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# On Quantum-MV algebras-Part II: Orthomodular Lattices, Softlattices and Widelattices

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**Abstract.** Orthomodular lattices generalize the Boolean algebras; they have arisen in the study of quantum logic. Quantum-MV algebras were introduced as non-lattice theoretic generalizations of MV algebras and as non-idempotent generalizations of orthomodular lattices.

In this paper, we continue the research in the "world" of involutive *algebras* of the form $(A, \odot, ^-, 1)$, with $1^- = 0$, 1 *being the last element*. We clarify now some aspects concerning the quantum-MV (QMV) algebras as non-idempotent generalizations of orthomodular lattices. We study in some detail the orthomodular lattices (OMLs) and we introduce and study two generalizations of them, the orthomodular softlattices (OMSLs) and the orthomodular widelattices (OMWLs). We establish systematically connections between OMLs and OMSLs/OMWLs and QMV, pre-MV, metha-MV, orthomodular algebras and ortholattices, orthosoftlattices/orthowidelattices - connections illustrated in 22 Figures. We prove, among others, that the transitive OMLs coincide with the Boolean algebras, that the OMSLs coincide with the OMLs, that the OMLs are included in OMWLs and that the OMWLs are a proper subclass of QMV algebras. The transitive and/or the antisymmetric case is also studied.

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**Keywords and Phrases:** m-MEL algebra, m-BE algebra, m-pre-BCK algebra, m-BCK algebra, MV algebra, Quantum-MV algebra, Pre-MV algebra, Metha-MV algebra, Orthomodular algebra, Ortholattice, Orthosoftlattice, Orthowidelattice, Boolean algebra

## 1 Introduction

The *algebraists* work usually with the commutative additive groups and with the positive (right) cone of a partially-ordered commutative group $(G, \leq, +, -, 0)$, where there are essentially a sum $\oplus = +$ and an element 0. Sometimes, the negative (left) cone is needed also, where there are essentially a product $\odot = +$ and an element $1 = 0$. They work with algebras that have associated an (pre-order) order relation, which usually does not appear explicitly in the definitions. The presence of the (pre-order) order relation implies the presence of the (generalized) duality principle. Thus, each algebra has a dual one, the (pre-order) order relation has a dual one. We have given names to the dual algebras [14], [16], [15]: "left" algebra and "right" algebra, names connected with the left-continuity of a t-norm and with the right-continuity of a t-conorm, respectively. Hence, the algebraists usually work with the commutative *right-unital magmas*.

By the contrary, the *logicians* work with the logic of *truth*, where the *truth* is represented by 1, and there is essentially one implication; we could name this logic "left-logic". One can imagine also a "right-logic", as

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a logic of *false*, where the *false* is represented by 0 and there is a "right-implication". Hence, the logicians usually work with the commutative *left-algebras of logic*.

In this paper, *regarding from (algebras of) logic side, we shall work with left-algebras (left-unital magmas) as principal algebras*, therefore, the unital magmas will be defined multiplicatively.

Thus, the commutative algebraic structures connected directly or indirectly with classical/ nonclassical logics belong to two parallel "worlds":

1. the "world" of *(left) algebras of logic*, where there are essentially one implication, $\to$ (two, in the non-commutative case), and an element 1 (that can be the last element); the algebras $(A, \to, 1)$, verifying the basic property (M): $1 \to x = x$, are called *M algebras* [16], [15]; an internal binary relation can be defined by: $x \leq y \overset{def.}{\Longleftrightarrow} x \to y = 1$ ($\leq$ can be a pre-order, an order, or even a lattice order); algebras belonging to this "world" are [17], [16], [15]: the bounded MEL, BE and aBE, pre-BCK algebras, BCK algebras, bounded BCK algebras, BCK(P) algebras, Hilbert algebras, Wajsberg algebras, implicative-Boolean algebras, etc. A "Big map" (hierarchy of algebras of logic) is presented in ([15], Figure 1).

2. the "world" of *(left) algebras*, where there are essentially a product, $\odot$, and an element 1 (that can be the last element); the algebras $(A, \odot, 1)$, verifying the corresponding basic properties (PU): $1 \odot x = x$ and (Pcomm): $x \odot y = y \odot x$, are called *commutative unital magmas*; in algebras with an additional operation, $(A, \odot, ^-, 1)$, an internal binary relation can be defined by: $x \leq_m y \Longleftrightarrow x \odot y^- = 0$ ($\leq_m$ can be a pre-order, an order, or even a lattice order), where 'm' comes from 'magma'; algebras belonging to this "world" are [16], [15]: the m-MEL, m-BE and m-aBE, m-pre-BCK algebras, m-BCK algebras, pocrims, (bounded) lattices, residuated lattices, BL algebras, MTL algebras, NM algebras, MV algebras, Boolean algebras, etc. A corresponding "Big map" (hierarchy of algebras) is presented in ([15], Figure 10).

Between the two parallel "worlds" there are some connections, as for example: the equivalence between BCK(P) algebras and pocrims, in the non-involutive case, and the definitional equivalence between Wajsberg algebras and MV algebras, in the involutive case ($(x^-)^- = x$). In [15], Theorems 9.1 and 9.3 connect the two "worlds" in the involutive case.

Beside the classical and non-classical logics, there exist the quantum logics. Examples of algebraic structures connected with quantum logics (= quantum structures/ algebras) are the bounded implicative (implication) lattices, the De Morgan algebras, the ortholattices, the orthomodular lattices, the quantum-MV algebras, etc.

*The ortholattice* is an important example of *sharp* structure (which satisfies the noncontradiction principle) from sharp quantum theory [4] (Birkhoff, 1967; Kalmbach, 1983).

Orthomodular lattices (particular ortholattices) generalize the Boolean algebras. They have arisen, cf. [25], "in the study of quantum logic, that is, the logic which supports quantum mechanics and which does not conform to classical logic. As noted by Birkhoff and von Neumann in 1936 [2], the calculus of propositions in quantum logic "is formally indistinguishable from the calculus of linear subspaces [of a Hilbert space] with respect to set products, linear sums and orthogonal complements" in the role of *and*, *or* and *not*, respectively. This has led to the study of the closed subspaces of a Hilbert space, which form an orthomodular lattice in contemporary terminology. As often happens in algebraic logic, the study of orthomodular lattices has tremendously developed, both for their interest in logic and for their own sake, see Kalmbach [23]".

Quantum-MV algebras (or QMV algebras) were introduced by Roberto Giuntini in [11] (see also [9], [8], [12], [10], [13], [7], [6]), as non-lattice theoretic generalizations of MV algebras and as non-idempotent generalizations of orthomodular lattices.

The connections between algebras of logic/ algebras and quantum algebras were not very clear. But, in papers [15], [20], [21], we established important connections, by redefining equivalently the bounded involutive lattices and De Morgan algebras as involutive *m-MEL algebras*, the ortholattices, the MV, the Boolean algebras and the quantum-MV algebras as involutive *m-BE algebras*, verifying some properties, and then putting all of them on the involutive "Big map"; thus, we have proved that the quantum algebras belong, in fact, to the "world" of *algebras* (involutive commutative unital magmas).

In this paper, we continue the research from [21], [18], based on [22], [20], [15], in the "world" of involutive *algebras* of the form $(A, \odot, ^-, 1)$, with $1^- = 0$, 1 *being the last element*. We clarify now some aspects concerning the quantum-MV algebras as non-idempotent generalizations of orthomodular lattices. We study the orthomodular lattices and we introduce and study two generalizations of them, the orthomodular softlattices and the orthomodular widelattices - in connection with the lattices/ ortholattices and their two generalizations, the softlattices/ orthosoftlattices and the widelattices/ orthowidelattices, generalizations introduced in [22]. Many results were obtained by the powerful computer program *Prover*9/*Mace*4 (version DEC. 2007) created by William W. McCune $(1953 - 2011)$ [24]. By lack of space, we shall not present here the examples we have. This paper, like [15], [20], [22], [21], [18], presents the facts in the same *unifying way*, which consists in fixing unique names for the defining properties, making lists of these properties and then using them for defining the different algebras and for obtaining results.

The paper is organized as follows.

In Section 2 (**Preliminaries**), we recall the notions and the results necessary for making the paper self-contained as much as possible.

In Section 3 (**Orthomodular lattices**), we study in some detail the orthomodular lattices (OMLs), that are QMV algebras. We establish connections between OMLs and QMV, pre-MV, metha-MV, orthomodular (OM) algebras and ortholattices (OLs), connections illustrated in Figures $3 - 8$. We prove that the antisymmetric OMLs and the transitive OMLs coincide with the Boolean algebras and that transitive OLs are included in transitive metha-MV algebras. We introduce the new notion of *modular algebra* and we prove that the modular algebras coincide with the modular ortholattices.

In Section 4 (**Orthomodular softlattices, widelattices**), based on the two generalizations of OLs: the orthosoftlattices (OSLs) and the orthowidelattices (OWLs), introduced in [22], we introduce and study, in separate subsections, two corresponding generalizations of OMLs: the orthomodular softlattices (OMSLs) and the orthomodular widelattices (OMWLs). We establish connections between OMSLs/OMWLs and QMV, pre-MV, metha-MV, OM algebras and OLs, OSLs/OWLs, connections illustrated in Figures $9 - 15/16 - 22$, respectively. We prove that the OMLs coincide with the OMSLs and that transitive OSLs are included in transitive metha-MV algebras. We also prove that the OMLs are included in OMWLs, which in turn are included in QMV algebras too, and that transitive OWLs are included in transitive metha-MV algebras, hence that transitive OMWLs are included in transitive QMV algebras.

## 2 Preliminaries

### 2.1 The "Big map" of algebras

Recall from [15] the following:

Let $\mathcal{A}^L = (A^L, \odot, ^- = ^{-^L}, 1)$ be an algebra of type $(2, 1, 0)$ and define $0 \overset{def.}{=} 1^-$. Define an *internal* binary relation $\leq_m$ on $A^L$ by: for all $x, y \in A^L$,

(m-dfrelP) $\quad x \leq_m y \overset{def.}{\Longleftrightarrow} x \odot y^- = 0.$

Consider the following list **m-A** of basic properties that can be satisfied by $\mathcal{A}^L$ [15]:

(PU) $\quad 1 \odot x = x = x \odot 1$ (unit element of product, the *identity*),
(Pcomm) $\quad x \odot y = y \odot x$ (commutativity of product),
(Pass) $\quad x \odot (y \odot z) = (x \odot y) \odot z$ (associativity of product);

(Neg1-0) $\quad 1^- = 0$,
(Neg0-1) $\quad 0^- = 1$;

(m-An) $(x \odot y^- = 0$ and $y \odot x^- = 0) \Longrightarrow x = y$ (antisymmetry),
(m-B) $[(x \odot y^-)^- \odot (x \odot z)] \odot (y \odot z)^- = 0$,
(m-BB) $[(z \odot x)^- \odot (y \odot x)] \odot (y \odot z^-)^- = 0$,
(m-*) $x \odot y^- = 0 \Longrightarrow (z \odot y^-) \odot (z \odot x^-)^- = 0$,
(m-**) $x \odot y^- = 0 \Longrightarrow (x \odot z) \odot (y \odot z)^- = 0$,
(m-L) $x \odot 0 = 0$ (last element),
(m-Re) $x \odot x^- = 0$ (reflexivity),
(m-Tr) $(x \odot y^- = 0$ and $y \odot z^- = 0) \Longrightarrow x \odot z^- = 0$ (transitivity),
etc.

Dually, let $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-R}, 0)$ be an algebra of type $(2, 1, 0)$ and define $1 \stackrel{def.}{=} 0^-$. Define an *internal* binary relation $\geq_m$ on $A^R$ by: for all $x, y \in A^R$,

(m-dfrelS) $x \geq_m y \stackrel{def.}{\Longleftrightarrow} x \oplus y^- = 1$.

The list of dual properties is omitted.

Recall from [15] the definitions of the following algebras needed in this paper (the dual ones are omitted):

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an algebra of type $(2, 1, 0)$ through this paper. Define $0 \stackrel{def.}{=} 1^-$ (hence (Neg1-0) holds) and suppose that $0^- = 1$ (hence (Neg0-1) holds too). We say that $\mathcal{A}^L$ is a [15]:

- *left-m-MEL algebra*, if (PU), (Pcomm), (Pass), (m-L) hold;
- *left-m-BE algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) hold;
- *left-m-pre-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) and (m-BB) hold;
- *left-m-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An) and (m-BB) hold.

Denote by **m-MEL**, **m-BE**, **m-pre-BCK**, **m-BCK** these classes of left-algebras, respectively.

In ([15], Figure 10), the "Big map", connecting the commutative unital magmas, including these algebras, was drawn.

We say that $\mathcal{A}^L$ is [15] *reflexive*, if $\leq_m$ is reflexive (i.e. (m-Re) holds); *transitive*, if $\leq_m$ is transitive (i.e. (m-Tr) holds); *antisymmetric*, if $\leq_m$ is antisymmetric (i.e. (m-An) holds). If **X** is a class of algebras, we shall denote by **tX** (**aX**, **atX**=**taX**) the subclass of all transitive (antisymmetric, transitive and antisymmetric, respectively) algebras of **X**.

We say that an algebra is *involutive*, if it verifies (DN) ($(x^-)^- = x$ or $x^= = x$). If **X** is a class of algebras, we shall denote by $\mathbf{X}_{(DN)}$ the subclass of all involutive algebras of **X**. By ([15], Theorem 6.12), in any involutive m-BE algebra we have the equivalences: (m-BB) ⇔ (m-B) ⇔ (m-**) ⇔ (m-*) ⇔ (m-Tr).

Note that: $\mathbf{m\text{-}pre\text{-}BCK}_{(DN)} = \mathbf{pre\text{-}m\text{-}BCK}_{(DN)}$ $(= \mathbf{m\text{-}tBE}_{(DN)})$.

Any left-m-BCK algebra is involutive, by ([15], Theorem 6.13). We write: $\mathbf{m\text{-}BCK} = \mathbf{m\text{-}BCK}_{(DN)}$ $(= \mathbf{m\text{-}taBE}_{(DN)})$. Note that a (involutive) m-BCK algebra satisfies all the properties in the list **m-A** of properties and, additionally, (DN) and other properties.

Note that the binary relation $\leq_m$ is only **reflexive** in $\mathbf{m\text{-}BE}_{(DN)}$, it is a **pre-order** in $\mathbf{m\text{-}pre\text{-}BCK}_{(DN)}$ and it is an **order** in **m-BCK**.

### 2.1.1 Involutive m-MEL algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. Because of the axiom (DN), we have introduced in [20] the new operation sum, $\oplus$, the dual of product, $\odot$, by: for all $x, y \in A^L$,

$$x \oplus y \stackrel{def.}{=} (x^- \odot y^-)^-. \tag{1}$$

Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-MEL algebra.

**Proposition 2.1.** *(See ([6], Proposition 2.1.2), in dual case, [9])* *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. We have:*

$$0 \oplus x = x = x \oplus 0, \quad i.e. \quad (SU) \quad holds, \tag{2}$$

$$x \oplus y = y \oplus x, \quad i.e. \quad (Scomm) \quad holds, \tag{3}$$

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \quad i.e. \quad (Sass) \quad holds, \tag{4}$$

$$x \oplus 1 = 1, \quad i.e. \quad (m - L^R) \quad holds; \tag{5}$$

$$(x \oplus y)^- = x^- \odot y^- \quad (De\ Morgan\ law\ 1), \tag{6}$$

$$(x \odot y)^- = x^- \oplus y^- \quad (De\ Morgan\ law\ 2),\ and\ hence \tag{7}$$

$$x \odot y = (x^- \oplus y^-)^-. \tag{8}$$

Beside the old, natural binary relation $\leq_m$ and its dual $\geq_m$, we have introduced in [20] a new binary relation:
(m-dfP) $x \ \leq^P_m \ y \overset{def.}{\Longleftrightarrow} x \odot y = x$ and, dually,
(m-dfS) $x \ \geq^S_m \ y \overset{def.}{\Longleftrightarrow} x \oplus y = x$.
By ([20], Proposition 3.11), $\leq^P_m$ is antisymmetric and transitive and $0 \leq^P_m x \leq^P_m 1$, for any $x$.

**Proposition 2.2.** *([20], Proposition 3.14)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. If (m-Pimpl) holds, then:*
*(1) the order relation $\leq^P_m$ is a lattice order (denoted by $\leq^O_m$),*
*(2) $x \leq^P_m y \Longleftrightarrow y \geq^S_m x$,*
*(3) $x \leq^P_m y \Longrightarrow y^- \leq^P_m x^-$.*

With the notations from this subsection, the definition of MV algebras [3], [5] becomes [15]:

**Definition 2.3.** *(The dual one is omitted)*
*A left-MV algebra is an algebra $\mathcal{A}^L = (A^L, \odot, ^- = ^{-L}, 1)$ of type $(2, 1, 0)$ verifying (PU), (Pcomm), (Pass), (m-L), (DN) and:*
*($\wedge_m$-comm) $(x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x$.*

We recall the following important remark, which was the motivation of paper [15]:
The left-MV algebra is just the involutive left-m-MEL algebra verifying ($\wedge_m$-comm).
We have denoted by **MV** the class of all left-MV algebras.

### 2.1.2 Involutive m-BE algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-BE algebra.

**Remark 2.4.** *(See ([15], Theorem 6.21 ) (The dual one is omitted)*
*Since ($\wedge_m$-comm) implies (m-Re), by ([15], (mB1)), it follows that* **any left-MV algebra is in fact an involutive left-m-BE algebra verifying ($\wedge_m$-comm)**. *And since ($\wedge_m$-comm) implies also (m-An) and (m-BB) ($\Longleftrightarrow$ ... (m-Tr)), by ([15], (mB2), (mCBN1)), respectively, it follows that* **any left-MV algebra is in fact a left-m-BCK algebra**, *i.e. we have:*

$$\mathbf{MV} \ \subset \ \mathbf{m-BCK} = \mathbf{m-BCK_{(DN)}} \ (= \mathbf{m-taBE_{(DN)}}).$$

We have introduced in [21], in an involutive left-m-MEL algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$, the following new operations:

$$x \wedge_m^M y \overset{def.}{=} (x^- \odot y)^- \odot y \overset{(Pcomm)}{=} y \odot (y \odot x^-)^- \quad and,\ dually, \tag{9}$$

$$x \vee_m^M y \overset{def.}{=} (x^- \wedge_m^M y^-)^- = [(x \odot y^-)^- \odot y^-]^- = y \oplus (y \oplus x^-)^- \tag{10}$$

and

$$x \wedge_m^B y \overset{def.}{=} (y^- \odot x)^- \odot x \overset{(Pcomm)}{=} x \odot (x \odot y^-)^- = y \wedge_m^M x \quad and,\ dually, \tag{11}$$

$$x \vee_m^B y \overset{def.}{=} (x^- \wedge_m^B y^-)^- = ((y \odot x^-)^- \odot x^-)^- = x \oplus (x \oplus y^-)^- = y \vee_m^M x. \tag{12}$$

**Proposition 2.5.** *(See [6], Proposition 2.1.2, in dual case) ([21], Proposition 3.2)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. We have:*

$$x \wedge_m^M 1 = x = 1 \wedge_m^M x, \quad x \wedge_m^M 0 = 0, \tag{13}$$

$$x \vee_m^M 0 = x = 0 \vee_m^M x, \quad x \vee_m^M 1 = 1, \tag{14}$$

$$(x \vee_m^M y)^- = x^- \wedge_m^M y^- \quad (De\ Morgan\ law\ 1), \tag{15}$$

$$(x \wedge_m^M y)^- = x^- \vee_m^M y^- \quad (De\ Morgan\ law\ 2),\ and\ hence \tag{16}$$

$$x \wedge_m^M y = (x^- \vee_m^M y^-)^-. \tag{17}$$

**Proposition 2.6.** *(See ([6], Proposition 2.1.2), in dual case) ([21], Proposition 3.3)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:*

$$if \quad x \odot y = 1, \quad then \quad x = y = 1; \tag{18}$$

$$if \quad x \wedge_m^M y = 1, \quad then \quad x = y = 1, \tag{19}$$

$$0 \wedge_m^M x = 0, \tag{20}$$

$$1 \vee_m^M x = 1, \tag{21}$$

$$x \wedge_m^M x = x, \quad x \vee_m^M x = x. \tag{22}$$

Beside the old, natural binary relation $\leq_m$ and its dual $\geq_m$, we have introduced in [21] two new binary relations: for all $x, y \in A^L$,
(m-dfWM) $x \ \leq_m^M \ y \overset{def.}{\Longleftrightarrow} x \wedge_m^M y = x$ and, dually,
(m-dfVM) $x \ \geq_m^M \ y \overset{def.}{\Longleftrightarrow} x \vee_m^M y = x$,
and
(m-dfWB) $x \ \leq_m^B \ y \overset{def.}{\Longleftrightarrow} x \wedge_m^B y = x$ ($\Longleftrightarrow y \wedge_m^M x = x$) and, dually,
(m-dfVB) $x \ \geq_m^B \ y \overset{def.}{\Longleftrightarrow} x \vee_m^B y = x$ ($\Longleftrightarrow y \vee_m^M x = x$).

**Proposition 2.7.** *([21], Proposition 3.6)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:*
*(1)* $x \leq_m y \Longleftrightarrow x \leq_m^B y$ *and, dually*
*(1')* $x \geq_m y \Longleftrightarrow x \geq_m^B y$.
*(2) If ($\wedge_m$-comm) holds (i.e. $x \wedge_m^M y = y \wedge_m^M x$), then*
$x \leq_m y$ ($\Longleftrightarrow x \leq_m^B y$) $\Longleftrightarrow x \leq_m^M y$.
*(2') If ($\wedge_m$-comm) holds, then ($\vee_m$-comm) holds (i.e. $x \vee_m^M y = y \vee_m^M x$) and*
$x \geq_m y$ ($\Longleftrightarrow x \geq_m^B y$) $\Longleftrightarrow x \geq_m^M y$.

**Remark 2.8.** *([21], Remark 3.7)*

*The equivalence $\leq_m \iff \leq_m^B$ implies that $\leq_m$ is an order relation if and only if $\leq_m^B$ is an order relation. But, it does not imply that if $\leq_m$ is a lattice order, then $\leq_m^B$ is a lattice order too with respect to $\wedge_m^B, \vee_m^B$ - see the examples in the last section.*

**Corollary 2.9.** *(See [6], Corollary 2.1.3 and [21], Corollary 3.9)*

*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, the binary relation $\leq_m^M$ is reflexive and antisymmetric and $0 \leq_m^M x \leq_m^M 1$, for all $x \in A^L$, where $0 \stackrel{def.}{=} 1^-$.*

## 2.2 Ortholattices, orthosoftlattices and orthowidelattices

**Definition 2.10.** *An algebra $\mathcal{A} = (A, \wedge, \vee)$ or, dually, $\mathcal{A} = (A, \vee, \wedge)$, of type $(2,2)$, will be said to be a (Dedekind) lattice, if the following properties hold [1]: for all $x, y, z \in A$,*

| | | |
|---|---|---|
| (m-Wid) | (idempotency of $\wedge$) | $x \wedge x = x$, |
| (m-Wcomm) | (commutativity of $\wedge$) | $x \wedge y = y \wedge x$, |
| (m-Wass) | (associativity of $\wedge$) | $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, |
| (m-Wabs) | (absorption of wedge over vee) | $x \wedge (x \vee y) = x$, and also |
| (m-Vid) | (idempotency of $\vee$) | $x \vee x = x$, |
| (m-Vcomm) | (commutativity of $\vee$) | $x \vee y = y \vee x$, |
| (m-Vass) | (associativity of $\vee$) | $(x \vee y) \vee z = x \vee (y \vee z)$, |
| (m-Vabs) | (absorption of vee over wedge) | $x \vee (x \wedge y) = x$, |

*where "W" comes from "wedge" (the LaTeX command for the meet symbol) and "V" comes from "vee" (the LaTeX command for the join symbol).*

*Moreover, if there exist $0, 1 \in A$ such that: for all $x \in A$,*
*(m-WU) $1 \wedge x = x$ and, dually,*
*(m-VU) $0 \vee x = x$,*
*then $\mathcal{A}$ is said to be a* bounded *(Dedekind) lattice (with last element 1 and first element 0) and is denoted by $\mathcal{A} = (A, \wedge, \vee, 0, 1)$ or, dually, by $\mathcal{A} = (A, \vee, \wedge, 0, 1)$.*

**Naming convention for the dual lattices:** $(A, \wedge, \vee)$ is the *left-lattice* and $(A, \vee, \wedge)$ is the *right-lattice* (names coming from the *left-continuity* of a t-norm and the *right-continuity* of a t-conorm; see more on *left- and right- algebras* in [14]).

We have analysed the ortholattices in [15], [20]. Recall the following definition:

**Definition 2.11.** *(See [25], [4]) (Definition 1) (The dual one is omitted)*

*A left-ortholattice, or a left-OL for short, is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, ^- = ^{-^L}, 0, 1)$ such that the reduct $(A^L, \wedge, \vee, 0, 1)$ is a bounded (Dedekind) left-lattice and the unary operation $^-$ satisfies (DN), (DeM1) ($(x \vee y)^- = x^- \wedge y^-$), (DeM2) ($(x \wedge y)^- = x^- \vee y^-$) and the complementation laws:*
*(m-WRe) $x \wedge x^- = 0$ (noncontradiction principle) and, dually,*
*(m-VRe) $x \vee x^- = 1$ (excluded middle principle).*

We have denoted by **OL** the class of all left-ortholattices.

Since, in a lattice, the absorption laws (m-Wabs) and (m-Vabs) are not independent (they imply the idempotency laws (m-Wid) and (m-Vid)), we have introduced in [22] the following two dual *independent absorption laws*:

(m-Wabs-i) $x \wedge (x \vee x \vee y) = x$ and, dually,
(m-Vabs-i) $x \vee (x \wedge x \wedge y) = x$ (dual laws of independent absorption).

We have proved that the system of eight axioms: **L8-i** = {(m-Wid), (m-Vid), (m-Wcomm), (m-Vcomm), (m-Wass), (M-Vass), (m-Wabs-i), (m-Vabs-i)} is equivalent with the "standard" system **L8** of axioms for lattices from Definition 2.10 ([22], Theorem 3.2).

We have then introduced in [22] the following two generalizations of lattices/ bounded lattices.

**Definition 2.12.** *(The dual ones are omitted) ([22], Definition 3.3)*
*(1) A left-softlattice is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee)$ of type $(2,2)$ such that the axioms (m-Wid), (m-Vid), (m-Wcomm), (m-Vcomm), (m-Wass), (m-Vass) are satisfied.*
*(2) A bounded left-softlattice is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, 0, 1)$ of type $(2,2,0,0)$ such that the reduct $(A^L, \wedge, \vee)$ is a left-softlattice and the elements 0 and 1 verify the axioms: for all $x \in A^L$,*
(m-WU) $1 \wedge x = x$, (m-VU) $0 \vee x = x$,
(m-WL) $0 \wedge x = 0$, (m-VL) $1 \vee x = 1$.

**Definition 2.13.** *(The dual ones are omitted) ([22], Definition 3.9)*
*(1') A left-widelattice is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee)$ of type $(2,2)$ such that the axioms (m-Wcomm), (m-Vcomm), (m-Wass), (m-Vass), (m-Wabs-i), (m-Vabs-i) are satisfied.*
*(2') A bounded left-widelattice is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, 0, 1)$ of type $(2,2,0,0)$ such that the reduct $(A^L, \wedge, \vee)$ is a left-widelattice and the elements 0 and 1 verify the axioms: for all $x \in A^L$,*
(m-WU) $1 \wedge x = x$, (m-VU) $0 \vee x = x$.

We have introduced in [22] the following two generalizations of OLs.

**Definition 2.14.** *(Definition 1) (The dual one is omitted) ([22], Definition 5.1)*
*A left-orthosoftlattice, or a left-OSL for short, is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, ^- = ^{-^L}, 0, 1)$ such that the reduct $(A^L, \wedge, \vee, 0, 1)$ is a bounded left-softlattice (Definition 2.12) and the unary operation $^-$ satisfies (DN), (DeM1), (DeM2) and (m-WRe), (m-VRe).*

**Definition 2.15.** *(Definition 1) (The dual one is omitted) ([22], Definition 5.6)*
*A left-orthowidelattice, or a left-OWL for short, is an algebra $\mathcal{A}^L = (A^L, \wedge, \vee, ^- = ^{-^L}, 0, 1)$ such that the reduct $(A^L, \wedge, \vee, 0, 1)$ is a bounded left-widelattice (Definition 2.13) and the unary operation $^-$ satisfies (DN), (DeM1), (DeM2) and (m-WRe), (m-VRe).*

We have denoted by **OSL** the class of all left-OSLs and by **OWL** the class of all left-OWLs.
Consider the following properties (the dual ones are omitted):
(m-Pimpl) $[(x \odot y^-)^- \odot x^-]^- = x$,
(G) $x \odot x = x$,
(m-Pabs-i) $x \odot (x \oplus x \oplus y) = x$.

**Proposition 2.16.** *([22], Proposition 3.15)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. Then,*

$$(m-Pimpl) \iff (G) + (m-Pabs-i).$$

We have obtained the following equivalent definitions.

**Definition 2.17.** *(Definition 2) (The dual ones are omitted)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. $\mathcal{A}^L$ is a:*
- *left-ortholattice (left-OL), if (m-Pimpl) holds ([20], Definition 4.15),*
- *left-orthosoftlattice (left-OSL), if (G) holds ([22], Definition 5.3),*
- *left-orthowidelattice (left-OWL), if (m-Pabs-i) holds ([22], Definition 5.8),*

*i.e.* **OL** $=$ **m-BE**$_{(DN)}$ + *(m-Pimpl)*, **OSL** $=$ **m-BE**$_{(DN)}$ + *(G)*, **OWL** $=$ **m-BE**$_{(DN)}$ + *(m-Pabs-i)*.

Hence, we have:

$$\mathbf{OL} = \mathbf{OSL} \cap \mathbf{OWL}, \tag{23}$$

i.e. we have the representation from Figure 1, useful in the sequel.

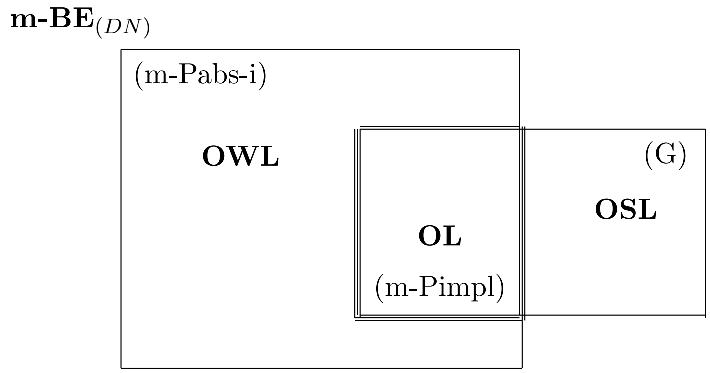


**Figure 1:** Resuming connection between **OSL**, **OWL** and **OL**

**Theorem 2.18.** *([20], Theorem 4.16) We have:* **aOL** = **OL** + *(m-An)* = **Boole**.

Finally, recall that [22]: **taOSL** = **Boole**.

### 2.3 Boolean algebras

**Definition 2.19.** *(Definition 1) (The dual one is omitted)*

*A left-Boolean algebra is a bounded (Dedekind) left-lattice that is distributive and complemented, i.e. is an algebra* $\mathcal{A}^L = (A^L, \wedge, \vee, ^- = ^{-^L}, 0, 1)$ *verifying: (m-Wid), (m-Wcomm), (m-Wass), (m-Wabs), (m-WU), (m-Wdis), (m-WRe) and, dually, (m-Vid), (m-Vcomm), (m-Vass), (m-Vabs), (m-VU), (m-Vdis), (m-VRe), where:*

*(m-Wdis)* $z \wedge (x \vee y) = (z \wedge x) \vee (z \wedge y)$,
*(m-Vdis)* $z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y)$.

We have denoted by **Boole** the class of all left-Boolean algebras.

Consider the following properties (the dual ones are omitted):

(m-Pdiv) $x \odot (x \odot y^-)^- = x \odot y$,
(m-Pdis) $z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y)$.

We have obtained the following equivalent definitions.

**Definition 2.20.** *(Definitions 2 and 3) (The dual ones are omitted)*

*Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra.* $\mathcal{A}^L$ *is a:*
*- left-Boolean algebra, if (m-Pdiv) holds ([20], Definition 4.19) or, equivalently,*
*- left-Boolean algebra, if (m-Pdis) holds ([20], Definition 4.21),*
*i.e.* **Boole** = **m-BE**$_{(DN)}$ + *(m-Pdiv)* = **m-BE**$_{(DN)}$ + *(m-Pdis)*.

### 2.4 QMV algebras. OM, PreMV, MMV algebras. MV algebras

Consider the following properties (the dual ones are omitted):

(Pqmv) $x \odot [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z)$,
(Pom) $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$ or, equivalently, $x \vee_m^M (x \odot y) = x$,
(Pmv) $x \odot ((x^- \odot y^-)^- \odot y^-)^- = x \odot y$ or, equivalently, $x \odot (x^- \vee_m^M y) = x \odot y$,
($\Delta_m$) $(x \wedge_m^M y) \odot (y \wedge_m^M x)^- = 0$.

**Definition 2.21.** *(The dual ones are omitted)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. $\mathcal{A}^L$ is a:*
- *left-quantum-MV algebra (left-QMV algebra), if (Pqmv) holds ([21], Definition 3.10),*
- *left-orthomodular algebra (left-OM algebra), if (Pom) holds ([21], Definition 4.1),*
- *left-pre-MV algebra (left-PreMV algebra), if (Pmv) holds ([21], Definition 4.1),*
- *left-metha-MV algebra (left-MMV algebra), if ($\Delta_m$) holds ([21], Definition 4.1).*

We have denoted by **QMV**, **OM**, **PreMV**, **MMV** the corresponding classes of left-algebras. Hence, we have:
$\mathbf{QMV} = \mathbf{m\text{-}BE}_{(DN)}$ + (Pqmv), $\mathbf{OM} = \mathbf{m\text{-}BE}_{(DN)}$ + (Pom),
$\mathbf{PreMV} = \mathbf{m\text{-}BE}_{(DN)}$ + (Pmv), $\mathbf{MMV} = \mathbf{m\text{-}BE}_{(DN)}$ + ($\Delta_m$).

**Theorem 2.22.** [21] *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*
*(1) (Pqmv) $\iff$ (Pmv) + (Pom), i.e.* $\mathbf{QMV} = \mathbf{PreMV} \cap \mathbf{OM}$,
*(2) (Pmv) $\Longrightarrow$ ($\Delta_m$), i.e.* $\mathbf{PreMV} \subset \mathbf{MMV}$,
*(3) (Pqmv) $\iff$ ($\Delta_m$) + (Pom), i.e.* $\mathbf{QMV} = \mathbf{MMV} \cap \mathbf{OM}$.

The connections between these algebras, and the transitive ones, were established in [21] (see Figure 2).

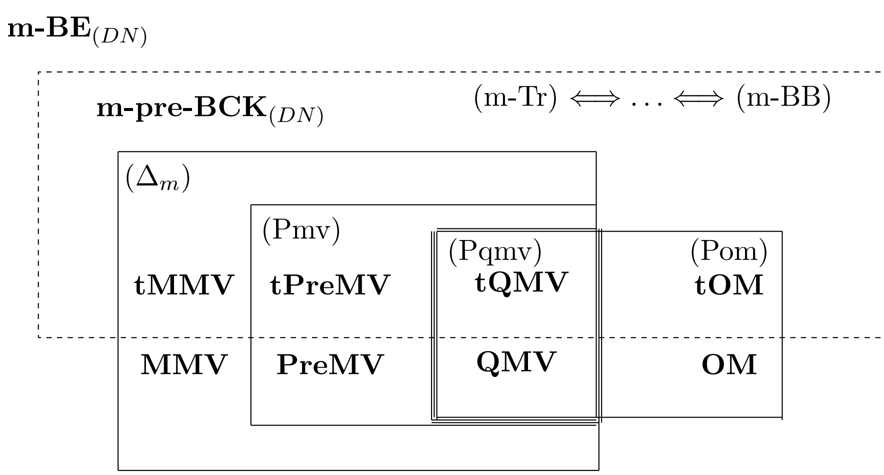


**Figure 2:** Resuming connections between **OM**, **PreMV**, **MMV**, **QMV** and (m-Tr)

**Proposition 2.23.** *([21], Proposition 3.22)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra verifying (G). Then:*
*(1) $\leq_m^P$ is reflexive also, hence it is an* **order relation**.
*(2) We have the equivalence:*
$(x \odot y = x \Longleftrightarrow)\ x \leq_m^P y \Longleftrightarrow x \leq_m^M y\ (\Longleftrightarrow x \wedge_m^M y = x)$.

**Theorem 2.24.** [21] *We have:*
$\mathbf{aPreMV} = \mathbf{aMMV} = \mathbf{aQMV} = \mathbf{atQMV} = \mathbf{taQMV} = \mathbf{MV}$ *and* $\mathbf{MV} \subset \mathbf{taOM}$.

Recall, finally, some properties of OM algebras.

**Proposition 2.25.** *([18], Proposition 3.1)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OM algebra. We have:*

$$x \odot (y \vee_m^M x^-) = x \odot y, \tag{24}$$

$$x \ \leq_m^M \ y \Longrightarrow y^- \ \leq_m^M \ x^- \quad (order-reversibility \ of \ ^-), \tag{25}$$

$$x \ \leq_m^M \ y \Longrightarrow x \oplus z \ \leq_m^M \ y \oplus z \quad (monotonicity \ of \ \oplus), \tag{26}$$

$$x \ \leq_m^M \ y \Longrightarrow x \odot z \ \leq_m^M \ y \odot z \quad (monotonicity \ of \ \odot). \tag{27}$$

**Corollary 2.26.** *([18], Corollary 3.7)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OM algebra. The binary relation $\leq_m^M$ is an order relation.*

## 3 Orthomodular lattices

Recall the following definition [25].

**Definition 3.1.** *(Definition 1) (The dual one is omitted)*
*A left-orthomodular lattice or an orthomodular left-lattice, or a left-OML for short, is a left-OL $\mathcal{A}^L = (A^L, \wedge, \vee, ^-, 0, 1)$ verifying: for all $x, y \in A^L$,*
*(Wom)* $(x \wedge y) \vee ((x \wedge y)^- \wedge x) = x$.

Denote by **OML** the class of all left-OMLs .

Following the equivalent Definition 2 of a left-OL (see Definition 2.17), we obtain immediately the equivalent definition:

**Definition 3.2.** *(Definition 2) (The dual one is omitted)*
*A left-orthomodular lattice (left-OML) is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying (m-Pimpl) and (Pom), i.e.*

$$\mathbf{OML} = \mathbf{m-BE_{(DN)}} \ + \ (m-Pimpl) \ + \ (Pom) \ = \ \mathbf{OL} \ \cap \ \mathbf{OM}. \tag{28}$$

Further, we shall work with Definition 2 of left-OMLs. Hence, we have the connections from Figure 3.

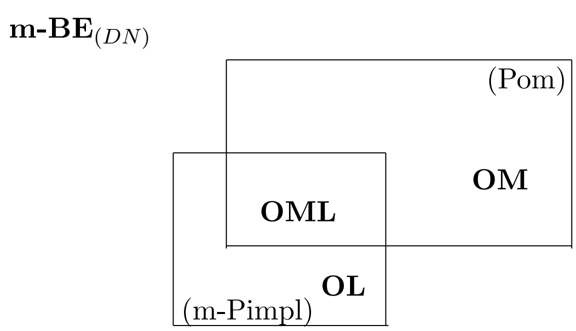


**Figure 3:** Resuming connections between **OL**, **OML** and **OM**

Recall ([6], Corollary 2.3.13) that:

$$\mathbf{OML} \ \subset \ \mathbf{QMV}, \tag{29}$$

the inclusion being strict, since there are examples of QMV algebras not verifying (m-Pimpl).

**Proposition 3.3.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OML. We have the equivalence:*

$$(x \odot y = x \overset{def.}{\Longleftrightarrow}) \ x \leq_m^P y \Longleftrightarrow x \leq_m^M y \ (\overset{def.}{\Longleftrightarrow} x \wedge_m^M y = x).$$

**Proof.** Suppose $x \leq_m^P y$, i.e. $x \odot y = x$. Then,
$x \wedge_m^M y \stackrel{(9)}{=} (x^- \odot y)^- \odot y = ((x \odot y)^- \odot y)^- \odot y$
$\stackrel{(DN)}{=} (((x^-)^- \odot y)^- \odot y)^- \odot y \stackrel{(9)}{=} (x^- \wedge_m^M y)^- \odot y$
$\stackrel{(16)}{=} ((x^-)^- \vee_m^M y^-)^- \odot y \stackrel{(m-Wcomm),(DN)}{=} y \odot (x \vee_m^M y^-)$
$\stackrel{(24)}{=} y \odot x = x \odot y = x$, since **OML** $\subset$ **OM**.

Conversely, suppose $x \leq_m^M y$, i.e. $x \wedge_m^M y = x$, i.e. $(x^- \odot y)^- \odot y = x$. Then,
$x \odot y = ((x^- \odot y)^- \odot y) \odot y \stackrel{(m-Wass)}{=} (x^- \odot y)^- \odot (y \odot y)$
$\stackrel{(G)}{=} (x^- \odot y)^- \odot y \stackrel{(9)}{=} x \wedge_m^M y = x$, since (m-Pimpl) implies (G), by Proposition 2.16. $\Box$

### 3.1 Connections between OML and PreMV, QMV, MMV, OM, OL

• **OML** + (Pmv) (Connections between **OML** and **PreMV**)

We establish the connections between the OMLs and the pre-MV algebras verifying (m-Pimpl).

**Proposition 3.4.** *(See Proposition 4.3)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) + (m-Pimpl) \Longrightarrow (Pmv).$$

**Proof.** Since (m-Pimpl) ($[(x \odot y^-)^- \odot x^-]^- = x$) is equivalent to $(x \odot y^-) \oplus x = x$, hence (by taking $X := x^-$) to $(x^- \odot y^-) \oplus x^- = x^-$, we obtain:
$x \odot (x^- \vee_m^M y) = x \odot ((x^- \odot y^-)^- \odot y^-)^- = (x^- \oplus ((x^- \odot y^-)^- \odot y^-)^=)^-$
$\stackrel{(DN)}{=} (x^- \oplus ((x^- \odot y^-)^- \odot y^-))^- \stackrel{(m-Pimpl),(Scomm)}{=} ((x^- \oplus (x^- \odot y^-)) \oplus ((x^- \odot y^-)^- \odot y^-))^-$
$\stackrel{(Sass),(Pcomm)}{=} (x^- \oplus ((y^- \odot x^-) \oplus ((y^- \odot x^-)^- \odot y^-)))^- \stackrel{(Pom)}{=} (x^- \oplus y^-)^- = x \odot y.$ $\Box$

Note that Proposition 3.4 says: **OML** $\subset$ **PreMV**, which follows by (29).

The following converse of Proposition 3.4 also holds:

**Proposition 3.5.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra, Then,*

$$(Pmv) + (m-Pimpl) \Longrightarrow (Pom).$$

**Proof. (Following a proof by** *Prover* 9 **of length** 25**, lasting** 0.11 **seconds)**

We know that (m-Pimpl) implies (G), and (G) implies:
$(a)$ $x \odot (y \odot x) = y \odot x$;
indeed, $x \odot (y \odot x) \stackrel{(Pcomm),(Pass)}{=} y \odot (x \odot x) \stackrel{(G)}{=} y \odot x$.
Then, (m-Pimpl) ($((x \odot y^-)^- \odot x^-)^- = x$) implies, taking $Y := y^-$ and using (DN) and (Pcomm):
$(b)$ $(x^- \odot (y \odot x)^-)^- = x$ and
$(b')$ $x^- \odot (x \odot y)^- = x^-$.

On the other hand, (Pmv) ($x \odot (y^- \odot (x^- \odot y^-)^-)^- = x \odot y$) implies, by (Pcomm):
$(c)$ $x \odot (y^- \odot (y^- \odot x^-)^-)^- = x \odot y$.

Now, by $(a)$, $(b)$ and $(c)$, we obtain:
$(d)$ $x \odot (x \odot (y \odot x)^-)^- = y \odot x$;
indeed, in $(c)$, take $Y := y \odot x$ and $X := x$, to obtain:
$(x)$ $x \odot ((y \odot x)^- \odot ((y \odot x)^- \odot x^-)^-)^- = x \odot (y \odot x) \stackrel{(a)}{=} y \odot x$;
since in $(x)$, $((y \odot x)^- \odot x^-)^- \stackrel{(Pcomm)}{=} (x^- \odot (y \odot x)^-)^- \stackrel{(b)}{=} x$,

it follows that $(x)$ becomes:
$(x')$ $x \odot ((y \odot x)^- \odot x)^- = y \odot x$, i.e. $(d)$ holds, by (Pcomm).

Now, by $(b')$, $(d)$, we obtain:
$(e)$ $(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-$;
indeed, in $(d)$, take $X := (x \odot y)^-$ and $Y := x^-$ to obtain:
$(y)$ $(x \odot y)^- \odot ((x \odot y)^- \odot (x^- \odot (x \odot y)^-)^-)^- = x^- \odot (x \odot y)^-$;
but, in $(y)$, $x^- \odot (x \odot y)^- \overset{(b')}{=} x^-$, hence $(y)$ becomes:
$(y')$ $(x \odot y)^- \odot ((x \odot y)^- \odot x^=)^- = x^-$, which becomes, by (DN):
$(y'')$ $(x \odot y)^- \odot ((x \odot y)^- \odot x)^- = x^-$, which becomes, by (DN) and (Pcomm):
$((x \odot y)^- \odot (x \odot (x \odot y)^-)^-)^- = x$, that is (Pom). $\Box$

Note that Proposition 3.5 says: **PreMV** $\cap$ **OL** $\subset$ **OM**.

By Propositions 3.4 and 3.5, we obtain:

**Theorem 3.6.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra, Then,*

$$(m - Pimpl) \implies ((Pom) \iff (Pmv))$$

*or*

$$(m - Pimpl) + (Pom) \iff (Pmv) + (m - Pimpl),$$

*i.e. OMLs coincide with pre-MV algebras verifying (m-Pimpl).*

Hence, Theorem 3.6 says:

$$\mathbf{OML} = \mathbf{PreMV} + (m - Pimpl) = \mathbf{PreMV} \cap \mathbf{OL}. \tag{30}$$

- **OML** + (Pqmv) (Connections between **OML** and **QMV**)

We establish now the connection between the OMLs and the QMV algebras verifying (m-Pimpl).

**Proposition 3.7.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OML. Then, $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pimpl).*
*(i.e. in an involutive m-BE algebra, (Pom) + (m-Pimpl) $\implies$ (Pqmv).)*

**Proof.** Since $\mathcal{A}^L$ is a left-OML, it is an involutive m-BE algebra verifying (m-Pimpl) and (Pom) (Definition 2). By Theorem 3.4, it verifies (Pmv) also. Hence, $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pimpl). $\Box$

Note that Proposition 3.7 says: **OML** $\subset$ **QMV**, which is (29). Note also that Proposition 3.4 follows from Proposition 3.7, since (Pqmv) $\implies$ (Pmv).

The following converse of Proposition 3.7 also holds.

**Proposition 3.8.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra verifying (m-Pimpl). Then, $\mathcal{A}^L$ is a left-OML.*
*(i.e. in an involutive m-BE algebra, (Pqmv) + (m-Pimpl) $\implies$ (Pom).)*

**Proof.** Since $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pimpl), it is an involutive m-BE algebra verifying (Pqmv) (hence (Pom), (Pmv) ) and (m-Pimpl). Hence, $\mathcal{A}^L$ is an involutive m-BE algebra verifying (m-Pimpl) and (Pom), i.e. it is a left-OML. $\Box$

Note that Proposition 3.8 says: **QMV** $\cap$ **OL** $\subset$ **OM**. Note also that Proposition 3.8 follows from Proposition 3.5.

By Propositions 3.7 and 3.8, we obtain:

**Theorem 3.9.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(m-Pimpl) \implies ((Pom) \Leftrightarrow (Pqmv))$$

*or*

$$(m-Pimpl) + (Pom) \iff (Pqmv) + (m-Pimpl)$$

*i.e. orthomodular lattices coincide with QMV algebras verifying (m-Pimpl).*

Hence, Theorem 3.9 says:

$$\mathbf{OML} = \mathbf{QMV} + (m-Pimpl) = \mathbf{QMV} \cap \mathbf{OL}. \tag{31}$$

By the previous results (28), (29), (30) and (31), we obtain the connections from Figure 4.

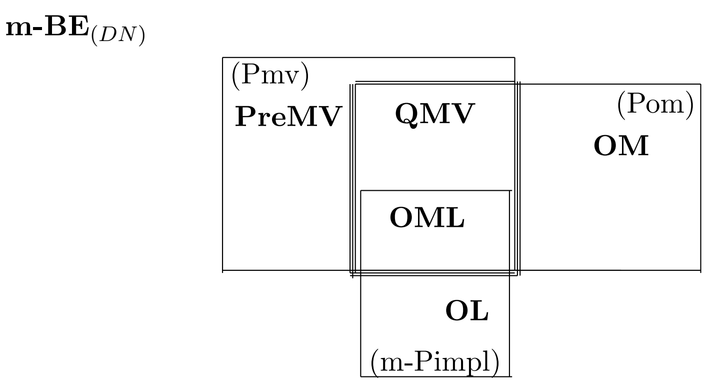


**Figure 4:** Resuming connections between **QMV**, **PreMV**, **OM**, **OL** and **OML**

- **OML** + $(\Delta_m)$ (Connections between **OML** and **MMV**)

**Proposition 3.10.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(Pom) + (m-Pimpl) \implies (\Delta_m).$$

**Proof.** By Proposition 3.4, (Pom) + (m-Pimpl) imply (Pmv) and (Pmv) implies $(\Delta_m)$. □

Note that Proposition 3.10 says: **OML** $\subset$ **MMV**. which follows also by (29). Note also that Proposition 3.7 follows also from Proposition 3.10, since (Pom) + $(\Delta_m)$ imply (Pqmv) and that Proposition 3.10 follows from Proposition 3.7, since (Pqmv) implies $(\Delta_m)$.

**Remark 3.11.** *The following converse of Proposition 3.10 ($(\Delta_m)$ + (m-Pimpl) $\Longrightarrow$ (Pom)) does not hold: there are examples of involutive m-BE algebras verifying $(\Delta_m)$ and (m-Pimpl) and not verifying (Pom).*

By the previous Remark, from the connections from Figure 4, we obtain the connections from Figure 5.

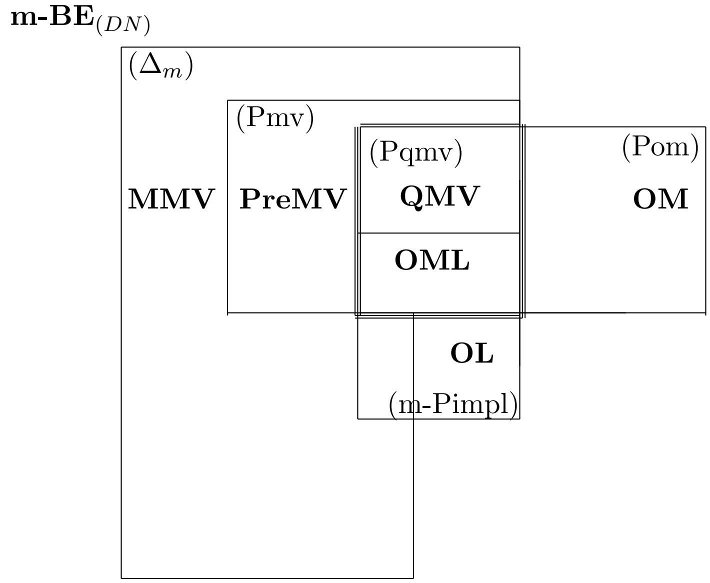


**Figure 5:** Resuming connections between **QMV**, **PreMV**, **MMV**, **OM**, **OL** and **OML**

**Remark 3.12.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OL (Definition 2). Note that:*
*- the initial binary relation, $\leq_m$ ($x \leq_m y \Longleftrightarrow x \odot y^- = 0$), is only reflexive ((m-Re) holds, by definition of m-BE algebra);*
*- the binary relation $\leq_m^M$ ($x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x$) is only reflexive and antisymmetric;*
*- the binary relation $\leq_m^P$ ($x \leq_m^P y \Longleftrightarrow x \odot y = x$) is a* **lattice order**, *with respect to $\wedge = \odot$, $\vee = \oplus$, denoted $\leq_m^O$, by Proposition 2.2.*

**Remark 3.13.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OML (Definition 2). Note that:*
*- The initial binary relation, $\leq_m$ ($x \leq_m y \Longleftrightarrow x \odot y^- = 0$), is only reflexive;*
*- The binary relation $\leq_m^M$ ($x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x$) is an* **order**, *by Corollary 2.26, but not a lattice order with respect to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative;*
*- The binary relation $\leq_m^P$ ($x \leq_m^P y \Longleftrightarrow x \odot y = x$) is a* **lattice order**, *with respect to $\wedge = \odot$, $\vee = \oplus$, denoted $\leq_m^O$, by Proposition 2.2;*
*- We have the equivalence $\leq_m^O \Longleftrightarrow \leq_m^M$, by Proposition 2.23; consequently, the tables of $\wedge$ and $\wedge_m^M$ are different, but they coincide for the comparable elements of $A^L$ (with respect to $\leq_m^O$ and $\leq_m^M$, respectively).*

## 3.2 The transitive and/or antisymmetric case

### 3.2.1 Antisymmetric orthomodular lattices: aOML = Boole

Denote by **aOML** the class of all antisymmetric left-OMLs. We prove that **aOML** does not exist as a proper class:

**Theorem 3.14.** *We have:*

$$\mathbf{aOML} = \mathbf{Boole}.$$

**Proof.** **aOML** = **m-BE**$_{(DN)}$ + (m-Pimpl) + (Pom) + (m-An) = **OL** + (Pom) + (m-An) = **Boole** + (Pom) = **Boole**, by Theorem 2.18. □

**Remark:** We have:

$$\mathbf{OML} \subset \mathbf{QMV} \quad \textit{and} \quad \mathbf{aOML} = \mathbf{Boole} \subset \mathbf{aQMV} = \mathbf{MV}.$$

### 3.2.2 Transitive orthomodular lattices: tOML = Boole

Denote by **tOML** the class of all transitive left-OMLs. We shall prove that **tOML** does not exist as a proper class (**tOML** = **Boole**, by Theorem 3.16).

**Theorem 3.15.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom)\ +\ (m-Pimpl)\ +\ (m-BB)\ \Longrightarrow\ (m-An).$$

**Proof. (By** $Prover9$**, in** $0.03$ **seconds, the length of the proof being** $32$**)**
Suppose: $(i)$ $c_1 \odot c_2^- = 0$ and $(j)$ $c_2 \odot c_1^- = 0$; we have to prove that $c_1 = c_2$.
First, (Pom): $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$ means
$[(x \odot y)^- \odot ((x \odot y)^- \odot x)^-]^- = x$, hence by (Pcomm), (DN):

$$(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-. \tag{32}$$

Second, (m-BB): $[(x \odot y)^- \odot (z \odot y)] \odot (z \odot x^-)^- = 0$, means, by (Pass):

$$(x \odot y)^- \odot [z \odot (y \odot (z \odot x^-)^-)] = 0. \tag{33}$$

Take $x := c_2^-$, $y := c_1$, $z := x$ in (33) to obtain:
$(c_2^- \odot c_1)^- \odot [x \odot (c_1 \odot (x \odot c_2)^-)] = 0$, hence by (i), (Neg0-1), (PU):
$x \odot (c_1 \odot (x \odot c_2)^-) = 0$, hence, by (Pass), (Pcomm):

$$c_1 \odot (x \odot (c_2 \odot x)^-) = 0. \tag{34}$$

Since (p-Pimpl) implies (G), then (G) $(x \odot x = x)$ implies $x \odot y = (x \odot x) \odot y \overset{(Pass)}{=} x \odot (x \odot y)$, hence we have:

$$x \odot (x \odot y) = x \odot y. \tag{35}$$

Take now $x := c_1$ in (34) to obtain: $c_1 \odot (c_1 \odot (c_2 \odot c_1)^-) = 0$, hence by (35) and (Pcomm):

$$c_1 \odot (c_1 \odot c_2)^- = 0. \tag{36}$$

Take now $x := c_1$, $y := c_2$ in (32) to obtain:
$(c_1 \odot c_2)^- \odot (c_1 \odot (c_1 \odot c_2)^-)^- = c_1^-$; then, by (36), (Neg0-1), (PU), we obtain:

$$(c_1 \odot c_2)^- = c_1^-, \quad hence \tag{37}$$

$$c_1 \odot c_2 = c_1. \tag{38}$$

Now, from (m-Pimpl): $[(x \odot y^-)^- \odot x^-]^- = x$, we obtain by (Pcomm) and for $y := y^-$:

$$[x^- \odot (y \odot x)^-]^- = x. \tag{39}$$

Take now $x := c_2$, $y := c_1$ in (39) to obtain: $[c_2^- \odot (c_1 \odot c_2)^-]^- = c_2$, hence, by (37), $[c_2^- \odot c_1^-]^- = c_2$, hence, by (Pcomm):

$$(c_1^- \odot c_2^-)^- = c_2, \quad hence \tag{40}$$

$$c_1^- \odot c_2^- = c_2^-. \tag{41}$$

Finally, take $x := c_1^-$, $y := c_2^-$ in (32) to obtain: $(c_1^- \odot c_2^-)^- \odot (c_1^- \odot (c_1^- \odot c_2^-)^-)^- = c_1$; hence, by (40), we obtain:
$c_2 \odot (c_1^- \odot c_2)^- = c_1$; hence, by (j), (Pcomm), (Neg0-1) and (PU), we obtain: $c_2 = c_1$. □

Note that Theorem 3.15 says: **tOML** $\subset$ **m-aBE**$_{(DN)}$. Hence, **tOML** $\subset$ **taOML**. But **taOML** = **aOML** + (m-Tr) = **Boole** + (m-Tr) = **Boole**, by Theorem 3.14. It follows that **tOML** = **Boole**. Thus, we have proved Theorem 3.16:

**Theorem 3.16.** *We have:*

$$\mathbf{tOML} \ = \ \mathbf{Boole}. \tag{42}$$

#### 3.2.3 The transitive and antisymmetric case

If we make the following table:

| No. | (m-Tr) | (m-Pimpl) | (Pqmv) | Type of m-$BE_{(DN)}$ algebra |
|---|---|---|---|---|
| (1) | 0 | 0 | 0 | proper m-$BE_{(DN)}$ |
| (2) | 0 | 0 | 1 | proper QMV |
| (3) | 0 | 1 | 0 | proper OL |
| (4) | 0 | 1 | 1 | proper OM |
| (5) | 1 | 0 | 0 | proper m-pre-$BCK_{(DN)}$ |
| (6) | 1 | 0 | 1 | proper tQMV |
| (7) | 1 | 1 | 0 | proper tOL |
| (8) | 1 | 1 | 1 | tOML = aOML = Boole |

then, we obtain the resuming connections from Figures 6 and 7.

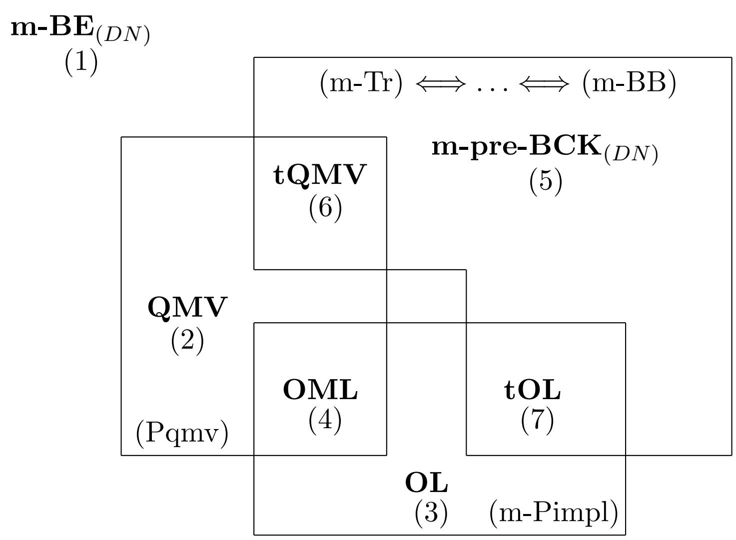


**Figure 6:** Resuming connections in **m-$BE_{(DN)}$**

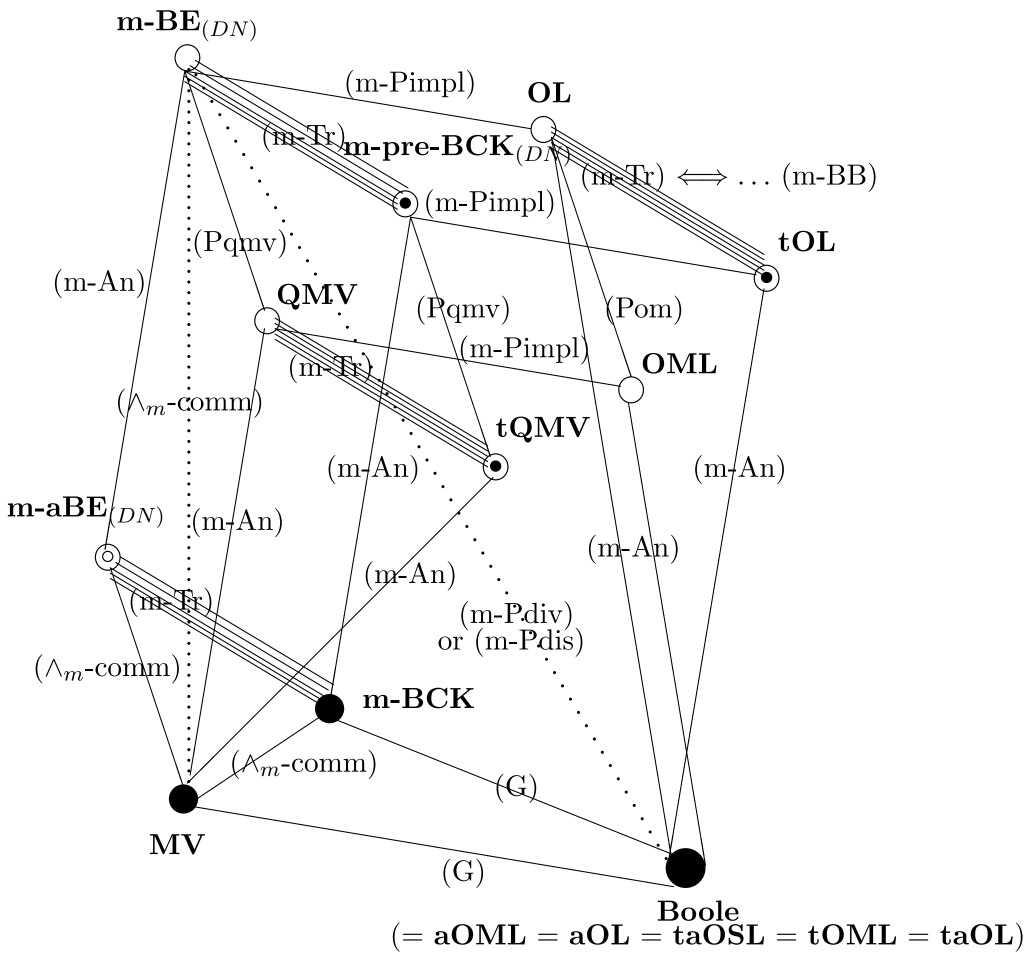


**Figure 7:** Resuming connections in this section

### 3.2.4 The transitive case: tOL $\subset$ tMMV

**Theorem 3.17.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(m - Pimpl) \ + \ (m - BB) \ \Longrightarrow \ (\Delta_m).$$

**Proof.** Since (m-Pimpl) implies (m-Pabs-i) and since, by ([22], Theorem 5.13), (m-Pabs-i) + (m-BB) imply $(\Delta_m)$, it follows that (m-Pimpl) + (m-BB) imply $(\Delta_m)$. □

Note that Theorem 3.17 says: **tOL** $\subset$ **MMV**, hence **tOL** $\subset$ **tMMV**, since (m-BB) $\Leftrightarrow$ (m-Tr). Now, by Theorems 3.16 and 3.17, from the connections from Figure 5, we obtain the connections from Figure 8.

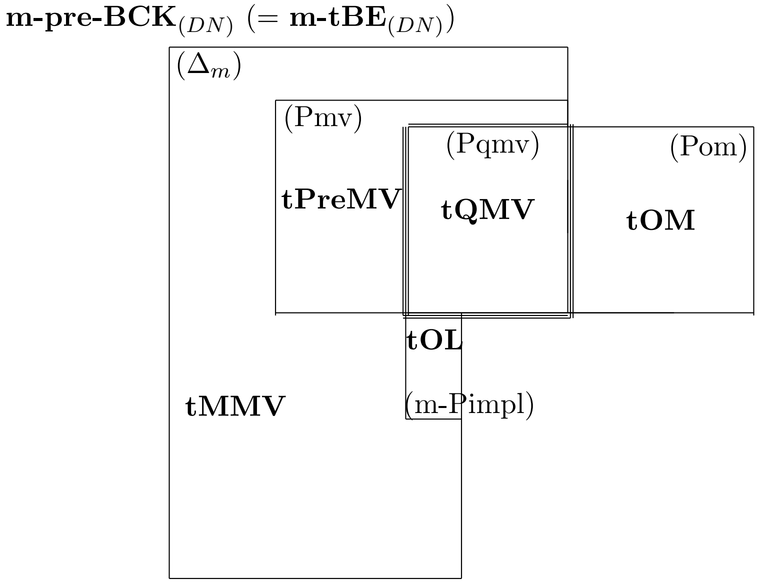


**Figure 8:** Resuming connections between **tQMV**, **tPreMV**, **tMMV**, **tOM** and **tOL**

### 3.3 Modular algebras: MOD $\subset$ OML

Recall the following definitions [25]:

$(i)$ A latice $(L, \wedge, \vee)$ is *modular*, if for all $x, y, z \in L$,
(Wmod) $x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)$ and, dually,

$(i')$ the dual latice $(L, \vee, \wedge)$ is *modular*, if for all $x, y, z \in L$,
(Vmod) $x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$.

**Definition 3.18.** *(Definition 1) (The dual case is omitted) [25]*
*A modular left-ortholattice is a left-OL $\mathcal{A}^L = (A^L, \wedge, \vee, ^-, 0, 1)$ whose lattice $(A^L, \wedge, \vee)$ is modular.*

We shall denote by **MODOL** the class of all modular left-ortholattices.
Recall also [25] that any modular ortholattice is an orthomodular lattice, i.e.

$$\textbf{MODOL} \subset \textbf{OML}. \tag{43}$$

Following the equivalent definition of OLs, we obtain the following equivalent definition.

**Definition 3.19.** *(Definition 2) (The dual one is omitted)*
*A modular left-ortholattice is an involutive left-m-BE algebra $(A^L, \odot, ^-, 1)$ verifying (m-Pimpl) and (Pmod), where: for all $x, y, z \in A^L$,*
*(Pmod) $x \odot (y \oplus (x \odot z)) = (x \odot y) \oplus (x \odot z)$, i.e.*
**MODOL** = **m-BE**$_{(DN)}$ + *(m-Pimpl) + (Pmod)* = **OL** + *(Pmod).*

Then, we introduce the following notion:

**Definition 3.20.**
*(i) A left-modular algebra or a modular left-algebra, or a left-MOD algebra for short, is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^- = ^{-^L}, 1)$ verifying: for all $x, y, z \in A^L$,*
*(Pmod) $x \odot (y \oplus (x \odot z)) = (x \odot y) \oplus (x \odot z)$.*
*(i') Dually, a right-modular algebra or a modular right-algebra, or a right-MOD algebra for short, is an involutive right-m-BE algebra $\mathcal{A}^R = (A^L, \oplus, ^- = ^{-^R}, 0)$ verifying: for all $x, y, z \in A^R$,*
*(Smod) $x \oplus (y \odot (x \oplus z)) = (x \oplus y) \odot (x \oplus z)$.*

We shall denote by **MOD** the class of all left-MOD algebras and by $\mathbf{MOD}^R$ the class of all right-MOD algebras. Hence, $\mathbf{MOD} = \mathbf{m\text{-}BE}_{(DN)}$ + (Pmod).

Then,

$$\mathbf{MODOL} = \mathbf{OL} + (Pmod) = \mathbf{OL} \cap \mathbf{MOD}. \tag{44}$$

**Proposition 3.21.** *(The dual one is omitted)*

*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pmod) \implies (Pom).$$

**Proof.** (**Following a proof by Prover9 of length 14, lasting 0.05 seconds**)

(Pmod), i.e. $x \odot (y \oplus (x \odot z)) = (x \odot y) \oplus (x \odot z)$, is equivalent with
$(a)$ $x \odot (y^- \odot (x \odot z)^-)^- = ((x \odot y)^- \odot (x \odot z)^-)^-$, i.e. with:
$(a')$ $((x \odot y)^- \odot (x \odot z)^-)^- = x \odot (y^- \odot (x \odot z)^-)^-$.

Then,
$((x \odot y)^- \odot (x \odot z)^-)^- \overset{(Pcomm)}{=} ((x \odot z)^- \odot (x \odot y)^-)^- \overset{(a')}{=} x \odot (z^- \odot (x \odot y)^-)^-$,
hence we obtain:
$(b)$ $x \odot (y^- \odot (x \odot z)^-)^- = x \odot (z^- \odot (x \odot y)^-)^-$.
Take now $Z := (x \odot y)^-$ in (b) to obtain:
$x \odot (y^- \odot (x \odot (x \odot y)^-)^-)^- = x \odot ((x \odot y)^= \odot (x \odot y)^-)^-$
$\overset{(DN)}{=} x \odot ((x \odot y) \odot (x \odot y)^-)^- \overset{(m-Re)}{=} x \odot 0^- \overset{(Neg0-1)}{=} x \odot 1 \overset{(PU)}{=} x$;
hence, we have:
$(c)$ $x \odot (y^- \odot (x \odot (x \odot y)^-)^-)^- = x$.

Now, (Pom), i.e. $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$, is equivalent with:
$(d)$ $((x \odot y)^- \odot ((x \odot y)^- \odot x)^-)^- = x$, which by (Pcomm) means:
$(d')$ $((x \odot y)^- \odot (x \odot (x \odot y)^-)-)^- = x$;
hence, we must prove that (d') holds.

Indeed, $((x \odot y)^- \odot (x \odot (x \odot y)^-)-)^- \overset{(a')}{=} x \odot (y^- \odot (x \odot (x \odot y)^-)^-)^- \overset{(c)}{=} x$, hence (d') holds, i.e. (Pom) holds. $\square$

Note that Proposition 3.21 says: $\mathbf{MOD} \subset \mathbf{OM}$.

**Proposition 3.22.** *(The dual one is omitted)*

*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pmod) \implies (m - Pimpl).$$

**Proof.** (**Following a proof by Prover9 of length 16, lasting 0.00 seconds**)

(Pmod), i.e. $x \odot (y \oplus (x \odot z)) = (x \odot y) \oplus (x \odot z)$, is equivalent with
$(a)$ $x \odot (y^- \odot (x \odot z)^-)^- = ((x \odot y)^- \odot (x \odot z)^-)^-$, i.e. with:
$(a')$ $((x \odot y)^- \odot (x \odot z)^-)^- = x \odot (y^- \odot (x \odot z)^-)^-$.

Now, take in (a') $Y := 1$ and $Z := y$ to obtain, by (PU), (Neg1-0), (Pcomm), (m-L):
$(x^- \odot (x \odot y)^-)^- = ((x \odot 1)^- \odot (x \odot z)^-)^- \overset{(a')}{=} x \odot (1^- \odot (x \odot z)^-)^- = x \odot (0 \odot (x \odot z)^-)^- = x \odot 0^- = x \odot 1 = x$,
hence:
$(b)$ $(x^- \odot (x \odot y)^-)^- = x$.

Note that (m-Pimpl), i.e. $((x \odot y^-)^- \odot x^-)^- = x$, follows from $(b)$, by (Pcomm). $\square$

Note that Proposition 3.22 says: $\mathbf{MOD} \subset \mathbf{OL}$, hence, $\mathbf{MOD} = \mathbf{OL} \cap \mathbf{MOD} \overset{(44)}{=} \mathbf{MODOL}$. Thus, we have:

$$\mathbf{MOD} = \mathbf{MODOL}. \tag{45}$$

By Propositions 3.21 and 3.22, we obtain obviously:

**Theorem 3.23.** *(The dual one is omitted)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pmod) \implies (Pom) + (m - Pimpl).$$

By above Theorem 3.23, which says: **MOD** $\subset$ **OM** $\cap$ **OL** $=$ **OML**, by (28), we reobtain immediately the recalled known result from (43): **MODOL** ($=$ **MOD**) $\subset$ **OML** ( $\subset$ **OL**).

Recall [25] that the inclusion is strict.

Since **OML** $\subset$ **QMV**, by (28), we obtain:

$$\mathbf{MOD}\ (= \mathbf{MODOL})\ \subset\ \mathbf{OML}\ \subset\ \mathbf{QMV}. \tag{46}$$

Hence, we have:

$$\mathbf{aMOD}\ =\ \mathbf{aOML}\ =\ \mathbf{Boole}\ \subset\ \mathbf{aQMV}\ =\ \mathbf{MV}\quad \textit{and} \tag{47}$$

$$\mathbf{tMOD}\ =\ \mathbf{tOML}\ =\ \mathbf{Boole}\ \subset\ \mathbf{tQMV}. \tag{48}$$

**Remark 3.24.** *Recall that any OL that is distributive is a Boolean algebra, by definitions. Consequently, any OML that is distributive is a Boolean algebra and any modular algebra that is distributive is a Boolean algebra.*

# 4 Orthomodular softlattices, widelattices

Starting from the two generalizations of ortholattices (**OL**): the orthosoftlattices (**OSL**) and the orthowidelattices (**OWL**) (Definition 2.17 and Figure 1), we introduce, in separate subsections, two corresponding generalizations of orthomodular lattices (OMLs): the orthomodular softlattices and the orthomodular widelattices.

## 4.1 Orthomodular softlattices: OMSL

We introduce the following notion.

**Definition 4.1.** *(Definition* 1*) (The dual one is omitted)*
*A left-orthomodular softlattice or an orthomodular left-softlattice, or a left-OMSL for short, is a left-OSL $\mathcal{A}^L = (A^L, \wedge, \vee, ^-, 0, 1)$ verifying: for all $x, y \in A^L$,*
*(Wom)* $(x \wedge y) \vee ((x \wedge y)^- \wedge x) = x$.

Denote by **OMSL** the class of all left-OMSLs. Following the equivalent Definition 2 of a left-OSL (see Definition 2.17), we obtain immediately an equivalent definition:

**Definition 4.2.** *(Definition* 2*) (The dual one is omitted)*
*A left-OMSL is a left-OSL verifying (Pom), i.e. is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying (G) and (Pom), i.e.*

$$\mathbf{OMSL}\ =\ \mathbf{m-BE_{(DN)}}\ +\ (G)\ +\ (Pom)\ =\ \mathbf{OSL}\ \cap\ \mathbf{OM}. \tag{49}$$

Further, we shall work with Definition 2 of left-OMSLs. Hence, we have the connections from Figure 9.

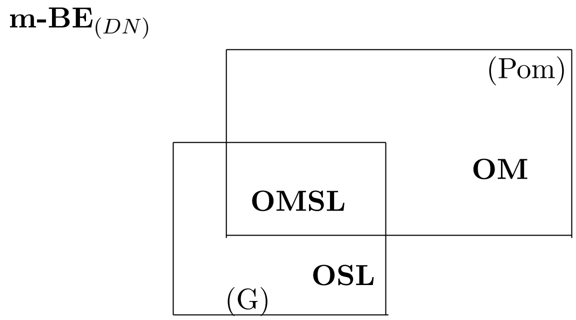


**Figure 9:** Resuming connections between **OSL**, **OMSL** and **OM**

Denote by **tOMSL** the class of all transitive left-OMSLs. We shall prove that **OMSL** and **tOMSL** do not exist (as proper classes) (**OMSL** = **OML**, by (53), hence **tOMSL** = **Boole**, by Theorem 3.16).

#### 4.1.1 Connections between OMSL and PreMV, QMV, MMV, OM, OSL

• **OMSL** + (Pmv) (Connections between **OMSL** and **PreMV**)

We establish the connections between the OMSLs and the pre-MV algebras verifying (G).

**Proposition 4.3.** *(See Proposition 3.4)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra, Then,*

$$(Pom) \ + \ (G) \implies (Pmv).$$

**Proof. (following a proof by *Prover*9, of length 24, lasting 0.36 seconds)**
First, from (Pom) $(((x \odot y)^- \odot (x \odot (x \odot y)^-)^-)^- = x)$, by (DN), we obtain:
$(a)$ $(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-$.
Then, $(a)$ implies:
$(b)$ $x^- \odot (x \odot y)^- = x^-$;
indeed, $x^- \odot (x \odot y)^- \overset{(Pcomm)}{=} (x \odot y)^- \odot x^-$
$\overset{(a)}{=} (x \odot y)^- \odot ((x \odot y)^- \odot (x \odot (x \odot y)^-)^-)$
$\overset{(Pass)}{=} ((x \odot y)^- \odot (x \odot y)^-) \odot (x \odot (x \odot y)^-)^-$
$\overset{(G)}{=} (x \odot y)^- \odot (x \odot (x \odot y)^-)^- \overset{(a)}{=} x^-$.
Then, $(b)$ implies $(c)$, by (DN):
$(c)$ $x \odot (x^- \odot y)^- = x$.
On the other hand, $(a)$ implies $(d)$, by interchanging $x$ with $y$:
$(d)$ $(y \odot x)^- \odot (y \odot (y \odot x)^-)^- = y^-$,
and $(d)$ implies $(e)$, by taking $X := x^-$ and by (Pcomm):
$(e)$ $(x^- \odot y)^- \odot (y \odot (x^- \odot y)^-)^- = y^-$.
Finally, $(c)$ and $(e)$ imply:
$(f)$ $x \odot y^- = x \odot (y \odot (x^- \odot y)^-)^-$;
indeed, $x \odot y^- \overset{(e)}{=} x \odot ((x^- \odot y)^- \odot (y \odot (x^- \odot y)^-)^-)$
$\overset{(Pass)}{=} (x \odot (x^- \odot y)^-) \odot (y \odot (x^- \odot y)^-)^-$
$\overset{(c)}{=} x \odot (y \odot (x^- \odot y)^-)^-$; thus, $(f)$ holds.

By taking $Y := y^-$ in $(f)$, we obtain, by (DN):
$x \odot y = x \odot (y^- \odot (x^- \odot y^-)^-)^-$, that is (Pmv). $\Box$

Note that Proposition 3.4 follows from Proposition 4.3, since (m-Pimpl) implies (G). Note also that Proposition 4.3 says: **OMSL** $\subset$ **PreMV**.

The following converse of Proposition 4.3 also holds:

**Proposition 4.4.** *(See Proposition 3.5)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(Pmv) \; + \; (G) \implies (Pom).$$

**Proof. (following a proof by** ***Prover9*****, of length 27, lasting 0.12 seconds)**

From (Pmv) ( $x \odot (y^- \odot (x^- \odot y^-)^-)^- = x \odot y$), by taking $Y := y^-$ and by (DN), we obtain:
$(a)$ $x \odot (y \odot (x^- \odot y)^-)^- = x \odot y^-$.

On the other hand, from (G) ($x \odot x = x$), we obtain:
$(b)$ $x \odot (x \odot y) = x \odot y$;
indeed, $x \odot (x \odot y) \overset{(Pass)}{=} (x \odot x) \odot y \overset{(G)}{=} x \odot y$; hence, by (Pcomm), we obtain:
$(b')$ $x \odot (y \odot x) = y \odot x$.

Now, from $(b)$ and $(a)$ we obtain:
$(c)$ $x \odot (x^- \odot y)^- = x$;
indeed, in $(a)$, take $X := x$ and $Y := x^- \odot y$ to obtain:
$(x)$ $x \odot ((x^- \odot y) \odot (x^- \odot (x^- \odot y))^-)^- = x \odot (x^- \odot y)^-$;
but, the part from (x): $x^- \odot (x^- \odot y) \overset{(b)}{=} x^- \odot y$, hence (x) becomes:
$(x')$ $x \odot ((x^- \odot y) \odot (x^- \odot y)^-)^- = x \odot (x^- \odot y)^-$,
which by (m-Re) and (Neg0-1) becomes:
$(x'')$ $x \odot 1 = x \odot (x^- \odot y)^-$,
which, by (PU), becomes $(c)$.

Now, from $(c)$, by (Pcomm), we obtain:
$(c')$ $x \odot (y \odot x^-)^- = x$ and
from $(c)$, by taking $X := x^-$ we obtain:
$(c'')$ $x^- \odot (x \odot y)^- = x^-$.

Now, from $(c')$ and $(a)$, we obtain:
$(d)$ $x \odot (x \odot (y \odot x)^-)^- = y \odot x$;
indeed, in $(a)$, take $X := x$ and $Y := (y \odot x^=)^-$ to obtain:
$(y)$ $x \odot ((y \odot x^=)^- \odot (x^- \odot (y \odot x^=)^-)^-)^- = x \odot (y \odot x^=)^=$;
but, the part from (y) $x^- \odot (y \odot x^=)^- \overset{(c')}{=} x^-$, hence (y) becomes, by (DN):
$(y')$ $x \odot ((y \odot x)^- \odot x^=)^- = x \odot (y \odot x)$;
but $(y')$, by (DN) and $(b')$ becomes:
$(y'')$ $x \odot ((y \odot x)^- \odot x)^- = y \odot x$;
and $(y'')$, by (Pcomm), becomes $(d)$.

Now, from $(c'')$ and $(d)$, we obtain:
$(e)$ $(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-$;
indeed, in $(d)$, take $X := (x \odot y)^-$ and $Y := x^-$ to obtain:
$(u)$ $(x \odot y)^- \odot ((x \odot y)^- \odot (x^- \odot (x \odot y)^-)^-)^- = x^- \odot (x \odot y)^-$;
but, the parts from (u) $x^- \odot (x \odot y)^- \overset{(c'')}{=} x^-$, hence $(u)$ becomes:
$(u')$ $(x \odot y)^- \odot ((x \odot y)^- \odot x^=)^- = x^-$,
which by (DN) and (Pcomm) becomes:
$(x \odot y)^- \odot (x \odot (x \odot y)^-)^- = x^-$, that is $(e)$.

Finally, from $(e)$, by (DN), we obtain:
$((x \odot y)^- \odot (x \odot (x \odot y)^-)^-)^- = x$, that is (Pom). $\square$

Note that Proposition 3.5 follows from Proposition 4.4, since (m-Pimpl) $\Longrightarrow$ (G). Note also that Proposition 4.4 says: **PreMV** $\cap$ **OSL** $\subset$ **OM**.

By Propositions 4.3 and 4.4, we obtain:

**Theorem 4.5.** *(See Theorem 3.6)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(G) \implies ((Pom) \Leftrightarrow (Pmv))$$

*or*

$$(G) + (Pom) \Longleftrightarrow (Pmv) + (G),$$

*i.e. OMSLs coincide with pre-MV algebras verifying (G).*

Hence, Theorem 4.5 says:

$$\mathbf{OMSL} = \mathbf{PreMV} + (G) = \mathbf{PreMV} \cap \mathbf{OSL}. \tag{50}$$

Note that Theorem 3.6 follows from Theorem 4.5, since (m-Pimpl) implies (G).

- **OMSL** + (Pqmv) (Connections between **OMSL** and **QMV**)

We establish now the connection between the OMSLs and the QMV algebras verifying (G).

**Proposition 4.6.** *(See Proposition 3.7)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OMSL. Then, $\mathcal{A}^L$ is a left-QMV algebra verifying (G).*
*(i.e. in an involutive m-BE algebra, (Pom) + (G) $\Longrightarrow$ (Pqmv).)*

**Proof.** Since $\mathcal{A}^L$ is a left-OMSL, it is an involutive m-BE algebra verifying (G) and (Pom) (Definition 2). By Proposition 4.3, it verifies (Pmv) also. Hence, $\mathcal{A}^L$ is a left-QMV algebra verifying (G). $\square$

Note that Proposition 4.6 says:

$$\mathbf{OMSL} \subset \mathbf{QMV}, \tag{51}$$

the inclusion being strict, since there are examples of QMV algebras not verifying (G). Note also that Proposition 3.7 follows from Proposition 4.6 and also that Proposition 4.3 follows from Proposition 4.6, since (Pqmv) implies (Pmv).

The following converse of Proposition 4.6 holds.

**Proposition 4.7.** *(See Proposition 3.8)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra verifying (G). Then, $\mathcal{A}^L$ is a left-OMSL.*
*(i.e. in an involutive m-BE algebra, (Pqmv) + (G) $\Longrightarrow$ (Pom).)*

**Proof.** Since $\mathcal{A}^L$ is a left-QMV algebra verifying (G), it is an involutive m-BE algebra verifying (Pmv), (Pom) and (G) (Definition 2). Hence, $\mathcal{A}^L$ is an involutive m-BE algebra verifying (G) and (Pom), i.e. it is a left-OMSL. $\square$

Note that Proposition 4.7 says: **QMV** $\cap$ **OSL** $\subset$ **OM**. Note also that Proposition 3.8 follows from Proposition 4.7, since (m-Pimpl) implies (G), and also that Proposition 4.7 follows from Proposition 4.4, since (Pqmv) implies (Pmv).

By Propositions 4.6 and 4.7, we obtain:

**Theorem 4.8.** *(See Theorem 3.9)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(G) \implies ((Pom) \Leftrightarrow (Pqmv))$$

*or*

$$(G) + (Pom) \iff (Pqmv) + (G)$$

*i.e. orthomodular softlattices coincide with QMV algebras verifying (G).*

Hence, Theorem 4.8 says:

$$\mathbf{OMSL} = \mathbf{QMV} + (G) = \mathbf{QMV} \cap \mathbf{OSL}. \tag{52}$$

Note that Theorem 3.9 follows from Theorem 4.8, since (m-Pimpl) implies (G).
By the previous results (49), (50), (51) and (52), we obtain the connections from Figure 10.

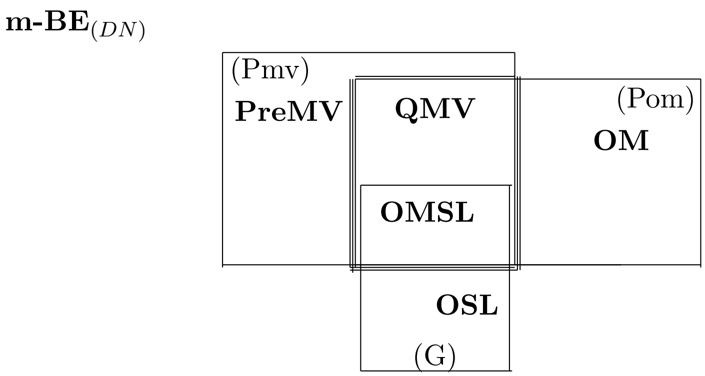


**Figure 10:** Resuming connections between **QMV**, **PreMV**, **OM**, **OSL** and **OMSL**

- **OMSL** + $(\Delta_m)$ (Connections between **OMSL** and **MMV**)

**Proposition 4.9.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra, Then,*

$$(Pom) + (G) \implies (\Delta_m).$$

**Proof.** By Proposition 4.3, (Pom) + (G) implies (Pmv) and (Pmv) implies $(\Delta_m)$. □

Note that Proposition 4.9 says: **OMSL** $\subset$ **MMV**. Note also that Proposition 3.10 follows from Proposition 4.9, since (m-Pimpl) implies (G), that Proposition 4.6 follows from Proposition 4.9, since (Pom) + $(\Delta_m)$ imply (Pqmv), and that Proposition 4.9 follows also from Proposition 4.6, since (Pqmv) implies $(\Delta_m)$.

**Remark 4.10.** *The following converse of Proposition 4.9 ($(\Delta_m)$ + (G) $\Longrightarrow$ (Pom)) does not hold: there are examples of involutive m-BE algebras verifying $(\Delta_m)$ and (G) and not verifying (m-Pimpl) and (Pom).*

By the previous Remark, from the connections from Figure 10, we obtain the connections from Figure 11.

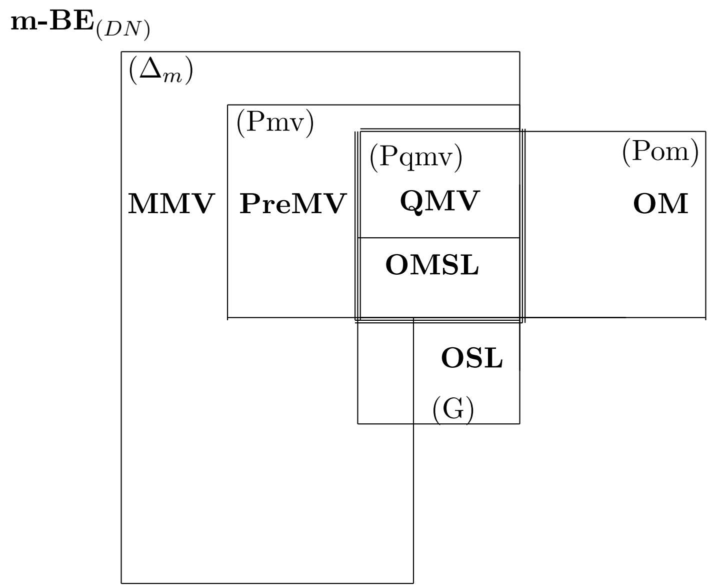


**Figure 11:** Resuming connections between **QMV**, **PreMV**, **MMV**, **OM**, **OSL** and **OMSL**

### 4.1.2 OMSL = OML

**Proposition 4.11.** *We have:*
*(mPom1) (Pom) + (Pcomm) + (Neg0-1) + (PU) + (DN)* $\Longrightarrow$ *(m-Re)* [21]
*(mPom2) (Pom) + (G) + (Pass) + (DN)* $\Longrightarrow$ *(m-Pimpl).*

**Proof.** (mPom2) : **(By** *Prover*9**, in** 0.01 **seconds, the length of the proof being** 15**)**
First, we have: $(a)\ x \odot y \overset{(G)}{=} (x \odot x) \odot y \overset{(Pass)}{=} x \odot (x \odot y)$.
Then, in $(a)$, take $X := (x\odot y)^-$ and $Y := ((x\odot y)^-\odot x)^-$ to obtain: $(b)\ X\odot Y = (x\odot y)^-\odot((x\odot y)^-\odot x)^- \overset{(Pom)}{=} x^-$. Then, $x^- = X \odot Y \overset{(a)}{=} X \odot (X \odot Y) \overset{(b)}{=} X \odot x^- = (x \odot y)^- \odot x^-$; hence, $((x \odot y)^- \odot x^-)^- = (x^-)^- \overset{(DN)}{=} x$, i.e. (m-Pimpl) holds. $\Box$

We know already, by Proposition 2.16, that:

**Proposition 4.12.** *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(m - Pimpl) \implies (G),$$

*i.e.* **OL** $\subset$ **OSL**.

**Proposition 4.13.** *Let* $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ *be an involutive left-m-BE algebra. Then,*

$$(Pom) + (G) \implies (m - Pimpl),$$

*i.e.* **OMSL** $\subset$ **OL**.

**Proof.** By (mPom2). $\Box$

By Propositions 4.12 and 4.13, we obtain:

**Theorem 4.14.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) \implies ((m - Pimpl) \Leftrightarrow (G))$$

*or*

$$(Pom) + (m - Pimpl) \iff (Pom) + (G).$$

By Theorem 4.14 and the equivalent definitions (Definition 2) of left-OMLs and of left-OMSLs, we obtain: **OML**= **OM** + (G) = **OSL** + (Pom) = **OSL** ∩ **OM** = **OMSL**, by (49). Hence, we have:

$$\mathbf{OMSL} = \mathbf{OML}. \tag{53}$$

By (28), (49) and (53), we obtain the connections from Figure 12.

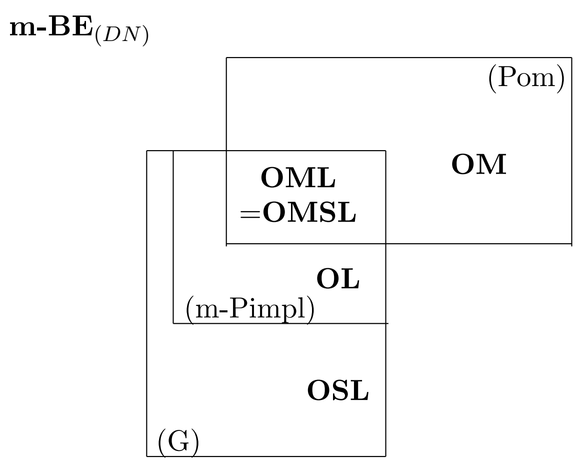


**Figure 12:** Resuming connections between **OML** = **OMSL**, **OL**, **OSL** and **OM**

Finally, since **OML** = **OMSL**, it follows, by Theorems 3.9 and 4.8:

**Corollary 4.15.** *We have:*

$$\mathbf{OML} = \mathbf{OMSL} = \mathbf{QMV} + (m - Pimpl) = \mathbf{QMV} \cap \mathbf{OL} = \mathbf{QMV} + (G) = \mathbf{QMV} \cap \mathbf{OSL}. \tag{54}$$

**Corollary 4.16.** *(See [6], Theorem 2.3.12)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. Consider the set of all idempotent elements of $A^L$ (i.e. elements verifying (G)):*

$$Id(A^L) = \{x \in A^L \mid x \odot x = x\}.$$

*Then, $(Id(A^L), \odot, ^-, 1)$ is a left-OML.*

**Proof.** Note that $(Id(A^L), \odot, ^-, 1)$ is a subalgebra of $\mathcal{A}^L$ verifying (G). Then apply above Corollary 4.15. □

Moreover,
- There are examples of involutive m-BE algebras verifying (G) and not verifying $(\Delta_m)$, (m-Pimpl) and (Pom);
- There are examples of involutive m-BE algebras verifying (m-Pimpl) and not verifying $(\Delta_m)$ and (Pom).

By the connections from Figures 4, 10 and 12, we obtain the connections from Figure 13.

By the connections from Figures 5, 11 and 13, we obtain the connections from Figure 14.

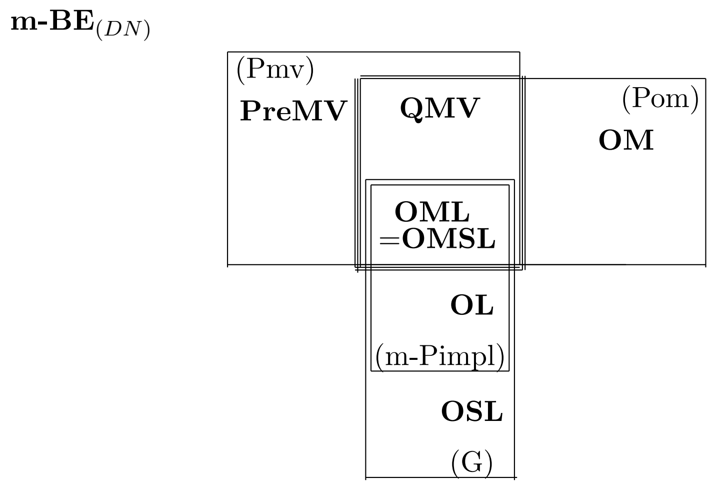


**Figure 13:** Resuming connections between **QMV**, **PreMV**, **OSL**, **OL**, **OM** and **OML** = **OMSL**

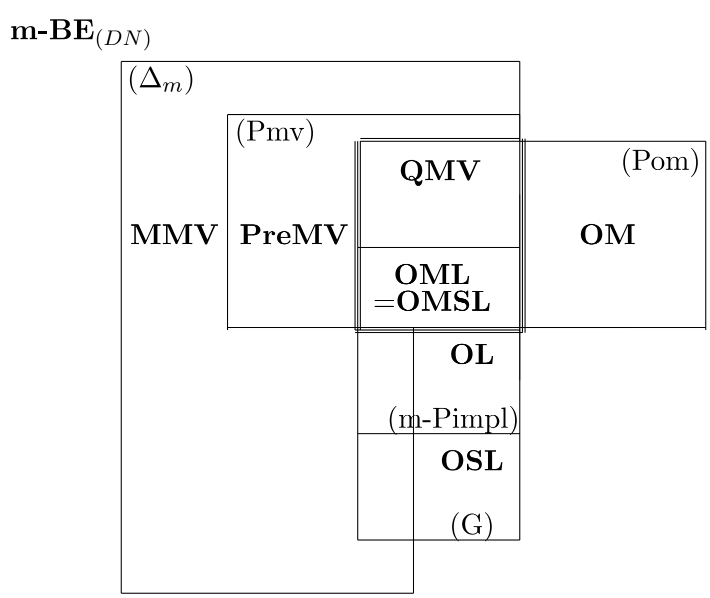


**Figure 14:** Resuming connections between **QMV**, **PreMV**, **MMV**, **OL**, **OSL** and **OML** = **OMSL**

#### 4.1.3 The transitive case: tOSL $\subset$ tMMV

**Theorem 4.17.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(G) \ + \ (m-BB) \implies (\Delta_m).$$

**Proof. (following a proof by** *Prover*9 **in** 10.75 **seconds, the length of the proof being** 28**)**

First, (G) ($x \odot x = x$) implies:

$$x \odot (x \odot y) = x \odot y. \tag{55}$$

Indeed, $x \odot (x \odot y) \stackrel{(Pass)}{=} (x \odot x) \odot y \stackrel{(G)}{=} x \odot y$.

Second, (m-BB) ($[(x \odot y)^- \odot (z \odot y)] \odot (z \odot x^-)^- = 0$) implies:

$$x \odot (y \odot ((x \odot z^-)^- \odot (z \odot y)^-)) = 0. \tag{56}$$

Indeed, interchange $x$ with $z$ in (m-BB) to obtain:
$(x)$ $[(z \odot y)^- \odot (x \odot y)] \odot (x \odot z^-)^- = 0$;
then, in $(x)$, apply (Pass) and (Pcomm) to obtain:
$(x')$ $[(x \odot y) \odot (x \odot z^-)^-] \odot (z \odot y)^- = 0$;
then apply (Pass) to obtain (56).

Also (m-BB) ($[(x \odot y)^- \odot (z \odot y)] \odot (z \odot x^-)^- = 0$) implies:

$$(x \odot y)^- \odot (z \odot (x \odot (z \odot y^-)^-)) = 0. \tag{57}$$

Indeed, interchange $x$ with $y$ in (m-BB) to obtain, by (Pcomm):
$[(x \odot y)^- \odot (z \odot x)] \odot (z \odot y^-)^- = 0$;
then apply (Pass) to obtain (57).

Now, from (57), we obtain:

$$x \odot (y \odot (x \odot (y^- \odot z)^-)^-) = 0. \tag{58}$$

Indeed, in (57) take $X := x$ and $Y := x^- \odot y$ to obtain:
$(y)$ $(x \odot (x^- \odot y))^- \odot (z \odot (x \odot (z \odot (x^- \odot y)^-)^-)) = 0$;
but, in $(y)$, $x \odot (x^- \odot y) \stackrel{(Pass)}{=} (x \odot x^-) \odot y \stackrel{(m-Re)}{=} 0 \odot y \stackrel{(Pcomm)}{=} y \odot 0 \stackrel{(m-L)}{=} 0$, hence $(y)$ becomes:
$(y')$ $0^- \odot (z \odot (x \odot (z \odot (x^- \odot y)^-)^-)) = 0$,
which by (Neg0-1) and (PU) becomes:
$(y'')$ $z \odot (x \odot (z \odot (x^- \odot y)^-)^-) = 0$;
now, in (y") take $X := y$, $Y := z$ and $Z := x$ to obtain:
$x \odot (y \odot (x \odot (y^- \odot z)^-)^-) = 0$, that is (58).

Now, from (58) and (55), we obtain:

$$x \odot (x \odot (y \odot x^-)^-)^- = 0. \tag{59}$$

Indeed, in (55) take $X := x$ and $Y := (x \odot (x^- \odot y)^-)^-$ to obtain:
$(u)$ $x \odot (x \odot (x \odot (x^- \odot y)^-)^-) = x \odot (x \odot (x^- \odot y)^-)^-$;
also in (58) take $X := x$, $Y := x$ and $Z := y$ to obtain:
$(v)$ $x \odot (x \odot (x \odot (x^- \odot y)^-)^-) = 0$;
then, $(u)$ becomes, by $(v)$:
$(u')$ $0 = x \odot (x \odot (x^- \odot y)^-)^-$,
which by (Pcomm) becomes (59).

Now, from (m-BB) and (59) we obtain:

$$x \odot (y \odot (x \odot ((z \odot x^-)^- \odot y))^-) = 0. \tag{60}$$

Indeed, in (m-BB) ($[(x \odot y)^- \odot (z \odot y)] \odot (z \odot x^-)^- = 0$) take $X := x \odot (y \odot x^-)^-$, $Y := z$ and $Z := x$ to obtain:
$(w)$ $[((x \odot (y \odot x^-)^-) \odot z)^- \odot (x \odot z)] \odot (x \odot (x \odot (y \odot x^-)^-)^-)^- = 0$;
but, in $(w)$, the part $x \odot (x \odot (y \odot x^-)^-)^- = 0$, by (59); hence, $(w)$ becomes:
$(w')$ $[((x \odot (y \odot x^-)^-) \odot z)^- \odot (x \odot z)] \odot 0^- = 0$,
which by (Neg0-1) and (PU) becomes:
$(w'')$ $((x \odot (y \odot x^-)^-) \odot z)^- \odot (x \odot z) = 0$,
which by (Pcomm), (Pass) becomes:
$(w''')$ $(x \odot z) \odot (x \odot ((y \odot x^-)^- \odot z))^- = 0$,
which by interchanging $y$ with $z$ and by (Pass) becomes:
$x \odot (y \odot (x \odot ((z \odot x^-)^- \odot y))^-) = 0$, that is (60).

Now, from (56) and (60), we obtain:

$$(x \odot (x \odot y^-)^-) \odot (y \odot (y \odot x^-)^-)^- = 0. \tag{61}$$

Indeed, in (60), take $X := x$, $Y := (x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-$ and $Z := z$ to obtain:
$(z)$ $x \odot (((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-) \odot (x \odot ((z \odot x^-)^- \odot Y))^-) = 0$,
where the part of $(z)$:
$A \overset{notation}{=} x \odot ((z \odot x^-)^- \odot Y) = x \odot ((z \odot x^-)^- \odot ((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-)) = 0$;
indeed, in (56) take $X := x$, $Y := (z \odot x^-)^-$ and $Z := y$ to obtain:
$x \odot ((z \odot x^-)^- \odot ((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-)) = 0$, i.e. $A = 0$;
hence, $(z)$ becomes:
$(z')$ $x \odot (((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-) \odot 0^-) = 0$;
then, by (Neg0-1) and (PU), $(z')$ becomes:
$(z'')$ $x \odot ((x \odot y^-)^- \odot (y \odot (z \odot x^-)^-)^-) = 0$,
and $(z'')$ by (Pass) and by taking $z = y$ becomes:
$(x \odot (x \odot y^-)^-) \odot (y \odot (y \odot x^-)^-)^- = 0$, that is (61).

Finally, from (61), by interchanging $x$ with $y$, we obtain:
$(y \odot (y \odot x^-)^-) \odot (x \odot (x \odot y^-)^-)^- = 0$, that is $(\Delta_m)$. □

Note that Theorem 4.17 says: **tOSL** $\subset$ **MMV**. Hence, **tOSL** $\subset$ **tMMV**.

Note also that Theorem 3.17 follows from Theorem 4.17, since (m-Pimpl) implies (G).

By Theorems 3.17 and 4.17 and by the connections from Figures 8 and 14, we obtain the connections from Figure 15.

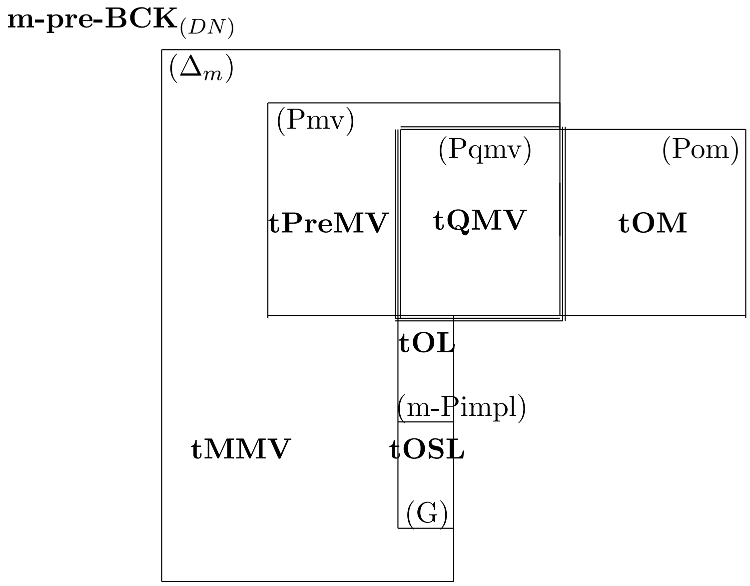


**Figure 15:** Resuming connections between **tQMV**, **tMMV**, **tOSL** and **tOL**

## 4.2 Orthomodular widelattices: OMWL

We introduce the following notion.

**Definition 4.18.** *(Definition 1) (The dual one is omitted)*
*A left-orthomodular widelattice or an orthomodular left-widelattice, or a left-OMWL for short, is a left-OWL verifying: for all $x, y \in A^L$,*
*(Wom)* $(x \wedge y) \vee ((x \wedge y)^- \wedge x) = x$.

Denote by **OMWL** the class of all left-OMWLs. Following the equivalent Definition 2 of a left-OWL (see Definition 2.17), we obtain immediately an equivalent definition:

**Definition 4.19.** *(Definition 2) (The dual one is omitted)*
*A left-OMWL is a left-OWL verifying (Pom), i.e. is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ verifying (m-Pabs-i) and (Pom), i.e.*

$$\mathbf{OMWL} = \mathbf{m-BE_{(DN)}} + (m-Pabs-i) + (Pom) = \mathbf{OWL} \cap \mathbf{OM}. \quad (62)$$

Further, we shall work with Definition 2 of OMWLs. Hence, we have the connections from Figure 16.

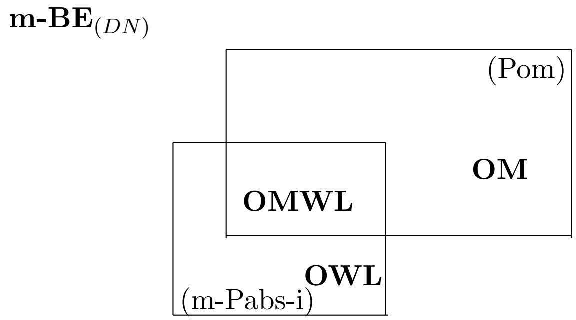


**Figure 16:** Resuming connections between **OWL**, **OMWL** and **OM**

### 4.2.1 Connections between OMWL and PreMV, QMV, MMV, OM, OWL

• **OMWL** + (Pmv) (Connections between **OMWL** and **PreMV**)

The next Proposition 4.21 (saying that (Pom) and (m-Pabs-i) imply (Pmv)) was proved by *Prover9* in 17.06 seconds and the proof produced by *Prover9* has the length 23. We divide the proof produced by *Prover9* into the proof of Lemma 4.20 and Proposition 4.21.

**Lemma 4.20.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra verifying (Pom) (i.e. an OM algebra). Then, we have:*

$$(x \odot y)^- \odot (y \odot (y \odot x)^-)^- = y^-, \tag{63}$$

$$(x \odot y)^- \odot [(y \odot (y \odot x)^-)^- \odot z] = y^- \odot z, \tag{64}$$

$$(x \odot (y \odot z))^- \odot (x \odot (y \odot (x \odot (y \odot z))^-))^- = (x \odot y)^-, \tag{65}$$

$$(x \odot y)^- \odot (z \odot (x \odot (x \odot y)^-)^-) = z \odot x^-, \tag{66}$$

$$(x \odot y^-)^- \odot [(y \odot z)^- \odot (x \odot (x \odot y^-)^-)]^- = ((y \odot z)^- \odot x)^-. \tag{67}$$

**Proof.** (63): From (Pom), by interchanging $x$ with $y$ and by (Pcomm).
(64): From (63), by "multiplying" by $z$.
(65): From (Pom), taking $X := x \odot y$ and $Y := z$ and by (Pass).
(66): By "multiplying" (Pom) by $z$, and by (Pcomm), (Pass).
(67): In (65), take $X := (y \odot z)^-$, $Y := x$, $Z := (y \odot (y \odot z)^-)^-$ to obtain:

$$[(y\odot z)^-\odot(x\odot(y\odot(y\odot z)^-)^-)]^-\odot[(y\odot z)^-\odot(x\odot[(y\odot z)^-\odot(x\odot(y\odot(y\odot z)^-)^-)]^-)]^- = ((y\odot z)^-\odot x)^-. \tag{68}$$

On the other hand, in (66), take $X := y$, $Y := z$, $Z := x$ to obtain:

$$(y \odot z)^- \odot (x \odot (y \odot (y \odot z)^-)^-) = x \odot y^-. \tag{69}$$

Now, from (68), by (69), we obtain:
$(x \odot y^-)^- \odot ((y \odot z)^- \odot (x \odot (x \odot y^-)^-))^- = ((y \odot z)^- \odot x)^-$, i.e. (67) holds. □

**Proposition 4.21.** *(See Proposition 3.4)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(Pom) + (m - Pabs - i) \implies (Pmv).$$

**Proof.** (**By** *Prover9*)

• First, from (m-Pabs-i) ($x \odot (x^- \odot (x^- \odot y^-))^- = x$), by taking $Y := y^-$, we obtain:

$$x \odot (x^- \odot (x^- \odot y))^- = x. \tag{70}$$

• Now, we prove:

$$x \odot (y \odot (y^- \odot ((x \odot y)^- \odot z))^-) = x \odot y. \tag{71}$$

Indeed, in (70), take $X := x \odot y$, Y:= $(y \odot (y \odot x)^-)^- \odot z$ to obtain:

$$(x \odot y) \odot ((x \odot y)^- \odot ((x \odot y)^- \odot [(y \odot (y \odot x)^-)^- \odot z]))^- = x \odot y. \tag{72}$$

Now, from (72), by (64), we obtain:

$$(x \odot y) \odot ((x \odot y)^- \odot (y^- \odot z))^- = x \odot y. \tag{73}$$

From (73), by (Pass), (Pcomm), we obtain:
$x \odot (y \odot (y^- \odot ((x \odot y)^- \odot z))^-) = x \odot y$, i.e. (71) holds.

• Now, we prove:

$$x \odot (y^- \odot (y \odot ((y \odot z)^- \odot x)^-)^-) = x \odot y^-. \tag{74}$$

Indeed, in (71), take $X := x$, $Y := y^-$, $Z := [(y \odot z)^- \odot (x \odot (x \odot y^-)^-)]^-$ to obtain:

$$x \odot (y^- \odot (y \odot ((x \odot y^-)^- \odot [(y \odot z)^- \odot (x \odot (x \odot y^-)^-)]^-))^-) = x \odot y^-. \tag{75}$$

From (75), by (67), we obtain:
$x \odot (y^- \odot (y \odot ((y \odot z)^- \odot x)^-)^-) = x \odot y^-$, i.e. (74) holds.

• Now, we prove:

$$x^- \odot (y \odot (y \odot x)^-)^- = x^- \odot y^-. \tag{76}$$

Indeed, in (74), take $X := (x \odot (x \odot y)^-)^-$, $Y := y$, $Z := x$ to obtain:

$$(x \odot (x \odot y)^-)^- \odot (y^- \odot [y \odot ((y \odot x)^- \odot (x \odot (x \odot y)^-)^-)^-]^-) = (x \odot (x \odot y)^-)^- \odot y^-. \tag{77}$$

In (63), take $X := y$, $Y := x$, to obtain:

$$(y \odot x)^- \odot (x \odot (x \odot y)^-)^- = x^-. \tag{78}$$

Then, from (77), by (78), we obtain:

$$(x \odot (x \odot y)^-)^- \odot (y^- \odot (y \odot x^=)^-) = (x \odot (x \odot y)^-)^- \odot y^-. \tag{79}$$

From (79), by (DN), we obtain:
$(x \odot (x \odot y)^-)^- \odot (y^- \odot (y \odot x)^-) = (x \odot (x \odot y)^-)^- \odot y^-$, hence, by (Pcomm), (Pass), we obtain:
$y^- \odot ((x \odot y)^- \odot ((x \odot y)^- \odot x)^-) = (x \odot (x \odot y)^-)^- \odot y^-$, hence by (Pom), we obtain:
$y^- \odot x^- = (x \odot (x \odot y)^-)^- \odot y^-$, hence, by interchanging $x$, $y$, we obtain:
$x^- \odot y^- = (y \odot (y \odot x)^-)^- \odot x^-$, hence, by (Pcomm), $x^- \odot (y \odot (y \odot x)^-)^- = x^- \odot y^-$, i.e. (76) holds.

• Now, finally, from (76), by $X := x^-$, $Y := y^-$ and (DN), (Pcomm), we obtain:
$x \odot ((x^- \odot y^-)^- \odot y^-)^- = x \odot y$, i.e. (Pmv) holds. □

Note that Proposition 3.4 follows from Proposition 4.21, since (m-Pimpl) implies (m-Pabs-i).

Note also that Proposition 4.21 says: **OMWL** $\subset$ **PreMV**.

**Remark 4.22.** *The following converse of Proposition 4.21 ((Pmv) + (m-Pabs-i) $\Longrightarrow$ (Pom)) does not hold: there are examples of involutive m-BE algebras verifying (Pmv) and (m-Pabs-i) and not verifying (Pom).*

• **OMWL** + (Pqmv) (Connections between **OMWL** and **QMV**)

We establish now the connection between the OMWLs and the QMV algebras verifying (m-Pabs-i).

**Proposition 4.23.** *(See Proposition 3.7)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OMWL. Then, $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pabs-i). (i.e. in an involutive m-BE algebras, (Pom) + (m-Pabs-i) $\Longrightarrow$ (Pqmv).)*

**Proof.** Since $\mathcal{A}^L$ is a left-OMWL, it is an involutive m-BE algebra verifying (m-Pabs-i) and (Pom) (Definition 2). By Proposition 4.21, it verifies (Pmv) also. Hence, $\mathcal{A}^L$ is a left-QMV algebra verifying (m-Pabs-i). □

Note that Proposition 4.23 says:

$$\mathbf{OMWL} \subset \mathbf{QMV}, \tag{80}$$

the inclusion being strict since there are examples of QMV algebras not verifying (m-Pabs-i). Note also that Propositions 3.7 and 4.21 follow from Proposition 4.23.

The following converse of Proposition 4.23 holds.

**Proposition 4.24.** *(See Proposition 3.8)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra verifying (m-Pabs-i). Then, $\mathcal{A}^L$ is a left-OMWL. (i.e. in involutive m-BE algebras, (Pqmv) + (m-Pabs-i) $\Longrightarrow$ (Pom).)*

**Proof.** Since $\mathcal{A}^L$ a left-QMV algebra verifying (m-Pabs-i), it is an involutive left-m-BE algebra verifying (Pqmv) (hence (Pmv), (Pom)) and (m-Pabs-i) (Definition 2). Hence, $\mathcal{A}^L$ is an involutive m-BE algebra verifying (m-Pabs-i) and (Pom), i.e. it is a left-orthomodular widelattice. □

Note that Proposition 4.24 says: **QMV** $\cap$ **OWL** $\subset$ **OM**. Note also that Proposition 3.8 follows from Proposition 4.24, since (m-Pimpl) $\Longrightarrow$ (m-Pabs-i).

By Propositions 4.23 and 4.24, we obtain:

**Theorem 4.25.** *(See Theorem 3.9)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(m-Pabs-i) \implies ((Pom) \Leftrightarrow (Pqmv))$$

*or*

$$(m-Pabs-i) + (Pom) \iff (Pqmv) + (m-Pabs-i),$$

*i.e. orthomodular widelattices coincide with QMV algebras verifying (m-Pabs-i).*

Note that Theorem 4.25 says:

$$\mathbf{OMWL} = \mathbf{QMV} + (m-Pabs-i) = \mathbf{QMV} \cap \mathbf{OWL}. \tag{81}$$

Note also that Theorem 3.9 follows from Theorem 4.25.

By (62), (81) and Remark 4.22, we obtain the connections from Figure 17.

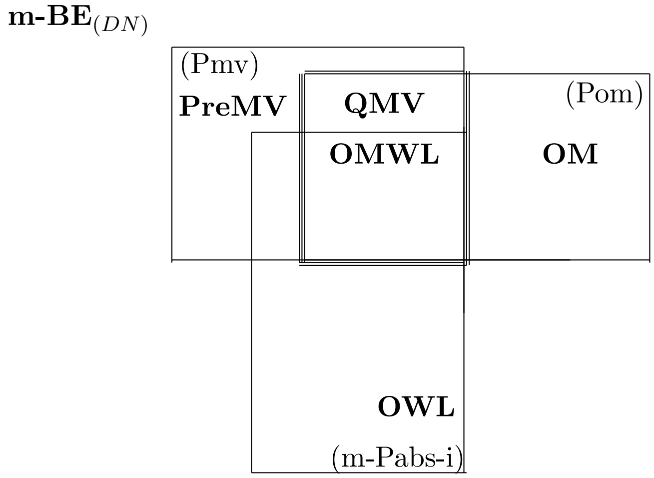


**Figure 17:** Resuming connections between **QMV**, **PreMV**, **OWL**, **OM** and **OMWL**

- **OMWL** + $(\Delta_m)$ (Connections between **OMWL** and **MMV**)

**Proposition 4.26.** *(See Proposition 3.10)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(Pom) \ + \ (m - Pabs - i) \implies (\Delta_m).$$

**Proof.** By Proposition 4.21, (Pom) + (m-Pabs-i) imply (Pmv) and (Pmv) implies $(\Delta_m)$, thus (Pom) + (m-Pabs-i) imply $(\Delta_m)$. □

Note that Proposition 4.26 says: **OMWL** $\subset$ **MMV**.

Note also that Proposition 3.10 follows from Proposition 4.26, since (m-Pimpl) implies (m-Pabs-i), that Proposition 4.23 follows also from Proposition 4.26, since (Pom) + $(\Delta_m)$ imply (Pqmv), and that Proposition 4.26 follows also from Proposition 4.23, since (Pqmv) implies $(\Delta_m)$.

**Remark 4.27.** *The following converse of Proposition 4.26 ($(\Delta_m)$ + (m-Pabs-i) $\implies$ (Pom)) does not hold: there are examples of involutive m-BE algebras verifying $(\Delta_m)$ and (m-Pabs-i) and not verifying (m-Pimpl) and (Pom).*

By the previous Remark and by the connections from Figure 17, we obtain the connections from Figure 18.

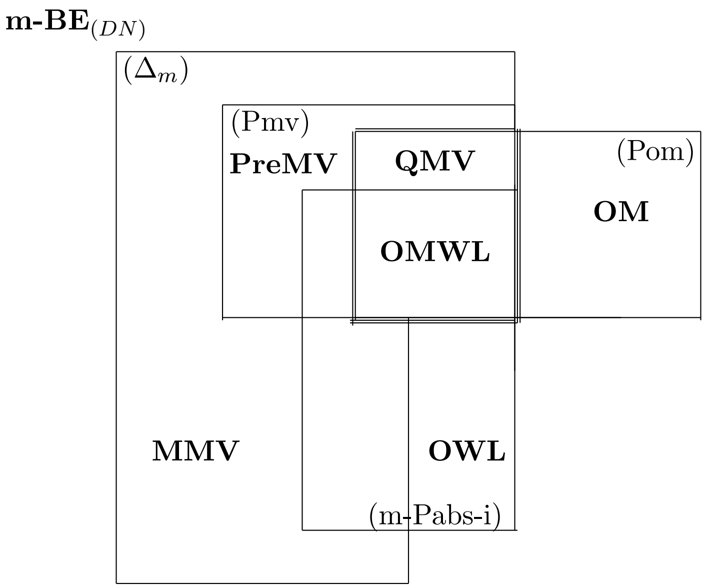


**Figure 18:** Resuming connections between **QMV**, **PreMV**, **MMV**, **OWL** and **OMWL**

### 4.2.2 OML $\subset$ OMWL

We know (by Proposition 2.16) that:

**Proposition 4.28.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(m - Pimpl) \implies (m - Pabs - i),$$

*i.e.* **OL** $\subset$ **OWL**.

**Proposition 4.29.** *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) + (G) \implies (m - Pabs - i).$$

**Proof.** By Proposition 4.13, (Pom) + (G) imply (m-Pimpl), and by Proposition 4.28, (m-Pimpl) implies (m-Pabs-i). $\square$

Note that Proposition 4.29 follows from Proposition 4.13.

Note also that Proposition 4.29 says: **OML** (= **OMSL** ) $\subset$ **OWL**, hence,

$$\textbf{OML} \ (= \textbf{OMSL}) \ \subset \ \textbf{OMWL}, \tag{82}$$

the inclusion being strict, since there are examples of OMWLs not verifying (G).

Note also that **OML** (= **OMSL**) $\subset$ **OMWL** means (see 23):

$$\textbf{OML} = \textbf{OMSL} \ \cap \ \textbf{OMWL}.$$

By (28), (62) and (82), we obtain the connections from the Figure 19.

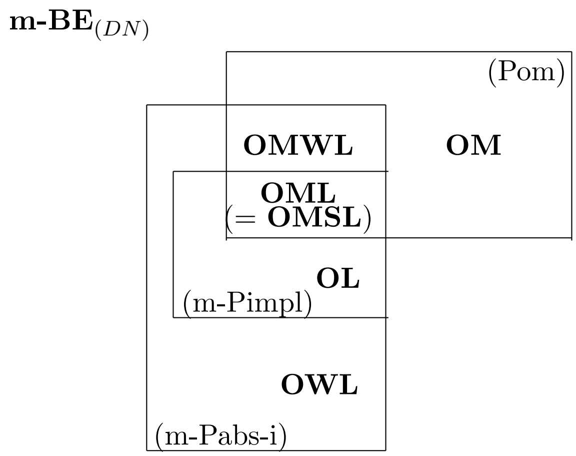


**Figure 19:** Resuming connections between **OMWL**, **OML**, **OL**, **OWL** and **OM**

Since **OML** = **OMSL** ⊂ **OMWL**, by Theorems 4.14 and 4.29, and **OMWL** ⊂ **QMV**, by (80), we obtain:

$$\mathbf{MOD} \subset \mathbf{OML} = \mathbf{OMSL} \subset \mathbf{OMWL} \subset \mathbf{QMV}.$$

By the connections from Figures 4, 17 and 19, we obtain the connections from Figure 20.

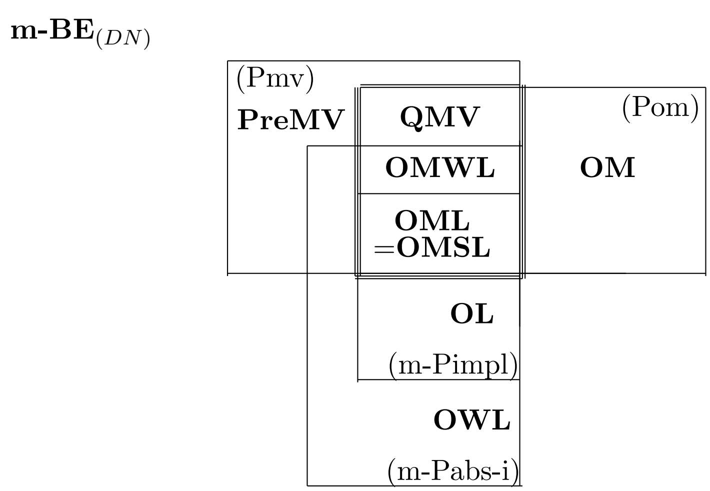


**Figure 20:** Resuming connections between **QMV**, **PreMV**, **OML**, **OWL**, **OL**, **OM** and **OMWL**

By the connections from Figures 5, 18 and 20, we obtain the connections from Figure 21.

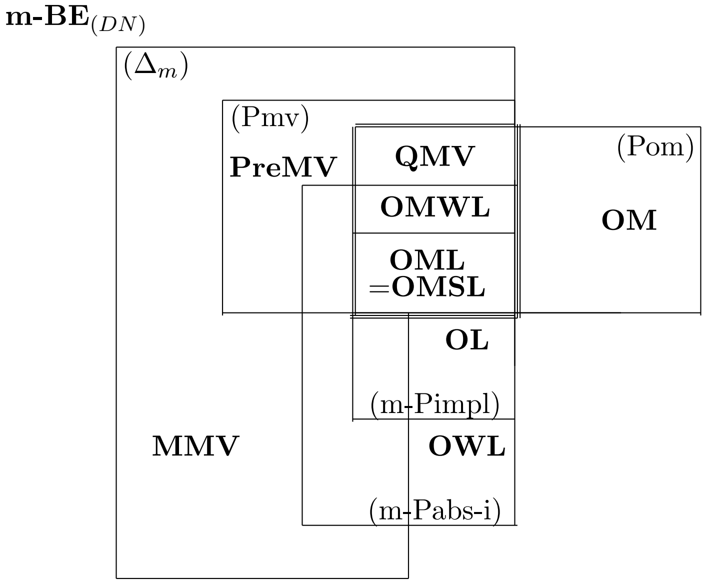


**Figure 21:** Resuming connections between **QMV**, **PreMV**, **MMV**, **OML**, **OL**, **OWL** and **OMWL**

#### 4.2.3 The transitive and/or antisymmetric case

**• The transitive case: tOWL ⊂ tMMV**

Denote by **tOMWL** the class of all transitive left-OMWLs.

**Theorem 4.30.** *(See Theorem 4.17)*
*Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$(m - Pabs - i) \ + \ (m - BB) \implies (\Delta_m).$$

Note that this theorem is Theorem 5.13 from [22], proved by *Prover*9. It says that: **tOWL** ⊂ **MMV**. Hence, **tOWL** ⊂ **tMMV**.

If, additionally, (Pom) holds, then, as expected: **tOMWL** ⊂ **tQMV**.

Note that Theorem 3.17 follows also from Theorem 4.30, since (m-Pimpl) implies (m-Pabs-i).

By (42), by Theorems 3.17 and 4.30 and the connections from Figure 21, we obtain the connections from Figure 22.

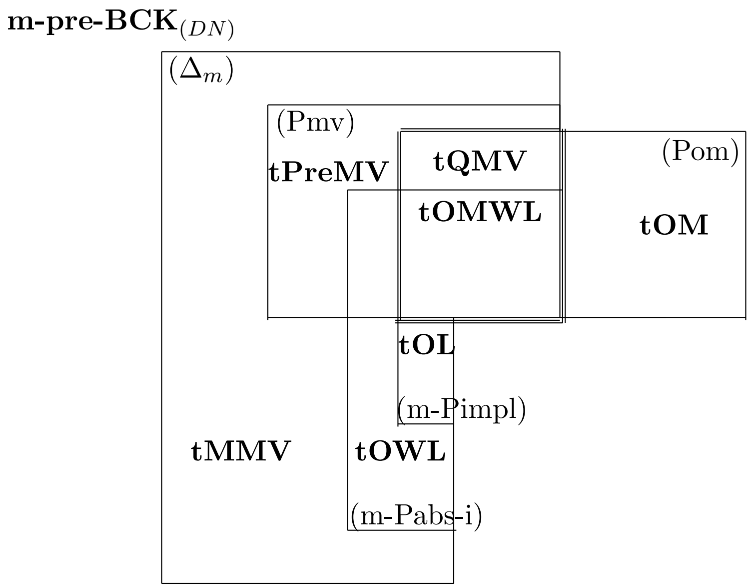


**Figure 22:** Resuming connections between **tQMV**, **tMMV**, **tOWL** and **tOL**

• **The transitive and the antisymmetric case**

Denote by **aOMWL** the class of all antisymmetric left-OMWLs.

**Theorem 4.31.** *We have:*

$$\mathbf{aOMWL} = \mathbf{taOMWL}.$$

**Proof.** Since **OMWL** $\subset$ **QMV**, by adding (m-An), we obtain: **aOMWL** $\subset$ **aQMV** $=$ **MV**, by Theorem 2.24, and since any MV algebra verifies (m-Tr), it follows that **aOMWL** $=$ **taOMWL**. □

While **tOMWL** $\subset$ **tOWL**, we obtain the following results.

**Theorem 4.32.** *We have:*
*(i)* **taOWL** $\subset$ **MV***; (ii)* **taOMWL** $\subset$ **MV***; (iii)* **taOWL** $=$ **taOMWL***.*

**Proof.** $(i)$ Since **tOWL** $\subset$ **tMMV**, by applying (m-An), we obtain:
**taOWL** $\subset$ **taMMV** $=$ **MV**, by Theorem 2.24.

$(ii)$ Since **tOMWL** $\subset$ **tQMV**, by applying (m-An), we obtain:
**taOMWL** $\subset$ **taQMV** $=$ **MV**, by Theorem 2.24.

$(iii)$ Since any MV algebra verifies (Pom), it follows by $(i)$ that **taOWL** $=$ **taOMWL**. □

**Theorem 4.33.** *We have:*

$$\mathbf{taOWL} = \mathbf{taOMWL} = \mathbf{aOMWL} \subset \mathbf{MV}.$$

**Proof.** By Theorems 4.31, 4.32. □

• **Final remarks** We have:

**tOMWL** $\subset$ **tQMV**

(m-An) $\downarrow$ $\downarrow$ (m-An)

**taOMWL** $=$ **taOWL** $\subset$ **MV**.

The tOMWLs (inside the tQMV algebras) will be deeply analysed in next paper [19], in connection with the taOWLs (inside the MV algebras).

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