speaking, this means that the definition of the membership function of a FS is not unique, depending on the observer's personal criteria. The only restriction is that this definition must be compatible to the common logic, because otherwise the corresponding FS does not give a reliable description of the corresponding real situation.

On the contrary, bivalent logic defines a bound, e.g. 1.80 m, above which people are considered to be tall and under which are considered to be short. Consequently, one with a height 1.79 m is considered to be short, whether another with a height 1.81 m is tall!

Bivalent logic is able to verify the validity/consistency of an argument only and not its truth. A deductive argument is always valid, even if its inference is false. A characteristic example can be found in the function of computers. A computer is unable to judge, if the input data inserted into it is correct, and therefore if the result obtained by elaborating this data is correct and useful for the user. The only thing that it guarantees is that, if the input is correct, then the output will be correct too. On the contrary, always under the bivalent logic approach, an inductive argument is never valid, even if its inference is true. To put it in a different way, if a property p is true for a sufficient large number of cases, the expression "the property p is possibly true in general" is not acceptable, since it does not satisfy the principle of the excluded middle.

People, however, always want to know the truth in order to organize better, or even to protect, their lives. Consequently, under this option, the significance of an argument has greater importance than its validity/precision. In Figure 2 [4], for example, the extra precision on the left makes things worse for the poor man in danger, who has to spend too much time trying to understand the data and misses the opportunity to take the much needed action of getting out of the way. On the contrary, the rough / fuzzy warning on the right could save his life.

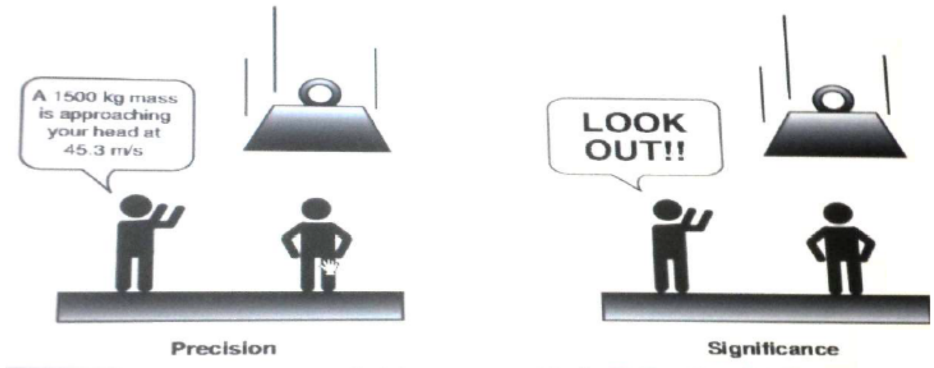

Figure 2: Validity/precision vs significance

Figure 2 illustrates very successfully the importance of FL for the real life situations. Real-world knowledge generally has a different structure and requires different formalization than the existing formal systems. FL, which according to Zadeh is "a precise logic of imprecision and approximate reasoning" [45], serves as a link between classical logic and human reasoning/experience, which is two incommensurable approaches. Having a much higher generality than bivalent logic, FL is capable of generalizing any bivalent logic-based theory.
They have appeared also whit strong voices of anger against FL, without bothering to present any logical arguments about it. Those voices, characterize FL as the tool for making the science unstable, or more emphatically as the "cocaine of science"! Such voices, however, frequently appear in analogous cases of the history of science and must be simply ignored.
Zadeh introduced further *fuzzy numbers (FNs)* [46] as a special form of FSs on the set of the real numbers. He defined the basic arithmetic operations on them in terms of his *extension principle*, which provides the means for any function mapping the crisp set X to the crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y [15]. FNs play an important role in fuzzy mathematics, analogous to the role of ordinary numbers in traditional mathematics. For general facts on FNs we refer to the book [13]. The present author has used in earlier works *triangular FNs (TFNs)*, the simplest form of FNs, as tools in assessment processes; e.g. [37, section 5].
Zadeh realized that FSs are connected to words (adjectives and adverbs) of the natural language; e.g. the adjective "tall" indicates the FS of the tall people, since "how tall is everyone" is a matter of degree. A grammatical sentence may contain many adjectives and/or adverbs, therefore it correlates a number of FSs. A synthesis of grammatical sentences, i.e. a group of FSs related to each other, forms what we call a *fuzzy system*. A fuzzy system provides empirical advice, mnemonic rules and common logic in general. It is not only able to use its own knowledge to represent and explain the real world, but can also increase it with the help of the new data; in other words, it learns from the experience. This is actually the way in which humans think. Nowadays, for example, a fuzzy system can control the function of an electric washing-machine or send signals for purchasing shares from the stock exchange, etc. [39]. Fuzzy systems are considered to be a part of the wider class of *Soft Computing*, which also includes probabilistic reasoning and *neural networks* (Figure 3) [27].

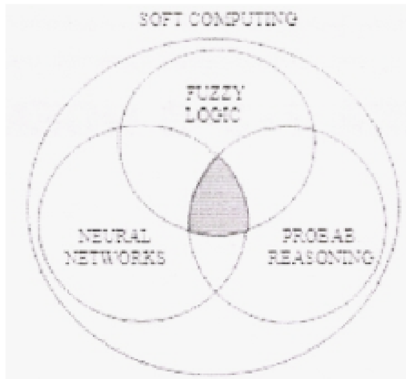

Figure 3: A graphical approach to the contents of Soft Computing

One may say that fuzzy systems and neural networks try to emulate the operation of the human brain. Neural networks have the ability to learn and also have a parallel structure that can rapidly process the information. In other words they concentrate on the structure of the human brain, i.e. on the "hardware", emulating its basic functions. On the other hand, fuzzy systems concentrate on the "software", emulating fuzzy and symbolic reasoning. Fuzzy systems make decisions based on the raw and ambiguous data given to them, whereas neural networks try to learn from the data, incorporating the same way involved in the biological neural networks.

Intersections in Figure 2 include *neuro-fuzzy systems* and techniques, probabilistic approaches to neural networks and Bayesian Reasoning. A neuro-fuzzy system is a fuzzy system that uses a learning algorithm derived from or inspired by neural network theory to determine its parameters (FSs and fuzzy rules) by processing data samples. Characteristic examples of such kinds of systems are the *Adaptive Neuro- Fuzzy Inference Systems (ANFIS)* [11].

3 Generalizations of Fuzzy Sets

As has been already mentioned in the previous section, the probability is suitable for managing the cases of uncertainty due to randomness. Fuzziness, on the other hand, treats as well the cases of vagueness. for the purpose of managing the existing real world uncertainty in a better way, a lot of research has been carried out during the last 60 years to improve/generalize the FS theory.

Zadeh, Sambuc, Jahn and Grattan Guiness introduced in 1975, independently from each other, the concept of the *interval-valued FS (IVFS)* [8]. The idea behind IVFs is that the membership degrees of the traditional FSs, as has been already explained in the previous section, can hardly be precise Thus, an IVFS, defined by a mapping from the universe U to the set of closed intervals in $[0, 1]$, replaces the membership degrees with closed sub-intervals of $[0, 1]$.

Similar to the concept of IVFS is the *hesitant FS (HFS)* introduced by Torra and Narukawa in 2009 [34]. The difference in the definition of a HFS with respect to an IVFS is that the *hesitant degree* $h(x)$ of an element x of U is not a single value like its membership degree, but a set of some values in $[0, 1]$. For example, if $U = \{a, b, c\}$, we could have $h(a) = \{0.2, 0.3\}$, $h(b) = \{0.75, 0.8, 0.82\}$ and $h(c) = \{0.9\}$.

Zadeh also introduced in 1975 the concept of *type-2 FS* [46], so that more uncertainty could be handled connected to the membership function. The membership function of a type-2 FS is three - dimensional, its third dimension is the value of the membership function at each point of its two-dimensional domain, which is called *Footprint of Uncertainty (FOU)*. The FOU is completely determined by its two bounding functions, a *lower* membership function and an *upper* membership function, both of which are ordinary FSs (otherwise called *type-1 FSs*). When no uncertainty exists about the membership function, then a type-2 FS reduces to a type-1 FS, in a way analogous to probability reducing to determinism when unpredictability vanishes. Zadeh in the same paper [46] generalized the type-2 FS to the *type-n FS* $n = 1, 2, 3, \ldots$. When Zadeh proposed the type-2 FS, however, the time was not right for researchers to drop what they were doing with type-1 FS and focus on type-2 FS. This changed in the late 1990s as a result of Prof. Jerry Mendel and his students' works on type-2 FS [22]. Since then, more and more researchers around the world have been writing articles about type-2 FSs and systems.

Another application of FS, inspired by Zadeh, is the process of *Computing with Words (CWW)*, a methodology in which the objects of computation are words and propositions drawn from a natural language [47]. The idea was that computers would be activated by words, which would be converted into a mathematical representation using FSs and that these FSs would be mapped by a CWW engine into some other FS, after which the latter would be converted back into a word. Much research is under way about CWW. As Mendel has argued [23], a type-2 fuzzy set should be used as a model for a word.

Ramot et al. [29] introduced in 2002 the notion of *Complex FS (CFS)* characterized by a complex-valued MF, whose range is extended from the traditional fuzzy range of $[0, 1]$ to the unit circle in the complex plane. More explicitly, the membership function of a CFS is of the form $m(x) = r(x)e^{i\theta(x)} = r(x)[\cos[\theta(x)] + i \sin[\theta(x)]$. In the above formula $r(x)$ is the *amplitude term* and $\theta(x)$ is the *phase term* of the membership function. The terms $r(x)$ and $\theta(x)$ are both real-valued and $r(x)$ is in $[0, 1]$ for all x in the universal set U. Since $m(x)$ is a periodic function, one may only consider $\theta(x)$ in $[0, 2\pi]$. When $\theta(x) = 0$ for all x in U, then $m(x)$ reduces to the membership function of an ordinary FS.

Kassimir Atanassov, Professor of Mathematics at the Bulgarian Academy of Sciences, introduced in 1986, as a complement of Zadehs membership degree $m(x)$, $x \in U$, the *degree of non-membership* $n(x)$. In a FS is always $m(x) + n(x) = 1$, but this need not be always true in real applications; e.g. see the example of section 2 with the rainy weather. Atanassov proposed the notion of *intuitionistic FS (IFS)* for more accurate quantification of the uncertainty [1].

An IFS A is formally defined as the set of the ordered triples

$$A = \{(x, m(x), n(x)) \; : \; x \in U, \quad 0 \leq m(x) + n(x) \leq 1\}. \tag{3}$$

One can write $m(x) + n(x) + h(x) = 1$, where $h(x)$ is called the *hesitation* or *uncertainty degree* of x. If $h(x) = 0$, then the corresponding IFS reduces to an ordinary FS. The characterization of intuitionistic is due to the fact that an IFS contains the intuitionistic idea, as it incorporates the degree of hesitation.

Most notions and operations concerning the crisp sets can be extended to IFS [1]. A *Pythagorean FS (PFS)*, introduced by Yager in 2013 [43], considers the membership degree $m(x)$ and non-membership degree $n(x)$ satisfying the condition $m^2(x) + n^2(x) \leq 1$. PFSs have a stronger ability than IFS to manage uncertainty in real-world decision-making problems [48].

The Romanian-American writer and mathematician Florentin Smarandache, Professor at the branch of Gallup of the New Mexico University, introduced in 1995 the degree of *indeterminancy/neutrality* membership of the elements of the universal set U in a subset of U and defined the concept of *neutrosophic set (NS)*, which generalizes the notions of FS and IFS [30].

A *single valued NS (SVNS)* A on U is of the form

$$A = \{(x, T(x), I(x), F(x)) \; : \; x \in U, \quad T(x), I(x), F(x) \in [0, 1], \quad 0 \leq T(x) + I(x) + F(x) \leq 3\}. \tag{4}$$

In (4) $T(x)$, $I(x)$, $F(x)$ are the degrees of truth, indeterminancy and falsity membership of x in A respectively, called the *neutrosophic components* of x. The etymology of the term "neutrosophy" comes from the adjective "neutral" and the Greek word "sophia" (wisdom) and means, according to Smarandache who introduced it, "the knowledge of neutral thought".

For example, let U be the set of the players of a football team and let A be the SVNS of the good players of U. Then each player x of U is characterized by a *neutrosophic triplet* (t, i, f) with respect to A, with t, i, f in $[0, 1]$. For instance, $(0.6, 0.2, 0.4) \in A$ means that there is a 60% probability for x to be in A, a 20% probability to be unknown if x is in A and a 40% probability for x to not be in A. In particular, $x(0, 1, 0) \in A$ means that we do not know absolutely nothing about x's affiliation with A.
Indeterminancy is understood to be in general everything which is between the opposites of truth and falsity [31]. One can find plenty of real examples of neutrosophic triplets, like (friend, neutral, enemy), (positive, zero, negative), (small, medium, high), (male, transgender, female), (win, draw, defeat), etc. This means that the previously given definition of SVNS is well placed.
In an IFS the inderterminancy is equal by default with the hesitancy, i.e. we have $I(x) = 1 - T(x) - F(x)$. Also, in a FS is $I(x) = 0$ and $F(x) = 1 - T(x)$, whereas in a crisp set is $T(x) = 1$ (or 0) and $F(x) = 0$ (or 1). In other words, crisp sets, FSs and IFSs are special cases of SVNSs.
When the sum $T(x) + I(x) + F(x)$ of the neutrosophic components of $x \in U$ in a SVNS A on U is < 1, then it leaves room for incomplete information about x, when is equal to 1 for complete information and when is greater than 1 for *parasconsistent* (i.e. contradiction tolerant) information about x. A SVNS may contain simultaneously elements leaving room for all the previous types of information.
When $T(x) + I(x) + F(x) < 1, \forall x \in U$, then the corresponding SVNS is usually referred as *picture FS (PiFS)* [39]. In this case $1 - T(x) - I(x) - F(x)$ is called the degree of *refusal membership* of x in A. The PiFSs based models are adequate in situations where we face human opinions involving answers of types yes, abstain, no and refusal Voting is a good example of such a situation.
The difference between the *general definition of a NS* and the previously given definition of a SVNS is that in the general definition $T(x)$, $I(x)$ and $F(x)$ may take values in the non-standard unit interval $]-0, 1+[$ (including values < 0 or > 1) [36]. This could happen in real world applications. For example, in a company with full-time work for its employees 40 hours per week an employee, upon his work, could belong by $\frac{40}{40} = 1$ to the company (full-time job) or by $\frac{30}{40} < 1$ (part-time job) or by $\frac{45}{40} > 1$ (over-time job). Assume further that a full-time employee caused a damage to his job's equipment, the cost of which must be taken from his salary. Then, if the cost is equal to $\frac{50}{40}$ of his weekly salary, the employee belongs this week to the company by $-\frac{10}{40} < 0$.
Most notions and operations concerning the crisp sets can be extended to NSs [31].

4 Alternative Theories Related to Fuzziness

In 1982 Julong Deng, Professor of the Huazhong University of Science and Technology, Wuhan, China, introduced the theory of *Grey System (GS)* [6] for handling the approximate data that are frequently appear in the study of large and complex systems, like the socio-economic, the biological ones, etc. The systems which lack information, such as structure message, operation mechanism and behaviour document, are referred to as GSs. Usually, on the grounds of existing grey relations and elements one can identify where "grey" means poor, incomplete, uncertain, etc. The GS theory was mainly developed in China and it has found many applications in agriculture, economy, management, industry, ecology and in many other fields of the human activity [7].
An effective tool of the GS theory is the use of *Grey Numbers (GNs)* that are indeterminate numbers defined in terms of the closed real intervals. More explicitly, a GN, say A, is of the form $A \in [a, b]$, where a, b are real numbers with $a \leq b$. In other words, the range in which A lies is known, but not its exact value. A GN may enrich its uncertainty representation with respect to the interval $[a, b]$ by a *whitenization function* $g : [a, b] \longrightarrow [0, 1]$, which defines the *degree of greyness* $g(x)$ for each x in $[a, b]$. The closer is $g(x)$ to 1, the

better x approximates the real value of A. The real number which is used as the crisp representative of the GN $A \in [a, b]$ is denoted by $W(A)$. When the distribution of A is unknown, i.e. no whitenization function has been defined for it, one usually takes $W(A) = \frac{a+b}{2}$ [5].
The arithmetic of the real intervals introduced by Moore et al. [25] has been used to define the basic arithmetic operations among the GNs. For general facts on GNs we refer to the book [19]. The present author has utilized in earlier works GNs as tools in assessment processes; e.g. see section 6 of [37])
A *rough set*, first described by the Polish computer scientist Zdzislaw Pawlak in 1991 [28] is a formal approximation of a crisp set in terms of a pair of sets which give the *lower* and the *upper approximation* of the original set. In the standard version of rough set theory the lower and upper-approximation sets are crisp sets, but in other variations, the approximating sets may be FSs. The theory of rough sets has found important applications to many scientific fields and in particular in Informatics.
In 1999 Dmtri Molodstov, Professor of the Computing Center of the Russian Academy of Sciences in Moscow, in order to overpass the existing difficulty for defining properly the membership functions of FSs, IFSs, NSs, etc. proposed the notion of *soft set* as a new mathematical tool for dealing with the uncertainty in a parametric manner [24].
Let E be a set of parameters, let A be a subset of E and let f be a mapping of A into the set $P(U)$ of all subsets of U. Then the soft set on U connected to A, denoted by (f, A), is defined as the set of the ordered pairs

$$(f, A) = \{(e, f(e)) \ : \ e \in A\}. \tag{5}$$

In other words, a soft set is a paramametrized family of subsets of U. Intuitively, it is "soft" because the boundary of the set depends on the parameters.
For example, let $V = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ be a set of cars and let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of the parameters e_1=high-speed, e_2 =automatic (gear-box), e_3 =hybrid (petrol and electric power), $e_4 = 4x4$ and e_5 =cheap. Consider the subset $A = \{e_1, e_2, e_3, e_5\}$ of E and assume that C_1, C_2, C_6 are the high-speed, C_2, C_3, C_5, C_6 are the automatic, C_3, C_5 are the hybrid cars and C_4 is the unique cheap car. Then a map $g : A \longrightarrow P(V)$ is defined by $g(e_1) = \{C_1, C_2, C_6\}$, $g(e_2) = \{C_2, C_3, C_5, C_6\}$, $g(e_3) = \{C_3, C_5\}$, $g(e_5) = \{C_4\}$ and the soft set

$$(g, A) = \{(e_1, \{C_1, C_2, C_6\}), (e_2, \{C_2, C_3, C_5, C_6\}), (e_3, \{C_3, C_5\}), (e_5, \{C_4\})\}. \tag{6}$$

A FS on U with membership function $y = m(x)$ is a soft set on U of the form $(f, [0, 1])$, where $f(\alpha) = \{x \in U : m(x) \geq \alpha\}$ is the corresponding $\alpha - cut$ of the FS, for each α in $[0, 1]$. Most notions and operations concerning the crisp sets can be extended to soft sets [20].
Soft sets have found important applications to several sectors of human activity [14, 33]. The present author has used recently soft sets as tools for assessment processes [40], as well as a combination of soft sets and GNs for developing a hybrid decision making method under fuzzy conditions [41].
The catalogue of the extensions of FS and of the related to the fuzziness theories does not end here. Several other alternatives have been proposed, many of them being hybrid constructions of the above mentioned primary approaches. For example, if in the definition of the soft set the set of all subsets of U is replaced by the set of all fuzzy subsets of U, one gets the notion of the *fuzzy soft set*. Also, the notion of neutrosophic set has been combined with that of the IVFS to form a new hybrid set called *interval valued neutrosophic set*, etc.

5 Discussion and Conclusion

The frequently existing in real life situations uncertainty is connected to the available information about the corresponding situation and appears in several types, like randomness, vagueness, imprecision, etc. The study performed in this work leads to the conclusion that there is no ideal model for managing the uncertainty; it all depends upon the form, the available data and the existing knowledge about the problem under solution. Probability treats efficiently the cases of randomness, FS describes vagueness, type-2 FS describes vagueness and imprecision by a 3-dimensional range of membership values, IFS is suitable for simulating imprecision in human thinking, NS can deal with vagueness, imprecision, ambiguity and inconsistency, etc. The combination of all these models, however, provides a sufficient framework for tackling the several types of uncertainty, but further research is needed to improving the existing methods, probably by using hybrid approaches.

Conflict of Interest: The author declares that there are no conflict of interest.

References

[1] K. T. Atanassov, Intuitionistic Fuzzy sets, *Fuzzy Sets and Systems*, 20(1) (1986), 87-96.

[2] Black, M., Vagueness, Phil. of Science, 4 (1937) 427-455. Reprinted in Int. J. of General Systems, 17 (1990), 107-128.

[3] S. L. Chang, Fuzzy topological spaces, *Journal of Mathematical Analysis and Applications*, 24(1) (1968), 182-190.

[4] B. C. Cuong, Picture Fuzzy sets, *Journal of Computer Science and Cybernetics*, 30(4) (2014), 409-420.

[5] F. Dernoncourt, Fuzzy logic: Between human reasoning and Artificial Intelligence, *Report, Ecole Normale Supperieure, Paris*, (2011). Retrieved from: https://www.researchgate.net/publication/235333084-Fuzzy-logic-between-human-reasoning-and-artificial-intelligence

[6] J. Deng, Control problems of grey systems, *Systems and Control Letters*, (1982), 288-294.

[7] J. Deng, Introduction to grey system theory, *The Journal of Grey System*, 1 (1989), 1-24.

[8] D. Dubois and H. Prade, Interval-Valued Fuzzy Sets, Possibility Theory and Imprecise Probability, *Proceedings EUSFLAT-LFA*, (2005), 314-319. Retrieved from: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.207.932rep=rep1type=pdf

[9] J. Fox, Towards a reconciliation of fuzzy logic and standard logic, *Int. J. of Man-Machine Studies*, 15 (1981), 213-220.

[10] S. Haack, Do we need fuzzy logic? *Int. J. of Man-Machine Studies*, 11 (1979), 437-445.

[11] J.-S. R. Jang, ANFIS: adaptive network-based fuzzy inference system, *IEEE Transactions on Systems, Man and Cybernetics*, 23(3) (1993), 665-685.

[12] E. T. Jaynes, Probability Theory: The Logic of Science, *Cambridge University Press, UK, 8th Printing*, (2011), (first published, 2003).

[13] A. Kaufmann and M. Gupta, Introduction to Fuzzy Arithmetic, *Van Nostrand Reinhold Company, New York*, (1991).

[14] A. Kharal and B. Ahmad, Mappings on Soft Classes, *New Mathematics and Natural Computation* , 7(3) (2011), 471-481.

[15] G. J. Klir and T. A. Folger, Fuzzy Sets, *Uncertainty and Information, Prentice-Hall, London*, (1988).

[16] S. Korner, Laws of Thought, *In Encyclopedia of Philosophy; Mac Millan: New York, NY, USA.*, 4 (1967), 414-417.

[17] B. Kosko, Fuzzy Thinking: The New Science of Fuzzy Logic, *Hyperion: New York*, (1993).

[18] B. Kosko, Fuzziness Vs Probability, *Int. J. of General Systems*, 17(2-3) (1990), 211-240.

[19] S. F. Liu and Y. Lin. (Eds.), Advances in Grey System Research, *Berlin-Heidelberg: Springer*, (2010).

[20] P. K. Maji, R. Biswas, and A. R. Roy, Soft Set Theory, *Computers and Mathematics with Applications*, 45 (2003), 555-562.

[21] E. H. Mamdani and S. Assilian, An experiment in linguistic synthesis with a fuzzy logic controller, *Int. J. of Man-Machine Studies*, 7(1) (1975), 1-13.

[22] J. M. Mendel, Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions, *Prentice-Hall, Upper-Saddle River, NJ*, (2001).

[23] J. M. Mendel, Fuzzy Sets for Words: a New Beginning, *Proc. IEEE FUZZ Conference, St. Louis, MO*, May 26-28, (2003), 37-42.

[24] D. Molodtsov, Soft Set Theory-First Results, *Computers and Mathematics with Applications*, 37(4-5) (1999), 19-31.

[25] R. A. Moore, R. B. Kearfort and M. J. Clood, Introduction to Interval Analysis, *2nd Printing, Philadelphia, SIAM*, (1995).

[26] D. Mumford, The Dawing of the Age of Stochasticity, *in V. Amoid, M. Atiyah, P. Laxand* & *B. Mazur (Eds.), Mathematics: Frontiers and Perspectives, AMS*, (2000), 197-218.

[27] A. P. Paplinski, Neuro-Fuzzy Computing, *Lecture Notes, Monash University, Australia*, (2005).

[28] Z. Pawlak, Rough Sets: Aspects of Reasoning about Data, *Kluer Academic Publishers, Dordrecht,* (1991).

[29] D. Ramot, R. Milo, M. Friedman and A. Kandel, Complex fuzzy set, *IEEE Transactions on Fuzzy Systems*, 10 (2002), 171-186.

[30] F. Smarandache, Neutrosophy/Neutrosophic probability, *set, and logic, Proquest, Michigan, USA*, (1998).

[31] F. Smarandache, Indeterminancy in Neutrosophic Theories and their Applications, *International Journal of Neutrosophic Science*, 15(2) (2021), 89-97.

[32] M. Sugeno, Industrial applications of fuzzy control, *Elsevier Science Pub. Co.*, (1985).

[33] B. K. Tripathy and K. R. Arun, Soft Sets and Its Applications, *J. S. Jacob (Ed.), Handbook of Research on Generalized and Hybrid Set Structures and Applications for Soft Computing, IGI Global, Hersey, PA*, (2016), 65-85.

[34] V. Torra and Y. Narukawa, On hesitant fuzzy sets and decision, *18th IEEE Int. Conference on Fuzzy Systems, Jeju island, Korea*, 544 (2009), 1378-1382.

[35] E. Van Broekhoven, and B. De Baets, Fast and accurate centre of gravity defuzzification of fuzzy systems outputs defined on trapezoidal fuzzy partitions, *Fuzzy Sets Syst.*, 157 (2006), 904-918.

[36] M. Gr. Voskoglou, Finite Markov Chain and Fuzzy Logic Assessment Models: Emerging Research and Opportunities, *Create Space Independent Publishing Platform, Amazon, Columbia, SC., USA.*, (2017).

[37] M. Gr. Voskoglou, Methods for Assessing Human-Machine Performance under Fuzzy Conditions, *Mathematics*, 7(3), article 230, (2019). DOI:10.3390/math7030230

[38] M. Gr. Voskoglou and E. Athanassopoulos, The Importance of Bayesian Reasoning in Everyday Life and Science, *Int. J. of Education, Development, Society and Technology*, 8(2) (2020), 24-33.

[39] M. Gr. Voskoglou, Fuzzy Control Systems, *WSEAS Transactions on Systems*, 19 (2020), 295-300. DOI: 10.37394/23202.2020.19.33

[40] M. Gr. Voskoglou, Use of Soft Sets and the Bloom's Taxonomy for Assessing Learning Skills, *Transactions on Fuzzy Sets and Systems*, 1(1) (2022), 106-113.

[41] M. Gr. Voskoglou, A Combined Use of Soft Sets and Grey Numbers in Decision Making, *Journal of Computational and Cognitive Engineering https://doi.org/10.47852/bonviewjcce2202237*, (2022).

[42] I. G. Umbers and P. J. King, An analysis of human decision-making in cement kiln control and the implications for automation, *Int. J. of Man-Mach. Stud.*, 12 (1980), 11-23.

[43] R. R. Yager, Pythagorean fuzzy subsets, *in Proceedings of Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada*, (2013), 57-61.

[44] L. A. Zadeh, Fuzzy Sets, *Information and Control*, 8 (1965), 338-353.

[45] L. A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, *IEEE Trans. Syst. Man Cybern.*, 3 (1980), 28-44.

[46] L. A. Zadeh, The Concept of a Linguistic Variable and its Application to Approximate Reasoning, *Information Science*, 8 (1975), 199-249.

[47] L. A. Zadeh, Fuzzy logic = computing with words, *IEEE Trans. on Fuzzy Systems*, 4 (1996), 103-111.

[48] Z. Zhang and Z. Hu, Extension of TOPSIS to Multiple Criteria Decision Making with Pythagorean Fuzzy Sets, *Int. J. of Information Systems*, 29(12) (2014), 1061-1078.

Michael Gr. Voskoglou
Department of Mathematics
School of Technological Applications
Patras, Greece
E-mail: voskoglou@teiwest.gr, mvoskoglou@gmail.com