Absorbing Markov Chain Models to Determine Optimum Process Target Levels in Production Systems with Rework and Scrapping

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Abstract

In this paper, absorbing Markov chain models are developed to determine the optimum process mean levels for both a single-stage and a serial two-stage production system in which items are inspected for conformity with their specification limits. When the value of the quality characteristic of an item falls below a lower limit, the item is scrapped. If it falls above an upper limit, the item is reworked.Otherwise, the item passes the inspection. This flow of material through the production system can be modeled in an absorbing Markov chain characterizing the uncertainty due to scrapping and reworking. Numerical examples are provided to demonstrate the application of the proposed model.

Keywords: Quality Inspection; Rework; Markovian Model; Expected Profit.

1. Introduction

The determination of optimum process mean is one of the most important decision-making problems encountered in industrial applications. Consider a certain production process, where an item is reworked if the value of its quality characteristic falls above an upper specification limit and it is scrapped when it falls below a lower specification limit.

A dimension in the surface finishing processes of metals is an example of this scenario. In this situation, on the one hand if the process mean is set too low, then the proportion of non- conforming items becomes high and the decision maker experiences high rejection costs associated with non- conforming items. On the other hand, if the process mean is set too high, then the proportion of reworking items becomes high, resulting in a higher reworking cost. This justifies the determination of the optimum process mean [1].

Al-Sultan and Pulak [2] proposed a model considering a production system with two stages in series to find the optimum mean values with a lower specification limit and application of a 100% inspection policy. Ferrell and Chhoker [4] proposed a method to determine the optimal acceptance sampling plans economically. Their Approach is based on the Taguchi loss function to quantify deviations between a quality characteristic and its target Level. Bowling et al [1] employed a Markovian model in order to maximize the total profit associated with a multi-stage serial production system. They tried to determine optimal process target levels, in which lower and upper specification limits are given at each stage. In addition, they assumed each quality characteristic is governed by a normal distribution and screening (100%) inspection is performed. Further, Pillai and Chandrasekharan [3] modeled the flow of material through the production system as an absorbing Markov chain considering scrapping and reworking. Their model promises better system design in material requirement planning, capacity requirement planning, and inventory control.

In the present research, similar to Bowling et al[1], the flow of a discrete production process is modeled into an absorbing Markov chain.

In other words, in this process, not all items reach the final stage due to scrapping and reworking. Hence, the absorbing Markov chain stochastic process model will be adopted. The data required for such a model are: (i) the probability which an item goes from one stage of production to the next, and (ii) the probability of reworking and scrapping items at various stages. At every stage of production, the item is inspected; if it does not conform to its specifications, it is either scrapped or reworked. The reworked item will be inspected again. However, there are two main differences between the current work and the one in Bowling et al [1]. The first

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relates to the assumption that in the present work the nonconforming items are repaired in a separate station. In other words, while in Bowling et al [1] the nonconforming items are repaired in the main station, in the current research they are repaired in a separate repair station. The second difference is the incorrectness of the objective function of the two-stage process in Bowling et al. (2004), where they multiplied the selling price of an item (SP) by $(1-f_{14})(1-f_{24})$. Knowing that in the derived Markovian model the $(1-f_{14})$ coefficient shows the percentage of conforming items, it is not necessary to multiply $(1-f_{14})SP$ by $(1-f_{24})$ again. In this research, we revise the objective function of Bowling et al [1], trying to offer a better solution.

The rest of the paper is organized as follows. We first present the required notations in section 2. The model development comes next in section 3. The numerical demonstration on the application of the proposed methodology is given in section 4. Finally, we conclude the paper in section 5.

2. Notations

The required notations are:

U_i : The upper specification limit in the i^{th} stage of the production, $i=1,2$

L_i : The lower specification limit in the i^{th} stage of the production, $i=1,2$

p_{ij} : The probability of going from state *i* to state *j* in a single step

f_{ij} : The long run probability of going from a non-absorbing state (*i*) to another absorbing state (*j*)

$E(PR)$: The expected profit per item

$E(RV)$: The expected revenue per item

$E(PC)$: The expected processing cost per item

$E(SC)$: The expected scrapping cost per item

$E(RC)$: The expected reworking cost per item

SP : The selling price of an item

PC_i : The processing cost of the i^{th} stage

SC_i : The scrapping cost of the i^{th} stage

RC_i : The reworking cost of the i^{th} stage

$\mathbf{P}$: The transition probability matrix

$\mathbf{Q}$: The transition probability matrix of going from a non-absorbing state to another non-absorbing state

$\mathbf{R}$: A matrix containing all probabilities of going from a non-absorbing state to another absorbing state (i.e., accepted or rejected item)

$\mathbf{I}$: The identity matrix

$\mathbf{O}$: A matrix with zero elements

$\mathbf{M}$: The fundamental matrix

$\mathbf{F}$: The absorption probability matrix

3. Model Development

Consider a serial production system in which items are 100% inspected in all stages. The item is then reworked, accepted or scrapped. As raw materials come into the production system and finally go out of it, a state in the Markovian model represents different conditions of the raw materials, i.e., reworking, scrapping, and accepting. In other words, an item can be in one of its three states modeled by a discrete random variable X. As time (t) goes on, the random variable X generates a random process $\{X(t):t>0\}$. This stochastic process with discrete state space and discrete values of the parameter t becomes a discrete time (first order) Markov chain when transition from one state to the next depends only on the current state. Among the states, some are transient and the others absorbing. A Markov chain with one or more absorbing states is known as absorbing Markov chain. When an item is in an absorbing state, it never leaves the state (Pillai and Chandrasekharan [3]).

The expected profit per item in the system under consideration can be expressed as follows:

$$E(PR)=E(RV)-E(PC) \\ -E(SC)-E(RC) \tag{1}$$

Then, in what follows a single-stage production system is first modeled. The two-stage modeling comes next.

3.1. The single-stage system

Consider a single-stage production system with the following states:

State 1: An item is being processed by the production system

State 2: An item is being reworked

State 3: An item is accepted to be finished work

State 4: An item is scrapped.

Then, the single-step transition probability matrix can be expressed as:

$$\mathbf{P}=\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & p_{12} & p_{13} & p_{14} \\ 2 & 0 & 0 & p_{23} & p_{24} \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array} \tag{2}$$

where p_{12} is the probability of reworking an item, p_{13} is the probability of accepting an item, and p_{14} is the probability of scrapping an item. Assuming that the quality characteristic of an item follows a normal distribution with mean μ_1 and standard deviation σ_1, these probabilities can be expressed as (Bowling et al. 2004):

$$p_{12} = \int_{U_1}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} dx_1 = 1-\Phi\left(U_1\right) \tag{3}$$

$$p_{13} = \int_{L_1}^{U_1} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} dx_1 = \Phi\left(U_1\right)-\Phi\left(L_1\right) \tag{4}$$

$$p_{14} = \int_{-\infty}^{L_1} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} dx_1 = \Phi\left(L_1\right) \tag{5}$$

Moreover, p_{23} and p_{24} denote the probabilities of accepting and scrapping a reworked item, respectively and the historical data from the production system can be used to determine the value of p_{23} and p_{24}. Note that in order to analyze the transition probability matrix $\mathbf{P}$ in an absorbing Markov chain, we rearranged it in the following form:

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix} \tag{6}$$

By determining the fundamental matrix $\mathbf{M}$ as:

$$\mathbf{M} = \left(\mathbf{I}-\mathbf{Q}\right)^{-1} = \begin{bmatrix} 1 & -p_{12} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & p_{12} \\ 0 & 1 \end{bmatrix} \tag{7}$$

the absorption probability matrix $\mathbf{F}$ can then be obtained as follows (Bowling et al. 2004):

$$\mathbf{F} = \mathbf{M} \times \mathbf{R} = \begin{bmatrix} 1 & p_{12} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} p_{13} & p_{14} \\ p_{23} & p_{24} \end{bmatrix} = \begin{bmatrix} p_{13}+p_{12}p_{23} & p_{14}+p_{12}p_{24} \\ p_{23} & p_{24} \end{bmatrix} = \begin{bmatrix} f_{13} & f_{14} \\ p_{23} & p_{24} \end{bmatrix} \tag{8}$$

where f_{13} and f_{14} are the probabilities of accepting and scrapping an item.
Now, the expected profit per item of equation (1) will be:

$$E\left(PR\right) = f_{13}SP - PC_1 - f_{14}SC_1 - p_{12}RC_1 \tag{9}$$

Substituting for f_{13} we have:

$$\begin{aligned} E\left(PR\right) &= SP\left(p_{13}+p_{12}p_{23}\right) - PC_1 \\ &\quad - SC_1\left(1-p_{13}-p_{12}p_{23}\right) - RC_1 p_{12} \\ &= \left(SP+SC_1\right)\left(p_{13}+p_{12}p_{23}\right) \\ &\quad - RC_1 p_{12} - PC_1 - SC_1 \end{aligned} \tag{10}$$

Further, substituting for $p_{12}, p_{13},$ and p_{14} the expected profit can be written in terms of the cumulative normal distribution as follows:

$$\begin{aligned} E\left(PR\right) &= \left(SP-SC_1\right)\left(\Phi\left(U_1\right)-\Phi\left(L_1\right)+\left(1-\Phi\left(U_1\right)\right)p_{23}\right) \\ &\quad - RC_1\left(1-\Phi\left(U_1\right)\right) - PC_1 - SC_1 \end{aligned} \tag{11}$$

The terms $\Phi\left(U_1\right)$ and $\Phi\left(L_1\right)$ are functions of the decision variable μ_1 and we desire to determine the optimal value of the process mean so that the objective function in (11) is maximized. This can be obtained using an ordinary numerical search algorithm, in which the intervals are first partitioned to some sub-intervals for each of which the objective function value is determined. Then, the maximum of these values is the near-optimal solution.

3.2. The Two-stage system

Consider a two-stage serial production system with the following states,

State 1: An item is being processed in the first stage of the production process
State 2: An item is being reworked in the first stage of the production process
State 3: An item is being processed in the second stage of the production process
State 4: An item is being reworked in the second stage of the production process
State 5: An item is accepted to be finished work
State 6: An item is scrapped

Then, assuming the quality characteristic of an item in the second stage follows a normal distribution with mean μ_2 and standard deviation σ_2, the single-step transition probability matrix can be expressed as follows:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & p_{12} & p_{13} & 0 & 0 & p_{16} \\ 0 & 0 & p_{23} & 0 & 0 & p_{26} \\ 0 & 0 & 0 & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & 0 & p_{45} & p_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{12}$$

where,

$$p_{12} = \int_{U_1}^{\infty} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} dx_1 = 1-\Phi(U_1) \qquad (13)$$

$$p_{13} = \int_{L_1}^{U_1} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} dx_1 = \Phi(U_1)-\Phi(L_1) \qquad (14)$$

$$p_{16} = \int_{-\infty}^{L_1} \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} dx_1 = \Phi(L_1) \qquad (15)$$

$$p_{34} = \int_{U_2}^{\infty} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2} dx_2 = 1-\Phi(U_2) \qquad (16)$$

$$p_{35} = \int_{L_2}^{U_2} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2} dx_2 = \Phi(U_2)-\Phi(L_2) \qquad (17)$$

$$p_{36} = \int_{-\infty}^{L_2} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2} dx_2 = \Phi(L_2) \qquad (18)$$

Moreover, p_{23} and p_{45} denote the probabilities of accepting a reworked item in stage one and two, respectively. Once again, the historical data available in the production system can be used to determine the values of p_{23} and p_{45}. The terms $\Phi(U_1)$, $\Phi(L_1)$, $\Phi(U_2)$, and $\Phi(L_2)$ are functions of the decision variables μ_1 and μ_2 that are the process means in stages 1 and 2, respectively.

Rearranging the $\mathbf{P}$ matrix and applying the method used for the single-stage system results in the following fundamental and absorption matrices:

$$\mathbf{M} = (\mathbf{I}-\mathbf{Q})^{-1} = \begin{bmatrix} 1 & -p_{12} & -p_{13} & 0 \\ 0 & 1 & -p_{23} & 0 \\ 0 & 0 & 1 & -p_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & p_{12} & p_{12}p_{23}+p_{13} & p_{34}(p_{12}p_{23}+p_{13}) \\ 0 & 1 & p_{23} & p_{23}p_{34} \\ 0 & 0 & 1 & p_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{F} = \mathbf{M}\times\mathbf{R}$$

$$= \begin{bmatrix} 1 & p_{12} & p_{12}p_{23}+p_{13} & p_{34}(p_{12}p_{23}+p_{13}) \\ 0 & 1 & p_{23} & p_{23}p_{34} \\ 0 & 0 & 1 & p_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & p_{16} \\ 0 & p_{26} \\ p_{35} & p_{36} \\ p_{45} & p_{46} \end{bmatrix} =$$

$$\begin{bmatrix} p_{35}(p_{12}p_{23}+p_{13})+p_{34}p_{45}(p_{12}p_{23}+p_{13}) & p_{16}+p_{12}p_{26}+p_{36}(p_{12}p_{23}+p_{13})+p_{34}p_{46}(p_{12}p_{23}+p_{13}) \\ p_{23}p_{35}+p_{23}p_{34}+p_{45} & p_{26}+p_{23}p_{36}+p_{23}p_{34}p_{46} \\ p_{35}+p_{34}p_{45} & p_{36}+p_{34}p_{46} \\ p_{45} & p_{46} \end{bmatrix} = \begin{bmatrix} f_{15} & f_{16} \\ f_{25} & f_{26} \\ f_{35} & f_{36} \\ p_{45} & p_{46} \end{bmatrix} \qquad (19)$$

where f_{15} and f_{16} are the probabilities of accepting and scrapping an item, and f_{36} is the probability of scrapping an item in stage 2.

The expected profit is obtained by determining the terms in equation (1) as follows.

$$E(RV) = (1-f_{16})SP \qquad (20)$$

$$E(PC) = PC_1 + \Pr(\text{item is processed in stage 2})PC_2 = PC_1 + (p_{12}p_{23} + p_{13})PC_2 \quad (21)$$

$$E(SC) = \Pr(\text{item is scrapped in stage 1})SC_1 + \Pr(\text{item is scrapped in stage 2})SC_2 = SC_1(p_{16} + p_{12}p_{26}) + SC_2(p_{12}p_{23} + p_{13})f_{36} \quad (22)$$

$$E(RC) = \Pr(\text{item is reworked in stage 1})RC_1 + \Pr(\text{item is reworked in stage 2})RC_2 = p_{12}RC_1 + p_{34}(p_{12}p_{23} + p_{13})RC_2 \quad (23)$$

Therefore, the expected profit per item for a two-stage serial production system is obtained as

$$E(PR) = (1 - f_{16})SP - PC_1 - (p_{12}p_{23} + p_{13})PC_2 - (p_{16} + p_{12}p_{26})SC_1 - (p_{12}p_{23} + p_{13})f_{36}SC_2 - RC_1p_{12} - p_{34}(p_{12}p_{23} + p_{13})RC_2 \quad (24)$$

As stated in the introduction, to determine the expected profit of a two-stage serial production system, Bowling et al. (2004) multiplied the selling price per item (SP) by the long-term probability of accepting items in stage 1 multiplied by the probability of accepting items in stage 2. This is not correct because the long-term probability of accepting items in stage 1 denotes the overall proportion of the items that have been accepted in all stages of the process. As a result, it is not required to multiply it by the probability of accepting items in stage 2.

4. Numerical Examples

In this section, we provide two numerical examples to illustrate the applications of the proposed model in both single-stage and two-stage processes.

4.1. A Numerical Example for a Single-Stage System

Consider a single-stage production system with the following characteristics: $SP = \$120$, $PC_1 = \$40$, $RC_1 = \$35$, $SC_1 = \$15$, $\sigma_1 = 1$, $p_{23} = 0.95$, $L_1 = 8$ and $U_1 = 12$. Note that p_{45} does not exist in a single stage model and the other probability terms are determined by equation (3), (4), and (5) using a numerical search method. Then, the expected profit per item is maximized at $\mu = 9.9$ with a value of $E(PR) = \$73.1$ The function $E(PR)$ defined in equation (11) is plotted for different values of the decision variable μ. Figure (1) shows the expected profit as a concave function of the process mean.

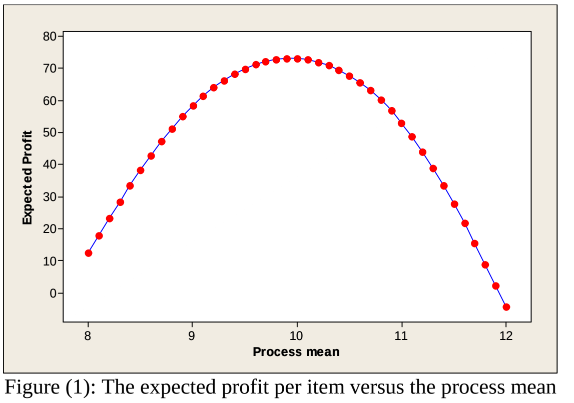

Figure (1): The expected profit per item versus the process mean

4.2. A Numerical Example for a Two-Stage System

Consider a two-stage production system and the following parameters: $SP = \$120$, $PC_1 = \$35$, $PC_2 = \$30$, $RC_1 = \$30$, $RC_2 = \$25$, $SC_1 = \$15$, $SC_2 = \$12$, $\sigma_1 = 1.0$, $\sigma_2 = 1.0$, $p_{23} = 0.95$, $p_{45} = 0.95$, $L_1 = 8$, $L_2 = 13$, $U_1 = 12$, and $U_2 = 17$. The other probability terms are first determined using equation (13) and through a numerical search method. Then, the expected profit is maximized at $\mu_1 = 9.8$ and $\mu_2 = 15$ with an expected profit per item of $E(PR) = \$19.06$. The function $E(PR)$ that is defined in equation (15) is plotted for different decision variables μ_1 and μ_2 in Figure (2). Once again, Figure (2) shows that the expected profit is a concave function of the process mean.

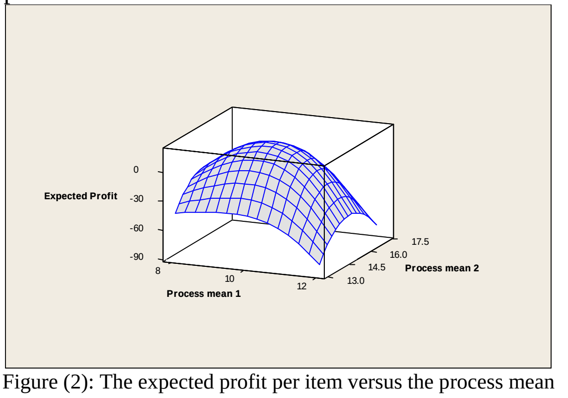

Figure (2): The expected profit per item versus the process mean

5. Conclusion

In this paper, absorbing Markov chain models were developed to determine the optimal process means that maximize the expected profit per item of both single-stage and two-stage production systems in which the items are %100 inspected to be classified as accepting, scrapping, and reworking ones. Two numerical examples were provided to illustrate the applications of the proposed models. The relationships between the process means and the expected profit per item have been given at the end.

6. References

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