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Characterization of Topological Fuzzy Sets in Hausdorff Spaces

Benard Okelo*, **Allan Onyango**

Abstract. In this paper, we have characterized big data fuzzy sets and shown that topological data points form singleton fuzzy sets which are closed. Besides, fuzzy sets of topological data points are compact and have at least one closed point. We have also shown that the fuzzy set of all condensation points of a fuzzy Hausdorff space is infinite and the cardinality of a topological data fuzzy set is also infinite and arbitrarily distributed in fuzzy Hausdorff spaces.

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1 Introduction

Studies on fuzzy sets have been carried out over a long period of time particularly in topological spaces with interesting results obtained and open problems indicated (see [1]-[16] and the references therein). From the beginning, fuzzy set theory has gained a lot of advancement in a variety of ways and in several fields [5]. Nice applications of fuzzy set theory have been seen in several disciplines like artificial intelligence, topological spaces, computer engineering, medical engineering, control and instrumentation engineering, risk theory, game theory, decision theory, expert systems, logical functions analysis, management systems science, operations research, face and pattern recognition among others [8]. With respect to mathematical developments, fuzzy set theory has led to a very high level of improvement in modern research with applications to real life problems [13]. This work describes the the pertinent logical framework of fuzzy set theory, together with very important significance of this theory to other methods and theories. Since the beginning of this area of study [15] by Lotfi Zadeh in 1965, several aspects have been considered in the study of fuzzy sets. These include: The intuitionist case [2], the empty set[3], singleton fuzzy sets among others[4]-[9]. These aspects have been utilized in several areas like logic [10]-[13], programming and decision making particularly in optimization and profit making in the business sector [14]. In this work, we consider fuzzy sets in topological spaces [8], particularly the Hausdorff space. We take advantage of the fact that for any two fuzzy sets say A, B in a Hausdorff space, $A \cap B = \emptyset$. This helps in classification of the fuzzy sets into different classes without overlaps. From early 90's, fuzzy set theory, neural circuits and programming of evolution acquired the title computational intelligence also known as soft computing [16]. There exists a very important relationship between these areas making them to be naturally equivalent in some sense. In this study, however, we particularly embark primarily on fuzzy Hausdorff spaces with applications to real life problems which are indispensable. For better under standing of this work, we give some preliminary notes which are very instrumental in the next section.

***Corresponding author:** Benard Okelo, Email: bnyaare@yahoo.com, ORCID: 0000-0003-3963-1910
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2 Preliminaries

We provide basic concepts which are useful in the sequel.

Definition 2.1. ([16], Definition 1) *Let X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. $\mu_{\tilde{A}}(x)$ is called the membership function (generalized characteristic function) which maps X to the membership space M. Its range is the subset of nonnegative real numbers whose supremum is finite. For $\sup \mu_{\tilde{A}}(x) = 1$ we have a normalized fuzzy set.*

Remark 2.2. *In Definition 2.1, the membership function of the fuzzy set is a crisp (real-valued) function. Zadeh [15] also defined fuzzy sets in which the membership functions themselves are fuzzy sets.*

Definition 2.3. ([17], Definition 3.2) *A type m fuzzy set is a fuzzy set whose membership values are type $m-1, m > 1$, fuzzy sets on $[0, 1]$.*

Remark 2.4. *For operations on fuzzy set see [16] and the references therein.*

Definition 2.5. ([6], Definition 2.3) *Let X be a fuzzy topological space and $\mathbf{H}$ be a nonempty fuzzy compact Hausdorff subspace of X. A point $\mathrm{a} \in \mathbf{H}$ is called a topological data point (TDP) if $\mathbf{H}^c \setminus \{\mathrm{a}\}$ is a compact fuzzy subspace of $\mathbf{H}$. The set of all topological data points is called a Topological Data Set (TDS). If this set is fuzzy then we call it a Fuzzy Topological Data Set (FTDS).*

Definition 2.6. ([11], Definition 1.5) *A nonempty fuzzy compact Hausdorff space $\mathbf{H}$ is called a TDP fuzzy space if every $\mathrm{a} \in \mathbf{H}$ is a TDP.*

Remark 2.7. *Let $\mathbf{H}$ be a fuzzy topological space. Then $\mathbf{H} = \mathbf{P} \dagger \mathbf{Q}$ means $\mathbf{P}$ and $\mathbf{Q}$ are nonempty fuzzy subsets of $\mathbf{H}$ such that $\mathbf{H} = \mathbf{P} \cup \mathbf{Q}$ and $\mathbf{P} \cap \bar{\mathbf{Q}} = \bar{\mathbf{P}} \cap \mathbf{Q} = \emptyset$.*

3 Topological data sets in fuzzy Hausdorff spaces

In this section, we characterize Topological Data Points in a fuzzy Hausdorff space. We begin with the following proposition.

Proposition 3.1. *Let $\mathbf{H}$ be a TDP fuzzy space and $\mathrm{a} \in \mathbf{H}$ such that $\mathbf{H}^c \setminus \{\mathrm{a}\} = \mathbf{P} \dagger \mathbf{Q}$. If $\{\mathrm{a}\}$ is open then $\mathbf{P}$ and $\mathbf{Q}$ are closed and if $\{\mathrm{a}\}$ is closed, then $\mathbf{P}$ and $\mathbf{Q}$ are open.*

Proof. Let $a \in \mathbf{H}$ be open and suppose that $\mathbf{P}$ is both open and closed in $\mathbf{H}^c \setminus \{a\}$. Without loss of generality, there exists a closed subset $\mathbf{R}$ of $\mathbf{H}$ such that $\mathbf{P} = \mathbf{R} \cap (\mathbf{H}^c \setminus \{a\}) = \mathbf{R}^c \setminus \{a\}$. Hence, $\mathbf{H}^c \setminus \{a\} = \mathbf{P} \dagger \mathbf{Q}$ meaning $\mathbf{Q}$ of $\mathbf{H}$ is closed and so is $\mathbf{P}$. Conversely, let $a \in \mathbf{H}$ be closed. Putting the same argument as the forward case, there exists an open set $\mathbf{Z}$ of $\mathbf{H}$ in which $\mathbf{P} = \mathbf{Z} \cap (\mathbf{H}^c \setminus \{a\}) = \mathbf{Z}^c \setminus \{a\}$. Therefore, $\mathbf{P} = \mathbf{R}^c \setminus \{a\} = \mathbf{Z}^c \setminus \{a\}$. Hence, $\mathbf{H}^c \setminus \{a\} = \mathbf{P} \dagger \mathbf{Q}$ meaning $\mathbf{Q}$ of $\mathbf{H}$ is open and so is $\mathbf{P}$. This completes the proof. □

This proposition leads to characterization of topological Data points in terms of compactness.

Lemma 3.2. *Let $\mathbf{H}$ be a TDP fuzzy space and $\mathrm{a} \in \mathbf{H}$. If $\mathbf{H}^c \setminus \{\mathrm{a}\} = \mathbf{P} \dagger \mathbf{Q}$ then $\mathbf{P} \cup \{\mathrm{a}\}$ is compact.*

Proof. Without loss of generality, let $\mathbf{W}$ and $\mathbf{V}$ be connected fuzzy subsets of $\mathbf{H}$ in which $\mathbf{P} \cup \{a\} = \mathbf{W} \dagger \mathbf{V}$. Let $a \in \mathbf{W}$. Then $\mathbf{V} \subseteq \mathbf{P}$. Now $(\overline{\mathbf{V}} \cup \overline{\mathbf{W}}) \cap \mathbf{V} = (\overline{\mathbf{Q}} \cap \mathbf{V}) \cup (\overline{\mathbf{W}} \cap \mathbf{V}) = \emptyset$. So $(\overline{\mathbf{V}} \cup \overline{\mathbf{W}}) \cap \mathbf{V} = \emptyset$ and consequently, $(\mathbf{Q} \cap \mathbf{W}) \cap \overline{\mathbf{V}} = \emptyset$ implying $\mathbf{H} = (\mathbf{Q} \cap \mathbf{W}) \dagger \mathbf{V}$. □

Example 3.3. Consider the points on n straight lines in the Euclidean plane with standard topology R^2. The union of these n straight lines is a compact TDP fuzzy space if and only if either all of them are concurrent or exactly $n-1$ of them are parallel.

Theorem 3.4. *Let* $\mathbf{H}$ *be a TDP fuzzy space and* $\mathrm{a} \in \mathbf{H}$. *If* $\mathbf{H}^c \setminus \{\mathrm{a}\} = \mathbf{P} \dagger \mathbf{Q}$ *and if every point of* $\mathbf{P}$ *is TDP in* $\mathbf{H}$ *then* $\mathbf{P}$ *has at least one closed point.*

Proof. Suppose that $\mathbf{P}$ is compact then by Proposition 3.1, $\mathbf{P} \cup \{a\}$ is compact. So $\{a\}$ is closed. Let $z_0 \in \mathbf{P}$. By Lemma 3.2, $\mathbf{H}^c \setminus \{z_0\} = \bigcup_{z=\mathbf{P}, z\neq z_0} \{\{a, z\} \cup (\mathbf{Q} \cup \{a\})\}$ is also compact. This contradicts the earlier hypothesis that a is TDP point of $\mathbf{H}$. □

Example 3.5. Consider the Euclidean plane with standard topology R^2. Let $X_0 = \{(x, 0) \in R^2 : x \leq 0\} \cup \{(x, 1) \in R^2 : x > 0\}$ and let for each positive integer n, $Y_n = \{(\frac{1}{n}, y) \in R^2 : 0 < y \leq 1\}$. Define $K = X_0 \cup (\cup_{n=1}^{\infty} Y_n)$. Then X is a TDP fuzzy space with at least one closed point.

Remark 3.6. *All TDP fuzzy spaces are connected spaces. However, a finite fuzzy topological space is not a TDP.*

Example 3.7. Consider the Khalimsky line given as follows. Let Z be the set of integers and let $D = \{\{2i-1, 2i, 2i+1\} : i \in Z\} \cup \{\{2i+1\} : i \in Z\}$. Then D is a base topology for Z. The set Z with this topology is a TDP fuzzy space which is connected.

At this juncture, we locate Topological Data points of Big Data fuzzy Sets in a fuzzy Hausdorff space. We state the following proposition.

Proposition 3.8. *Let* $\mathbf{H}$ *be a TDP fuzzy space. The set* $\mathbf{A}_0$ *of all condensation points of* $\mathbf{H}$ *is a fuzzy TDS which is infinite.*

Proof. Let $a_1, a_2,...$ be a sequence of distinct condensation points in $\mathbf{H}$. By induction, we have a condensation point a_0 in $\mathbf{A}_0 \subseteq \mathbf{H}$. But a_0 is a TDP of $\mathbf{H}$. So we have open TDS $\mathbf{W}_1$ and $\mathbf{V}_1$ of $\mathbf{H}$ such that $\mathbf{H}^c \setminus \{a_1\} = \mathbf{W}_1 \dagger \mathbf{V}_1$. Suppose that $a_1, a_2, ..., a_n$ are in $\mathbf{H}$ and open subsets $\mathbf{W}_i$ and $\mathbf{V}_i (i \in \mathbf{N})$ are picked such that $\mathbf{H}^c \setminus \{a_1\} = \mathbf{W}_i \dagger \mathbf{V}_i$, where $i = 1, ..., n$. Clearly, by induction and considering $\mathbf{W}_{i+1}$ and $\mathbf{V}_{i+1}$, the set $\mathbf{A}_0$ of all condensation points of $\mathbf{H}$ is infinite. □

The above Proposition 3.8 takes us to characterization of the size of the fuzzy sets. We give the size of the fuzzy data set in the next lemma.

Corollary 3.9. *Let* X *be a fuzzy topological space and* $\mathbf{H}$ *be a TDP fuzzy subspace of* X. *Then* $Card\mathbf{H} = \infty$.

Proof. By Hausdorff Maximal Principle (HMP) and by Proposition 3.8, the proof is complete. □

Next, we establish the distribution patterns of the topological Data Points within a fuzzy Hausdorff space in the following theorem.

Theorem 3.10. *All TDS in TDP fuzzy space are arbitrarily distributed if they are* $\boldsymbol{T}_2$. *Moreover, each FTDS has at least two TDPs with closed subsets of FTDS which are singletons.*

Proof. Let $\mathbf{H}$ be a TDP fuzzy space with two subsets $\mathbf{H}_1$ and $\mathbf{H}_2$. Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be TDP fuzzy subspaces of $\mathbf{H}$. Then it implies that if $\mathbf{H}_1$ and $\mathbf{H}_2$ are both empty then trivially we are done. Let $\mathbf{H}_1$ and $\mathbf{H}_2$ be non-empty. It remains to show that $\mathbf{H}_1 \cap \mathbf{H}_2 = \emptyset$ and hence it is $\mathbf{T}_2$. To see this, consider $a_1 \in \mathbf{H}_1$ and $a_2 \in \mathbf{H}_2$ such that $a_2 \notin \mathbf{H}_1$ and $a_1 \notin \mathbf{H}_2$. Clearly, $\mathbf{H}_1 \cap \mathbf{H}_2 = \emptyset$, hence it is Hausdorff. Now we show that $\mathbf{H}$ has at least two TDPs. Let $\mathbf{H}$ be such that it has at most one TDP. Let $a_1 \in \mathbf{H}$ and $\mathbf{H} \setminus \{a_1\} = \mathbf{P}_0 \dagger \mathbf{Q}_0$ for some $\mathbf{P}_0, \mathbf{Q}_0$, which are subsets of $\mathbf{H}$. Since $\mathbf{H}$ has only one TDP then either $\mathbf{P}_0$ or $\mathbf{Q}_0$ has TDPs. By proposition 4.8, $\mathbf{P}_0$ has some condensation point of $\mathbf{P}$ say a. Let $\mathbf{H} \setminus \{a\} = \mathbf{P} \dagger \mathbf{Q}$. Without loss of generality, let $a \in \mathbf{Q}$. By Hausdorff Maximal Principle, there is an optimal chain $\mathbf{C}$ in $\mathbf{S}$ of $\mathbf{H}$ such that for some $\mathbf{U}_\alpha$ of $\mathbf{S}$, $\bigcup_{\alpha\in\Lambda} \mathbf{U}_\alpha \in \mathbf{H}$. Hence, $\mathbf{H}$ is compact. Let $\mathbf{V} = \bigcup_{\alpha\in\Lambda} \mathbf{U}_\alpha$, then by Lemma 4.9, we can get at least two points of $\mathbf{H}$ which give a subcover for $\mathbf{H}$. Since the subcovers are open by Heine-Borel Property, each set forms a singleton set. □

Example 3.11. Consider the Euclidean plane with standard topology R^2. Let $X_1 = \{(x, y) \in R^2 : x \leq 0 \text{ and } |y| = 1\}$ and let $X_2 = \{(x, y) \in R^2 : x > 0 \text{ and } y = \sin\frac{1}{x}\}$. Define $X = X_1 \cup X_2$. Then X is a FTDS with at least two TDPs with closed subsets of FTDS which are singletons.

4 Conclusion

In this work, we have characterized big data fuzzy sets and shown that topological data points form singleton fuzzy sets which are closed. Besides, fuzzy sets of topological data points are compact and have at least one closed point. We have also shown that the fuzzy set of all condensation points of a fuzzy Hausdorff space is infinite and the cardinality of a fuzzy topological data set is also infinite and arbitrarily distributed in fuzzy Hausdorff spaces. For further research, this work can be extended by characterizing topological data point and sets in soft fuzzy Haursdorff spaces.

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Benard Okelo
Department of Pure and Applied Mathematics
Jaramogi Oginga Odinga University of Science and Technology
Bondo, Kenya
E-mail: bnyaare@yahoo.com

Allan Onyango
Department of Pure and Applied Mathematics
Jaramogi Oginga Odinga University of Science and Technology
Bondo, Kenya
E-mail: onyango.onyangoa@gmail.com