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Modified Linear Approximation for Assessment of Rigid Block Dynamics

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Abstract

This study proposes a new linear approximation for solving the dynamic response equations of a rocking rigid block. Linearization assumptions which have already been used by Hounser and other researchers cannot be valid for all rocking blocks with various slenderness ratios and dimensions; hence, developing new methods which can result in better approximation of governing equations while keeping simplicity is necessary. In this paper, a new linear approximation is derived for solving the deferential equations of a rocking block in order to include wider range of blocks with various slenderness. The proposed method is verified by numerical solutions of the governing equations utilizing two methods of: average acceleration and fourth order Runge-Kutta. Verifications revealed more reasonable accuracy of the proposed method in comparison with the current linearization assumptions.
Keywords: Rigid; Block; Rocking; Linearization; Equipment

1. Introduction

During earthquakes, many rigid structures such as nonstructural components, electrical substations, industrial equipments, and concrete radiation shields may be affected by ground motions and set into the rocking motions and toppling. The large amounts of nonstructural damages mostly due to the toppling of the rigid nonstructural components in the past earthquakes highlighted the need for modification of the current building codes for such components [1]. For rocking response analysis of such rigid components, a model composed of a rigid block in a rigid base has been widely used. The wide application and simplicity of this model has stimulated many researchers to use such a model [2-8]. An early analytical study in this field is Housner in which, minimum acceleration amplitude of a half-sine pulse required for toppling of a rigid block was derived analytically [8]. Housner concluded that toppling of a given block depends on the product of the acceleration amplitude of the pulse by its duration. Alsam et al analytically and experimentally showed that the response of a rigid block under seismic excitation is in line with response to single pulse excitation [6]. This research was taken up and continued by Spanos and Koh in which "safe" and "unsafe" regions were introduced [9]. Tso and Wong studied steady-state rocking responses of a rigid block analytically and experimentally and concluded inconsistencies between analytical and experimental results [10, 11] that further clarified Hogan by introducing the concepts of orbital stability and Poincaré maps [4,5]. Based on Alsam et al and Iwan and Chen, Makris studied [2,6,12] rocking responses of a rigid block using equivalent pulse of near-field earthquakes and declared that Housner's derivation for the required minimum amplitude of a half-sine pulse for overturning of a rigid block is un-conservative. For rocking response analysis, the appropriate rocking governing equations can be solved numerically by direct integration or analytically by incorporating some linearization into the governing equations. Incorporation of such a linearization may cause some inconsistencies between real response and approximated response by using linearization. The wide applications of analytical solutions in derivation of valuable outcomes, such as minimum toppling

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acceleration, justifies more efforts for more accurate derivation of analytical solutions. Whereas the major assumption in derivation of analytical solution is slenderness of the blocks, this assumption limits the applicability of such derivations. The purpose of this research is developing analytical solutions based on modified linearization method that can match numerical solutions better without incorporating complex terms and can be applicable for wider ranges of blocks.

2. Governing Equation

In this study, a free-standing object is modelled as a rectangular rigid body subjected to the horizontal excitation as shown in Fig. 1.

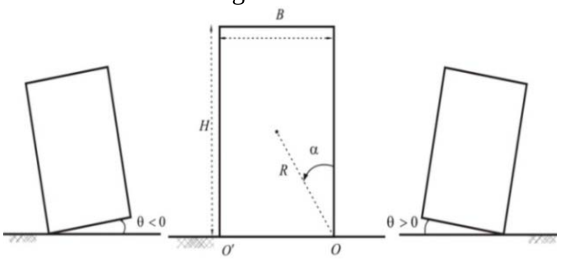


Fig. 1. Rigid Block on rigid base

Assuming that the coefficient of friction is large enough, no sliding will occur between block and base. In such a case, depending on the ground acceleration, block may move rigidly with the base or set into the rocking motion. It is assumed that the block will oscillate rigidly about centres of $O\,(\theta \succ 0)$ and $O'\,(\theta \prec 0)$. By taking moment about rotation centres of O and O', the dynamic governing equations can be written as the following:

$$I\ddot{\theta} = -MgRSin(\alpha-\theta) - M\ddot{u}_g RCos(\alpha-\theta)$$
$$\theta_{cr} \succ \theta \succ 0 \qquad (1)$$
$$I\ddot{\theta} = +MgRSin(\alpha+\theta) - M\ddot{u}_g RCos(\alpha+\theta)$$
$$-\theta_{cr} \prec \theta \prec 0 \qquad (2)$$

Where:

M : Mass of block

I : Moments of inertia about point O or O' and can be defined as: $I = 4MR^2/3$

g : Gravity acceleration

θ_{cr} : Critical rotation of block ($\theta_{cr} = \alpha$)

Housner [8] took into account simplifying assumptions for derivation of analytical solutions from equation(1) and (2) as the following:

$$Cos(\alpha \pm \theta) \cong 1 \qquad Sin(\alpha \pm \theta) \cong \alpha \pm \theta \qquad (3)$$

By using the above assumptions and also assuming $\ddot{u}_g = a_p \sin(\omega t + \psi)$, governing equation (1) and (2) can be rewritten as follows:

$$\ddot{\theta} - p^2\theta = -\frac{a_p}{g} Sin(\omega t + \psi)\, p^2 - p^2\alpha$$
$$\theta_{cr} \succ \theta \succ 0 \qquad (4)$$
$$\ddot{\theta} - p^2\theta = \frac{a_p}{g} Sin(\omega t + \psi)\, p^2 + p^2\alpha$$
$$-\theta_{cr} \prec \theta \prec 0 \qquad (5)$$

Where:

p : System parameter and can be defined as $p = \sqrt{\frac{3g}{4R}}$

ψ : Phase difference which can be defined as following:

$\psi = \sin^{-1}(\alpha g / a_p)$

The analytical solution of equation of (4) and (5) can be formed as follows (Makris[2]);

$$\theta = A_1 Sinh(pt) + A_2 Cosh(pt) + \alpha + \frac{1}{1+\frac{\omega^2}{p^2}} \frac{a_p}{g} Sin(\omega t + \psi)$$
$$\theta_{cr} \succ \theta \succ 0 \qquad (6)$$
$$\theta = A_3 Sinh(pt) + A_4 Cosh(pt) - \alpha + \frac{1}{1+\frac{\omega^2}{p^2}} \frac{a_p}{g} Sin(\omega t + \psi)$$
$$-\theta_{cr} \prec \theta \prec 0 \qquad (7)$$

In which:

$$A_1 = A_3 = \frac{\dot{\theta}_0}{p} - \frac{\frac{\omega}{p}}{1+\frac{\omega^2}{p^2}} \frac{a_p}{g} Cos(\psi) \qquad (8)$$
$$A_2 = \theta_0 + \alpha - \frac{1}{1+\frac{\omega^2}{p^2}} \frac{a_p}{g} Sin(\psi) \qquad (9)$$
$$A_4 = \theta_0 - \alpha - \frac{1}{1+\frac{\omega^2}{p^2}} \frac{a_p}{g} Sin(\psi) \qquad (10)$$

The above linearization and appropriate derivations are in line with many published researches in assessment of various aspects of rigid block dynamics (e.g. [2-8]), but because of Housner's initiation for using such an assumption hereinafter it will be called "Housner linearization".

3. Modified Approximation

In the new modified linearization, as mentioned before, it is tried to use better approximation of governing equations while keeping simplicity. The new assumptions can be expressed as follows:

$$Cos(\alpha \pm \theta) \cong Cos(\alpha) \qquad (11)$$

$$Sin(\alpha \pm \theta) = Sin(\alpha)Cos(\theta) \pm Cos(\alpha)Sin(\theta) \cong Sin(\alpha) \pm \theta Cos(\alpha)$$
(12)

Substituting equations (11) and (12) in the governing equations of (1) and (2), these equations can be rewritten as:

$$\ddot{\theta} - p^2 Cos(\alpha)\theta = \frac{a_g}{g} Sin(\omega t + \psi') p^2 Cos(\alpha) - p^2 Sin(\alpha).\frac{Cos(\alpha)}{Cos(\alpha)}$$

$$\theta_{cr} \succ \theta \succ 0 \qquad (13)$$

$$\ddot{\theta} - p^2 Cos(\alpha)\theta = \frac{a_g}{g} Sin(\omega t + \psi') p^2 Cos(\alpha) + p^2 Sin(\alpha).\frac{Cos(\alpha)}{Cos(\alpha)}$$

$$-\theta_{cr} \prec \theta \prec 0 \qquad (14)$$

Where:

$$\psi' = \sin^{-1}(g / a_p .B / H) = \sin^{-1}(g / a_p \tan(\alpha))$$

Finally, the governing equation can be formed as follows:

$$\ddot{\theta} - p'^2\theta = \frac{a_p}{g} Sin(\omega t + \psi') p'^2 - p'^2 \tan(\alpha)$$

$$\theta_{cr} \succ \theta \succ 0 \qquad (15)$$

$$\ddot{\theta} - p'^2\theta = \frac{a_p}{g} Sin(\omega t + \psi') p'^2 + p'^2 \tan(\alpha)$$

$$-\theta_{cr} \prec \theta \prec 0 \qquad (16)$$

In which:

$$p' = p\sqrt{Cos(\alpha)}$$

By solving equations of (14) and (15) analytically, response equations can be formed as follows:

$$\theta = A_1' Sinh(p't) + A_2' Cosh(p't) + \tan(\alpha) + \frac{1}{1+\frac{\omega^2}{p'^2}} \frac{a_p}{g} Sin(\omega t + \psi')$$

$$\theta_{cr} \succ \theta \succ 0 \qquad (17)$$

$$\theta = A_3' Sinh(p't) + A_4' Cosh(p't) - \tan(\alpha) + \frac{1}{1+\frac{\omega^2}{p'^2}} \frac{a_p}{g} Sin(\omega t + \psi')$$

$$-\theta_{cr} \prec \theta \prec 0 \qquad (18)$$

$$A_1' = A_3' = \frac{\dot{\theta}_0}{p'} - \frac{\frac{\omega}{p'}}{1+\frac{\omega^2}{p'^2}} \frac{a_p}{g} Cos(\psi') \qquad (19)$$

$$A_2' = \theta_0 + \tan(\alpha) - \frac{1}{1+\frac{\omega^2}{p'^2}} \frac{a_p}{g} Sin(\psi') \qquad (20)$$

$$A_4' = \theta_0 - \tan(\alpha) - \frac{1}{1+\frac{\omega^2}{p'^2}} \frac{a_p}{g} Sin(\psi') \qquad (21)$$

By comparison of the equations (7) to (21), it is clear that the new linearization substantially affects the system parameter of p and changes it to p' ($p\sqrt{\cos(\alpha)}$). As concluded by Yim, the hyperbolic divergent nature of rocking response is one the factors which causes sensitivity of response to initial conditions and system parameters [7]. Considering Yim's conclusion, changing parameters of p into p' that incorporates hyperbolic terms may make major differences between the proposed linearization and Housner linearization.

This matter is also valid for changing constants of $A_{1,2,,3,4}$ into $A'_{1,2,3,4}$ that multiply into hyperbolic terms. In the following sections, the efficiency of the proposed linearization will be examined against numerical results.

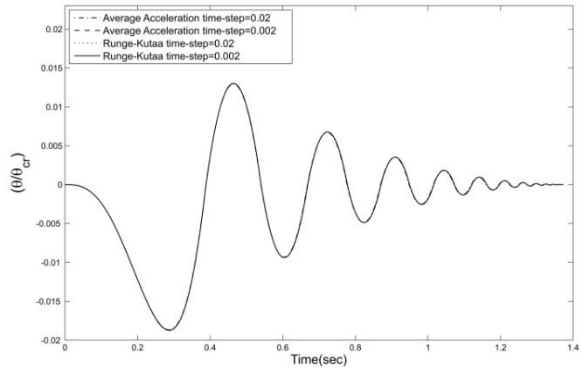
4. Numerical Solution

Whereas dynamic behaviour of a rigid block is geometrically nonlinear and also chaotic, more care must be taken in choosing the numerical algorithm [1,2,3,4]. Yim et al utilized the fourth order Runge-Kutta method in assessment of rigid block dynamics under seismic excitation with small time steps in order to satisfy numerical stability [7]. The fourth order Runge-Kutta was examined by Xiefor nonlinear damped and un-damped doffing oscillator [12]. Although Xie declared that this method usually is not used in finite element codes for transient dynamic analysis but conclusions of his study showed far more accurate results of this method in comparison with Hoboult, Wilson-θ, and central difference method when time step is small enough. In utilizing implicit methods in solving nonlinear problems, Batheas a general recommendation proposed using a combination of unconditionally stable methods (e.g. average acceleration) and modified Newton-Raphson. In this research, two solvers of fourth order Runge-Kutta and average acceleration using modified Newton Raphson were utilized and as shown in Fig (2), results show a good compatibility of these two methods [13].

Besides the choice of a suitable solver, another important issue in assessment of dynamic response of a rigid block, as can be seen in equations (1) to (21), is finding contact time of the block to its base. This time, the governing equation suddenly switches from one set to another set of equations with new initial velocity condition; therefore, a locating event algorithm, as pointed out by Bernal, is needed [14].

In this research, fractional time stepping algorithm is utilized as already used by Mahin and Nau in developing nonlinear response spectra [15,16]. The schematic procedure of this method according to the Mahin research can be seen in Fig (3) [15].

This procedure will repeat for required fractions till the event with prescribed criteria is detected. Allen and Duan assumed $|\theta| \prec 10^{-6}$ criteria for detecting contact; in this research, the same criteria has been taken into account for detecting the event [17]. For better interpretation of the comparison of the results, this algorithm was not only used for numerical solutions but also in analytical solutions.



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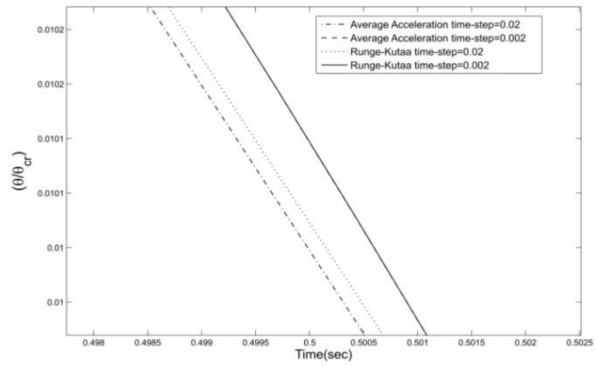


(b)

Fig. 2. Comparison of the rocking time histories using Average-acceleration and fourth order Runge-Kutta for:
$B = 1m, H = 3m, a/g = 0.4$, $\omega = 2\pi$

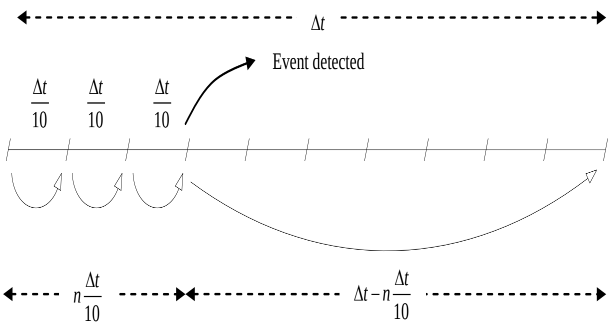
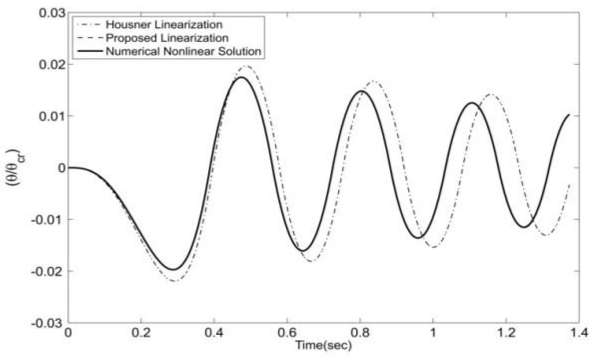


Fig. 3. Schematic procedure of fractional time stepping according Mahin research [15]

5. Comparison of Results

In order to compare the modified linearization and Housner linearization, first, a comparison of time histories for arbitrary blocks under half-sine and full-sine pulse was done and following that, a sensitivity analysis with selection of maximum rotation ratio (θ/θ_{cr})as the comparison criterion was performed. The aim of this sensitivity analysis is examining generality of modified linearization efficiency in a wide range of excitations and block sizes.

The results of the first section (comparison of time histories) are presented in Fig (4) for half-sine excitation and in Fig(5) for full-sine excitation. As it is obvious in Fig (4), the differences between numerical results using Housner linearization increased with the decrease in aspect ratio (H/B), although the consistency between numerical results and modified linearization was kept reasonably. This observation is also valid for full-sine pulse excitation and as can be seen in some cases like Fig (5-b) and Fig (5-c) Housner results lead to overturning conclusions tough the numerical and modified linearization results still report survival of block under such an excitation.



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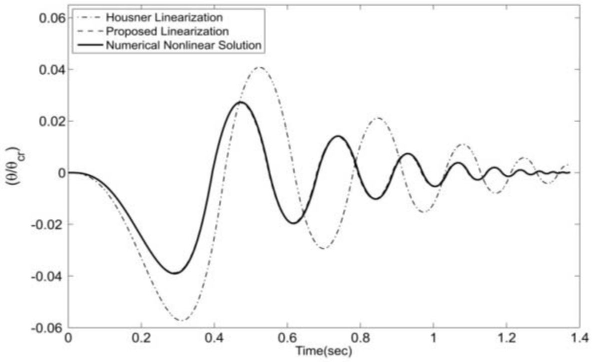

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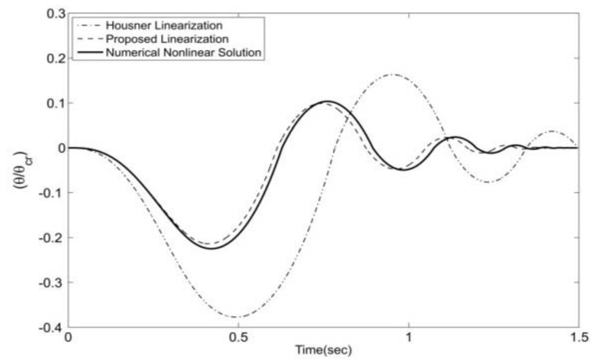


(c)

Fig. 4. Comparison of results under half-sine excitation:
(a) $B = 0.5m, H = 3m, a/g = 0.2$
(b) $B = 0.5m, H = 1.5m, a/g = 0.4$
(c) $B = 0.5m, H = 1m, a/g = 0.7$, in all cases $\omega = 2\pi$

In sensitivity analysis, three aspect ratios(H / B) of 2:1 3:1 and 4:1 were selected and in each aspect ratio two block widths of 0.5m and 1m for incorporating size effects in sensitivity analysis were selected. In selection of pulse amplitudes, whereas minimum acceleration for initiation of rocking mode is $(B/H)g$,an aptitude bigger than this value is needed for conclusion of rocking motion;

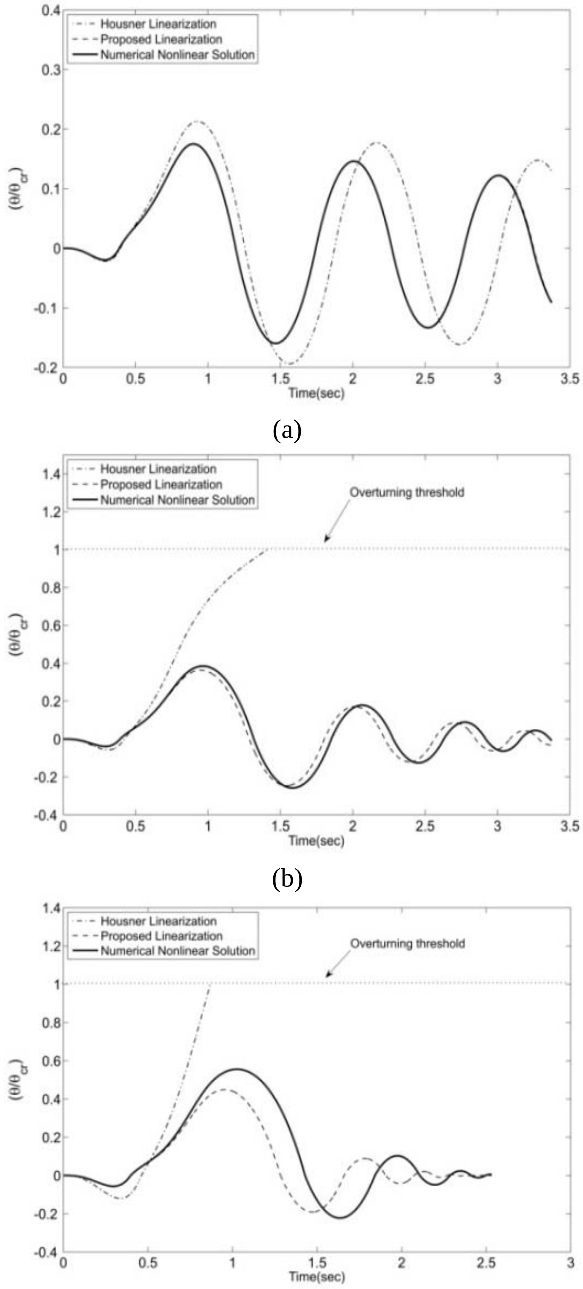


Fig. 5. Comparison of results under full-sine excitation:
(a) $B = 0.5m, H = 3m, a/g = 0.2$
(b) $B = 0.5m, H = 1.5m, a/g = 0.4$
(c) $B = 0.5m, H = 1m, a/g = 0.6$, in all cases $\omega = 2\pi$

in this analysis, amplitudes of: $a_p = (B/H + 0.05)g$ and $a_p = (B/H + 0.1)g$ for half-sine pulse excitation and $a_p = (B/H + 0.1)g$ and $a_p = (B/H + 0.125)g$ for full-sine pulse excitation were selected arbitrarily.

In each selected block with prescribed excitation amplitude, the excitation frequency (ω) sweeps ranges from $0.1p$ to $4p$ with increments of $0.1p$. The final results of these four excitations are presented in Fig (6) to Fig (9). As it can be seen in Figures (6) to (9), in high frequency excitation, all methods (i.e. numerical analysis, Housner and modified linearization) lead to close estimation of maximum rotation though in low frequency(e.g. $\omega / p \approx 1$) the deviation between Housner linearization and numerical analysis is noticeable; inversely, the modified linearization kept its consistency with numerical results reasonably.

While comparing the two blocks with different aspect ratio, the block with a smaller aspect ratio presented more deviation between Housner linearization and Modified Linearization method.

Although sensitivity analysis repeated for a vast range of pulse amplitudes rather than the above mentioned excitations, all conclusions were in line with the results and conclusions presented in this paper.

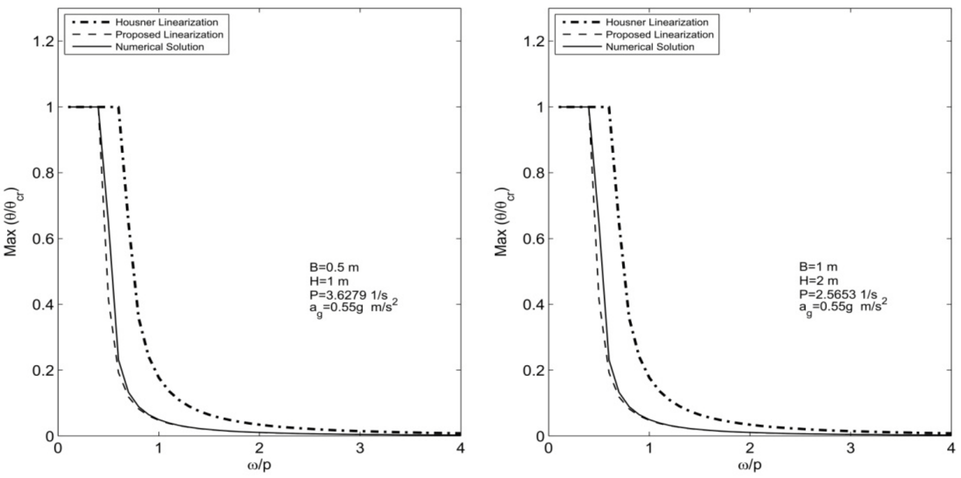
6. Conclusion

In this study, a new linearization method targeting at the reduction of inconsistencies between numerical results and analytical results was introduced. Although the fundamentals of the new linearization seem simple, it was showed that they can cause noticeable modifications in final results. This study is only one-step-forward and for better approval of modified linearization efficiency more follow up analyses are recommended. Based on this study and other analyses not presented in the current paper, the following conclusions can be made:

a) The system parameter of p does not seem as a suitable parameter for introducing a block characteristic and it seems that the parameter of $p\sqrt{\cos(\alpha)}$ can make a better consistency with the nature of block and may lead to closer solutions with numerical methods when incorporated in the final equation..

b) In the fragility analysis based on “safe” and “unsafe” regions using Housner linearization, which has been used widely in the literature, can cause misleading results and as shown in this paper, in some cases (even aspect ratio of 1:3), results report toppling; hence, numerical results do not show toppling.

c) In the application of dynamical response of a rigid block in seismic vulnerability assessment of nonstructural components, facilities, equipments and so on, researchers may face varieties of slenderness ratios that can be assessed by using modified linearization and wider range of blocks with results closer to the numerical solutions.



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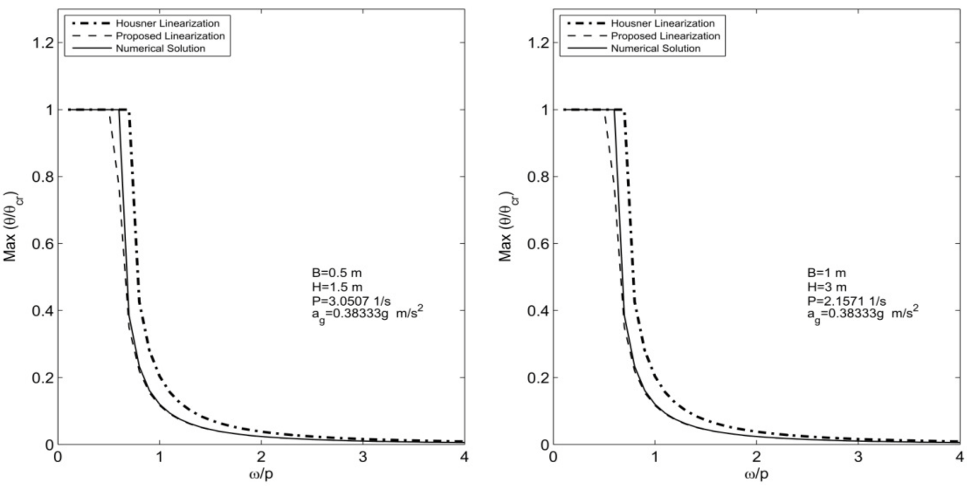

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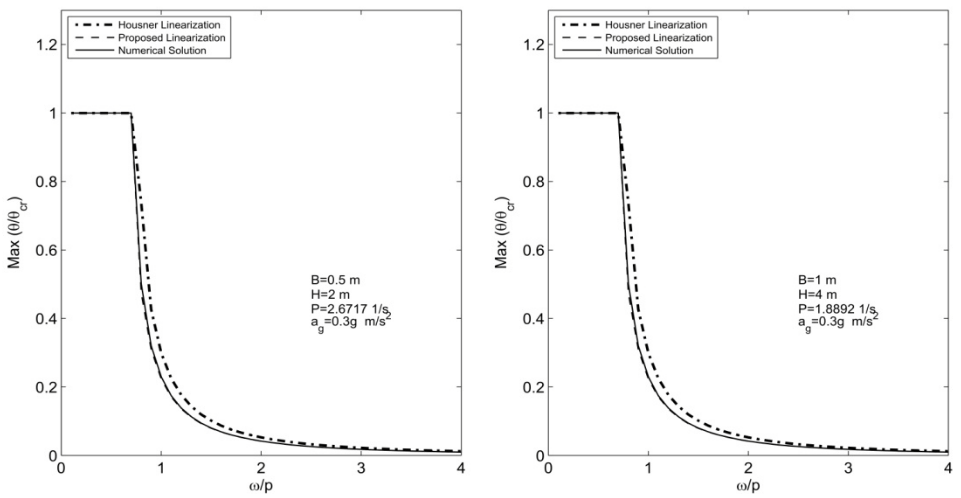


(c)

Fig. 6. Comparison of maximum rotation which are calculated using various methods under half-sine pulse excitation , (a):aspect ration=2:1, (b) aspect ratio=3:1, (c):aspect ratio=4:1, in each section left figures are for blocks with width of 0.5m and right figures for width of 1 m , $a_p = (B/H + 0.05)g$

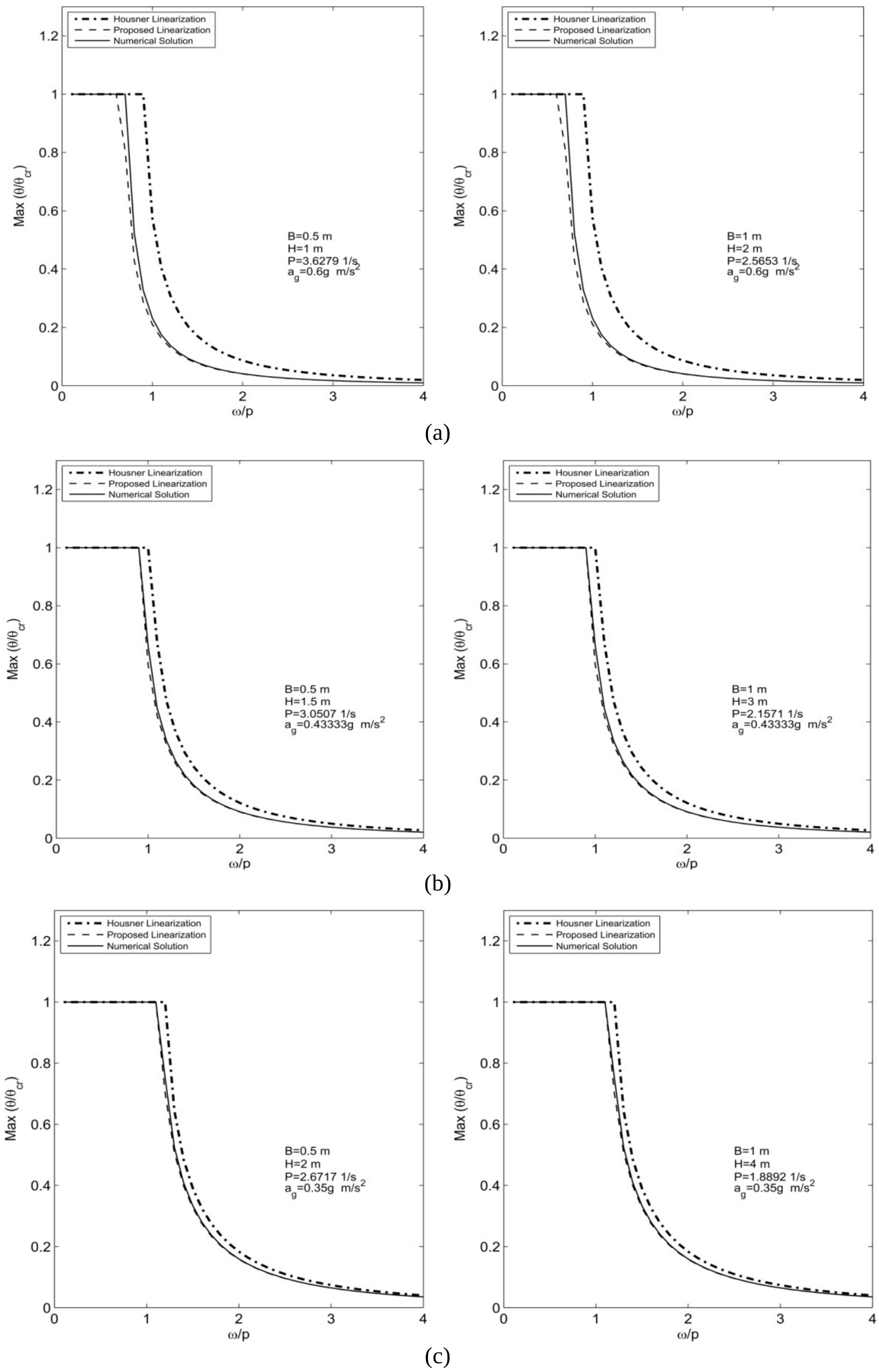
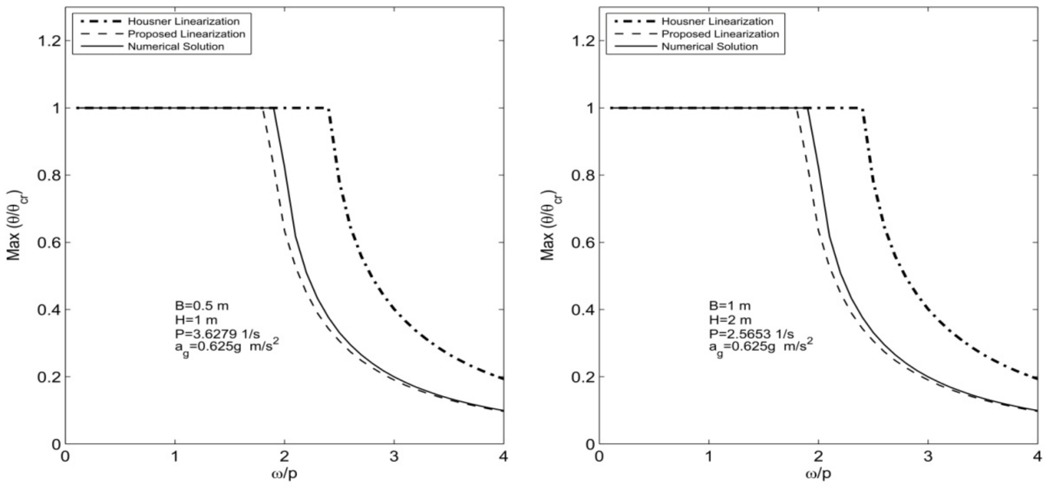


Fig. 7. Comparison of maximum rotation which are calculated using various methods under half-sine pulse excitation , (a):aspect ration=2:1, (b) aspect ratio=3:1, (c):aspect ratio=4:1, in each section left figures are for blocks with width of 0.5m and right figures for width of 1 m, $a_p = (B/H + 0.1)g$



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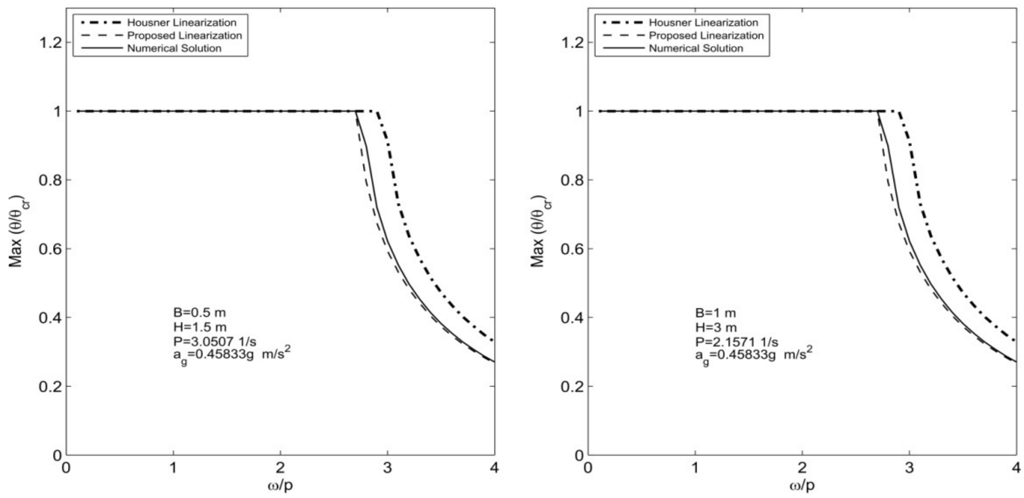

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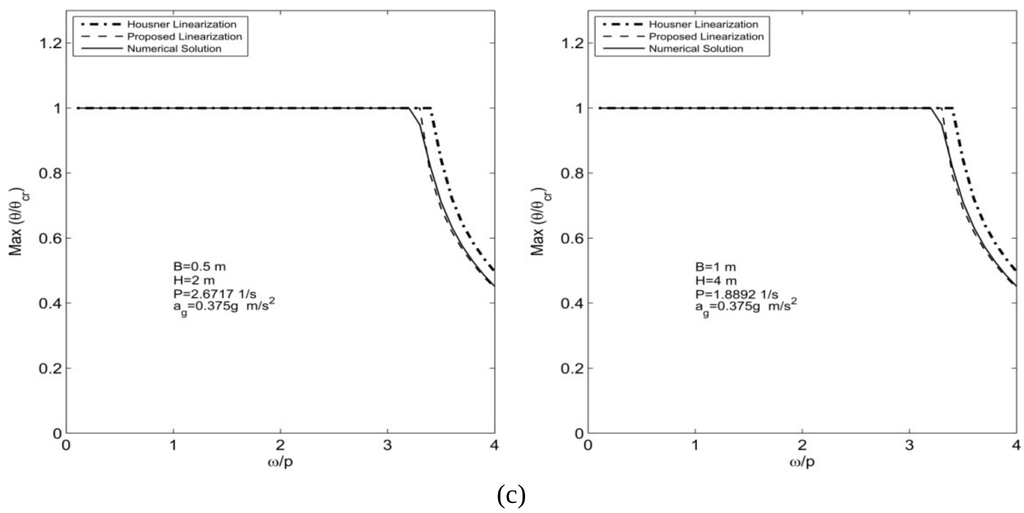

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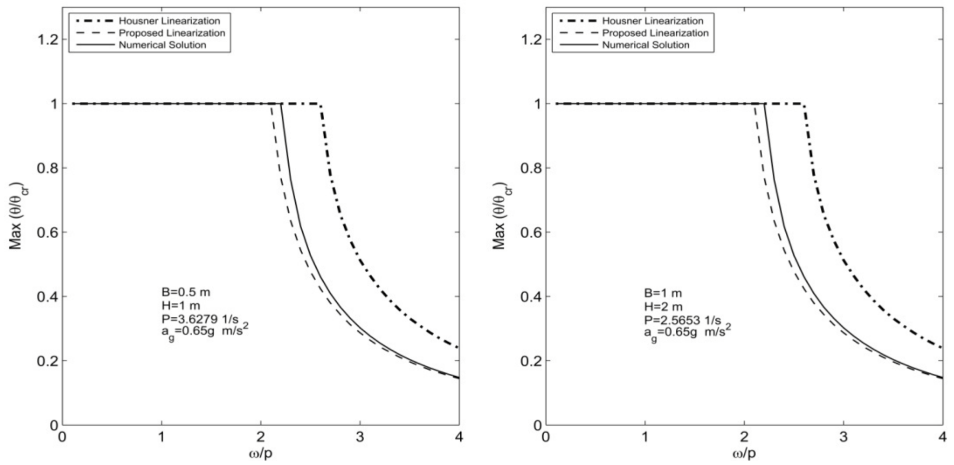
Fig. 8. Comparison of maximum rotation which are calculated using various methods under full-sine pulse excitation: (a):aspect ration=2:1, (b) aspect ratio=3:1, (c):aspect ratio=4:1, in each section left figures are for blocks with width of 0.5m and right figures for width of 1 m, $a_p = (B/H + 0.125)g$



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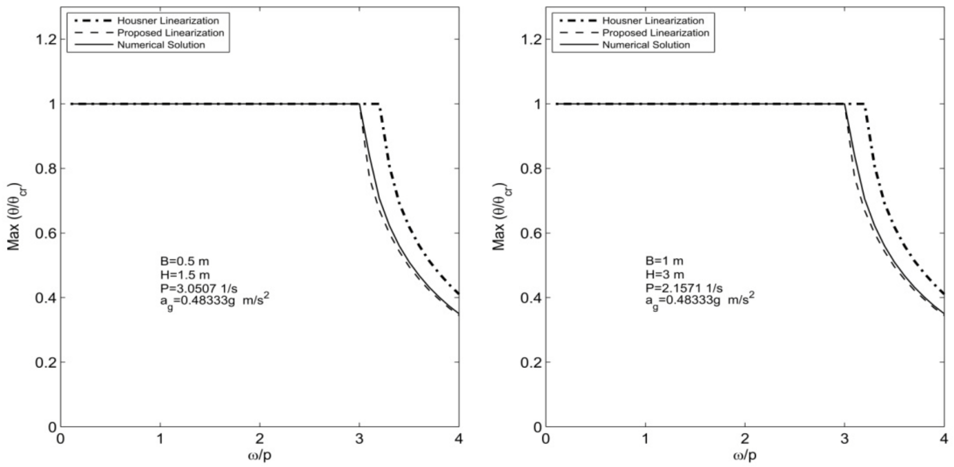

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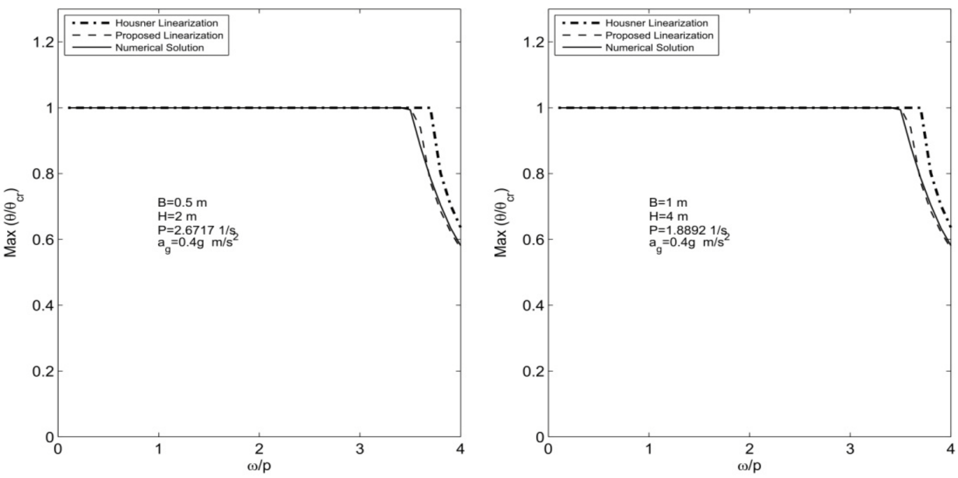

(c)

Fig. 9. Comparison of maximum rotation which are calculated using various methods under full-sine pulse excitation: (a):aspect ration=2:1, (b) aspect ratio=3:1, (c):aspect ratio=4:1, in each section left figures are for blocks with width of 0.5m and right figures for width of 1 m, $a_p = (B/H + 0.125)g$

References

[1] Singh MP, Mreschi LM, Suarez LE, Matheu EE. Seismic design forces I: Rigid non-structural components 2006; Journal of Structural Engineering, 132(10), 1524-1532

[2] Makris N, Roussos YS. Rocking response of rocking blocks under near-source ground motion 2000; Geotechnique, 50(3), 243-262

[3] Yim C, Lin H. Nonlinear Impact and chaotic response of slender rocking objects, Journal of Engineering Mechanics 1991, 117(9), 2079-2100.

[4] Hogan SJ. The many steady state responses of a rigid block under harmonic forcing 1990, Earthquake Engineering and Structural dynamics 1990, 19, 1057-1071.

[5] Hogan SJ, On the dynamics of rigid-block motion under harmonic forcing 1989, Proceeding Royal Society, B. London, 425, 441-476.

[6] Aslam M, Godden WG, Scalise DT. Earthquake rocking response of rigid bodies, Journal of engineering Mechanics Division 1980, 106(2), 377-392.

[7] Yim C., Chopra AK, Penzien J. Rocking response of rigid blocks to earthquakes, Earthquake Engineering and Structural Dynamics 1980, 8, 565-587.

[8] Housner GW. The behavior of inverted pendulum structures during Earthquakes, Bulletin of Seismological Society of America 1963, 53(2), 404-417.

[9] Spanos PD, Koh AS. Rocking of rigid blocks due to harmonic shaking 1984, Journal of Engineering Mechanics Division. 110, 1627-1642.

[10] Tso WK, Wong CM. Steady state rocking response of rigid blocks Part 1: Analysis 1989; Earthquake Engineering and Structural dynamics, 18, 89-106.

[11] Wong CM, Tso WK. Steady state rocking response of rigid blocks Part 2: Experiment 1989; Earthquake Engineering and structural dynamics, 18, 107-120.

[12] Iwan WD, Chen XD. Important near-field ground motion from the Landers earthquake 1994, Proceeding of 10th European Conference in Earthquake Engineering, Balkema, Rotterdam, 1, 229-234

[13] Bathe KJ. Finite element procedures, Prentice-Hall Inc, U.S.A, New Jersey, 1996

[14] Xie YM. An assessment of time integration schemes for non-linear dynamics equations 1996, Journal of Sound and Vibration, 192(1), 321-331.

[15] Mahin S.A., Boroschek R., Zeris C., Engineering interpretation of the responses of three instrumented buildings in San Jose, proc.SIMP89, 1989.

[16] Nau JM. Computation of inelastic spectra 1983, Journal of Engineering Mechanics Division, 109(1), 279-288.

[17] Allen RH, Duan X. Effects of linearization on rocking block toppling 1995; Journal of Structural Engineering, 121(7), 1146-1149