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Research Article

Continuous benchmarking using a time-framed inverse data envelopment analysis: The case of Iranian economic sectors

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Abstract

This paper develops a continuous benchmarking method using the inverse data envelopment analysis (InDEA) problem. It develops a continuous time-based framework capable of handling InDEA models. By this proposed approach, we find the optimal required input level for producing a given expected benchmark and preserving the efficiency scores over time. This approach provides a useful tool specifically for decision makers in the process of planning and budgeting based on performance. If the decision maker aims to produce a specific level of benchmark, then our approach helps them find the required input level over time. In fact, it helps us to find when and how much input is required for producing a given benchmark level. Compared with existing literature in classical InDEA, the proposed models give better solutions, namely, more produced outputs and less consumed inputs. These models determine not only the input-output level but also the best time of input consumption or output production. We applied our models for a decade efficiency analysis, more sensitivity, benchmarking, and planning analysis of selected Iranian economic sectors.

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1. Introduction

Data envelopment analysis (DEA) is a mathematical programming-based approach to assess the performance of homogeneous decision making units (DMUs) with multiple inputs and outputs. This method initially proposed by (Abraham Charnes, Cooper, & Rhodes, 1978) and has been extended in different paths over the last decades. Besides theoretical developments, this method has been extensively used in different sectors. See, for instance, (Emrouznejad, Parker, & Tavares, 2008) and (Emrouznejad & Yang, 2018)) for a review of some applications. Classical DEA models and most of the extended models have a linear structure and as (Cook & Seiford, 2009) mentioned, one of the attractive characteristics of the DEA model that makes it a widely used technique in different sectors is its simplicity to solve. In one of the theoretical developments of classical DEA models (X. S. Zhang & Cui, 1999) proposed a new research line in the DEA literature called inverse DEA (InDEA). This method was extended by (Wei, Zhang, & Zhang, 2000) and (Yan, Wei, & Hao, 2002). The research question in the InDEA problem is as follows: if a DMU perturbs its input level, then how much output can be produced using a new level of inputs with the current output efficiency score? (M. Zhang & Cui, 2016) extended this question by estimating expected outputs and preserving the input efficiency. On the other side, if a DMU perturbs its output level, then how much input is needed to produce a new level of outputs with the current input efficiency level? (Amin & Al-Muharrami, 2016) dealt with negative data in the inverse DEA problem. They proposed some models for identifying the levels of required inputs and outputs from merging units to realize an efficiency target. (Lertworasirikul, Charnsethikul, & Fang, 2011) considered the variable returns to scale (VRS) properties like (Banker, Charnes, & Cooper, 1984) for the production technology in the inverse DEA problem. However, there exist some drawbacks in their model that were pointed out and revised by (Ghiyasi, 2015).

(Ghiyasi, 2017b) proposed a cost-based inverse DEA model that incorporates price information. (Amin & Emrouznejad, 2007) applied the inverse linear programming for the DEA models, specifically for the additive DEA model. They yield to a faster method for solving DEA models using inverse linear programming. Besides theoretical developments, the InDEA problem has proved as a useful approach in various real-life applications as it enriched classical DEA models as a tool for making predictions. An application of the InDEA models in banking mergers was proposed by (Gattoufi, Amin, & Emrouznejad, 2014). (Amin, Emrouznejad, & Gattoufi, 2017a) proposed an InDEA model for modelling generalized firms' restructuring as well as anticipating the minor and major consolidation for a merger in a market. (Amin, Emrouznejad, & Gattoufi, 2017b) dealt with restructuring problems in an InDEA framework and developed a novel InDEA-based methodology, called Generalized Inverse DEA, for modeling the generalized restructuring. (Goshu, Matebu, & Kitaw, 2017) provided a productivity analysis framework for analyzing the performance of manufacturing companies. (Chen, Wang, Lai, & Feng, 2017) proposed a new InDEA model considering undesirable outputs. (Ghiyasi, 2017a) proposed an InDEA model for pollution generation in production technologies. (Fazelimoghadam, Ershadi, & Akhavan Niaki, 2020) used DEA methods for finding the efficient solution near the optimal solutions within the solutions gauged by a multi-objective particle swarm optimization. (Safaie & Nasri, 2022) used a robust DEA model for designing a failure mode and effect analysis with an application in an automobile oil filter. There are two main streams in the DEA literature for dealing with efficiency analysis over time, which are windows DEA analysis and the Malmquist productivity index. A DEA window analysis originally proposed by (A. Charnes, Clark, Cooper, & Golany, 1984) works on the principle of moving averages and is useful to determine the efficiency trends of a DMU over time. Malmquist index, initially proposed by (Malmquist, 1953), which measures productivity change over time and has been widely developed and applied in different sectors (See for instance (Asmild, 2015)). (Emrouznejad & Thanassoulis, 2005) developed a window-based model considering the changes in stock as a particular cause of inter-temporal input–output dependence and presented an envelopment dynamic DEA model to calculate the technical dynamic efficiency measures. (Jahanshahloo, Soleimani-Damaneh, & Ghobadi, 2015) developed a specific time based inverse DEA model based on the work of (Emrouznejad & Thanassoulis, 2005), assuming inter-temporal dependence of the dataset. (Emrouznejad, Amin, Ghiyasi, & Michali, 2023) reviewed the origin, theoretical development, application and future research line of the inverse DEA since this method was developed.

The current paper develops a generalized multi-period InDEA model considering time changes in the process of input-output estimation. The proposed models determine not only the input-output level but also the best time of input consumption or output production. In all proposed models in the current paper, the main assumption of InDEA problem is that preserving the relative efficiency is guaranteed. The main advantage of the proposed models is their ability to deal with the input-output estimation over time. This enables decision makers to find the best (optimal) amount of inputs for each unit over time. This yields to determining the best (optimal) time of consuming inputs for producing a desired level of production. Furthermore, the proposed models are production-based models, and they are generalized in terms of returns to scale; and any type of returns to scale assumption can be considered based on the characteristics of the production technology. The rest of the paper is organized as follows. Section 2 provided the basic classical DEA and InDEA models. Section 3 develops the input-oriented generalized multi-period InDEA model. The output-oriented generalized multi-period InDEA model is proposed in the section 4. Section 5 performs an application of the proposed models for regular efficiency analysis and more sensitivity analysis using the proposed model for selected Iranian economic sectors during 2003-2013.

2. Basic models

2.1. Classical DEA model

Suppose there is a set of n DMUs consuming m dimensional input vector of $x_j = (x_{1j}, x_{2j}, \dots, x_{mj}) \in \mathbb{R}_+^m$ to produce s dimensional output vector of $y_j = (y_{1j}, y_{2j}, \dots, y_{sj}) \in \mathbb{R}_+^s$. The general production technology can be seen as T={(x,y): x can produce y}. Considering the input-output set one may specify the following production technology: $T = \{(x, y): x \geq \sum_j \lambda_j x_j, y \leq \sum_j \lambda_j y_j, \lambda \in ג\}$, where ג specifies the returns to scale properties of the production. Considering the Farrelle's distance measure of $\min\{\theta: (\theta x_o, y) \in T\}$ (Farrell, 1957) for the production technology of T we get the following linear programming model for measuring the relative efficiency of DMU_o, that is, the DMU under assessment:

$$\begin{aligned} \theta_o &= Min\ \theta \\ \sum_{j=1}^{n} \lambda_j x_{ij} &\leq \theta x_{io};\ 1 \leq i \leq m \\ \sum_{j=1}^{n} \lambda_j y_{rj} &\geq y_{ro};\ 1 \leq r \leq s \\ \lambda &\in ג \end{aligned} \tag{1}$$

Solving the above model for all DMUs results in a list of relative efficiency scores based on the input-output parameters.

Definiation1. The DMU_o is called (weakly) efficient if the optimal value of linear programming (1) is unity ($\theta_o = 1$).

2.2. Classical inverse DEA model

Having the input-output set, the DEA models seek the efficiency score of DMUs. In a quite different path, the InDEA models try to answer another question. Assume DMU_o perturbs its output from y_{ro} to $\beta_{ro} = y_{ro} + \Delta y_{ro}$. The question is how much input is required for producing a new output level of DMU_o with its current efficiency score of θ_o. This question is answered by the following multiple objective linear programming (MOLP) in the InDEA literature.

$$\begin{aligned} \theta_o &= Min\ (\alpha_1, \alpha_2, \dots, \alpha_m) \\ \sum_{j=1}^{n} \lambda_j x_{ij} &\leq \theta(x_{io} + \Delta x_{io});\ 1 \leq i \leq m \\ \sum_{j=1}^{n} \lambda_j y_{rj} &\geq y_{ro} + \Delta y_{ro} = \beta_{ro};\ 1 \leq r \leq s \\ \lambda &\in ג \end{aligned} \tag{2}$$

Definition 2. Suppose (λ, α) is a feasible solution MOLP model (2) if there is no feasible solution $(\bar{\lambda}, \bar{\alpha})$ for this model such that $\bar{\alpha} < \alpha$, then we say the (λ, α) is a weak efficient solution of that model.

It is a well-known fact in the InDEA literature that weak efficient solutions of MOLP model (2) preserve the efficiency score of DMUs(see, for instance, (Ghiyasi, 2015)).

3. Multi-period InDEA

In this section, we look at the InDEA over time, and no technological change is considered in each time period. Consider the input and output vector of an arbitrary DMU_j as $x_i^t \epsilon \mathbb{R}_+^m$ and $y_r^t \epsilon \mathbb{R}_+^s$ at time period $t = 1,2, \dots, \bar{t}$. Assume DMU_o, for instance, perturbs its output level from y_{ro}^t to $\beta_{ro}^t = y_{ro}^t + \Delta y_{ro}^t$ at time period t. Now the question is how much (minimum) input is required for producing the new level of output for DMU_o at the current efficiency level of each period. One trivial answer to this question is estimating the required input of each period separately and then adding them to find the total required input level. To that end, we need to solve the following MOLP model at each period.

$$\begin{aligned} &Min\ (\alpha_1^t, \alpha_2^t,, \dots, \alpha_m^t) \\ &\sum_{j=1}^{n} \lambda_j x_{ij}^t \leq \theta_o^t(x_{io}^t + \Delta x_i^t);\ 1 \leq i \leq m \\ &\sum_{j=1}^{n} \lambda_j y_{rj}^t \geq y_{ro}^t + \Delta y_{ro}^t = \beta_{ro}^t;\ 1 \leq r \leq s \\ &\lambda \in ג^t \end{aligned} \tag{3}$$

, where θ_o^t is the efficiency score of DMU_o at time period of $t = 1,2, \dots, \bar{t}$.

Both MOLP model (2) and MOLP model (3) have m objective functions, $m+s+1$ constraints and $m+n$ variables. Similar to the static environment, it is not difficult to see that weak efficient solutions of MOLP (3) at each time period of $t = 1,2, \dots, \bar{t}$. preserve time associated relative efficiency of DMU_o, that is, $\theta_o^t, t = 1,2, \dots, \bar{t}$.. This fact is proved in the following theorem.

Theorem1. If $(\lambda^t, \bar{\alpha}^t)$ is a weak efficient solution of MOLP model (3), then the new input-output level of $(\bar{\alpha}^t, \beta_o^t)$ preserve the relative efficiency of DMU_o at time period of $t = 1,2, \dots, \bar{t}$ and the total required i-th input level for producing the new output level is $\sum_{t=1}^{\bar{t}} \bar{\alpha}_{io}^t = \sum_{t=1}^{\bar{t}} x_{io}^t + \bar{\Delta} x_{io}^t$.

Proof. For the fixed time period $t = 1,2, \dots, \bar{t}$ let $(\lambda^t, \bar{\alpha}^t)$ be a weak efficient solution of MOLP model (3) then it satisfies the associated constraints as follows:

$$\begin{aligned} &\sum_{j=1}^{n} \bar{\lambda}_j x_{ij}^t \leq \theta_o^t(x_{io}^t + \bar{\Delta} x_{io}^t) = \theta_o^t \alpha_{io}^t;\ 1 \leq i \leq m; \\ &\sum_{j=1}^{n} \bar{\lambda}_j y_{rj}^t \geq y_{ro}^t + \Delta y_{ro}^t = \beta_{ro}^t;\ 1 \leq r \leq s \\ &\bar{\lambda} \in ג \end{aligned} \tag{4}$$

Now let us check the efficiency score of perturbed DMU_o, that is, $(\bar{\alpha}^t, \beta_o^t)$. The following model estimates the relative efficiency of this DMU:

$$\theta_o^{tN} = Min\ \theta \tag{5}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^t \le \theta \bar{\alpha}_i^t;\ 1 \le i \le m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj}^t \ge \beta_{ro}^t;\ 1 \le r \le s$$
$$\lambda \in \gimel$$

, where θ_o^{tN} is the efficiency of perturbed DMU$_o$ at time period of t. Regarding to the constraint set of (4) we see that $(\bar{\lambda}, \theta_o^t)$is a feasible solution of model (5) that implies $\theta_o^{tN} \le \theta_o^t$. Now, by contradiction if $\theta_o^{tN} \le\ne \theta_o^t$ then we have:

$$\sum_{j=1}^{n} \lambda_j^* x_{ij}^t \le \theta_o^{tN} \bar{\alpha}_{io}^t;\ 1 \le i \le m$$
$$\sum_{j=1}^{n} \lambda_j^* y_{rj}^t \ge \beta_{ro}^t;\ 1 \le r \le s$$
$$\lambda^* \in \gimel$$

This suggests a feasible solution for the MOLP model of (3), where $(\lambda^*, \theta_o^{tN})$ is the optimal solution of model (5). Thus, we have:

$$\sum_{j=1}^{n} \lambda_j^* x_{ij}^t \le \theta_o^{tN} \bar{\alpha}_{io}^t < \theta_o^t \bar{\alpha}_{io}^t;\ 1 \le i \le m$$
$$\sum_{j=1}^{n} \lambda_j^* y_{rj}^t \ge \beta_{ro}^t;\ 1 \le r \le s$$
$$\lambda^* \in \gimel$$

, which implies

$$\sum_{j=1}^{n} \lambda_j^* x_{ij}^t < \theta_o^t \bar{\alpha}_{io}^t;\ 1 \le i \le m$$
$$\sum_{j=1}^{n} \lambda_j^* y_{rj}^t \ge \beta_{ro}^t;\ 1 \le r \le s$$
$$\lambda^* \in \gimel$$

And this contradicts the weak efficiency of the $(\lambda^t, \bar{\alpha}^t)$.

The above statement can be done for all time periods $t = 1,2,\dots,\bar{t}$. Now $\bar{\alpha}_{io}^t = x_{io}^t + \bar{\Delta} x_{io}^t$is the required input level that produces the given output level and preserves the relative efficiency of DMU$_o$ at time period of t. In order to preserve the efficiency scores in all periods and produce the total output level of all periods *separately*, we need $\sum_{t=1}^{\bar{t}} \bar{\alpha}_{io}^t = \sum_{t=1}^{\bar{t}} x_{io}^t + \bar{\Delta} x_{io}^t$ as required input level.□

DMUs over time may have different performances, which means different capabilities in output production and input consumption. They may have regress or progress in some period of time, and this trend may change during the time period. In order to reach the decision maker's aim that is producing a given level of output while preserving the efficiency scores, some input level may be required. The required input level may change based on the performance of DMUs. However, the goal is to reach the decision maker's aim with a minimum level of input. Thus, it is important to know when to consume the new input level. The following model determines not only how much input is required but also when is the best time for using inputs to reach the decision maker's aim, namely producing expected output with a minimum level of inputs.

$$Min\ \mathbf{1}^T(\alpha_1^t, \alpha_2^t,, \dots, \alpha_m^t) = \mathbf{1}^T(x_{1o}^t + \Delta x_{1o}^t, x_{2o}^t + \Delta x_{2o}^t, \dots, x_{mo}^t + \Delta x_{mo}^t,) = \left(\sum_{t=1}^{\bar{t}} x_{io}^1 + \Delta x_i^1, \sum_{t=1}^{\bar{t}} x_{io}^2 + \Delta x_i^2, \dots \sum_{t=1}^{\bar{t}} x_{io}^{\bar{t}} + \Delta x_i^{\bar{t}}\right) \quad (6)$$
$$\sum_{j=1}^{n} \lambda_j x_{ij}^t \le \theta_o^t (x_{io}^t + \Delta x_i^t);\ 1 \le i \le m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj}^t \ge y_{ro}^t + \Delta y_{ro}^t = \beta_{ro}^t;\ 1 \le r \le s$$
$$\lambda^t \in \gimel^t$$

Please note that each of m objective functions of model (6) is an $\bar{t}$-dimensional vector associated with all periods, that is, $(x_{io}^t + \Delta x_{io}^t) \in \mathbb{R}_+^{\bar{t}},\ 1 \le i \le m$. Thus, compared with the MOLP model of (3) there exist $m \times \bar{t}$objective functions, $(m+s+1) \times \bar{t}$constraints and $(m+n) \times \bar{t}$variables in MOLP model (6).

Theorem2. Weak efficient solutions of MOLP (6) preserve the efficiency score DMU$_o$ for *all* time period of $t = 1,2,\dots,\bar{t}$.

Proof. Consider an *arbitrary* time period of $t \in \{1,2,\dots,\bar{t}\}$ and take a weak efficient solution of MOLP model (6) like $(\bar{\lambda}, \bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_m) \in \mathbb{R}_+^{(m+1)\times t}$. Thus, $\forall t \in \{1,2,\dots,\bar{t}\}$ we have:

$$\sum_{j=1}^{n} \bar{\lambda}_j x_{ij}^t \le \theta_o^t \bar{\alpha}_i^t;\ 1 \le i \le m$$
$$\sum_{j=1}^{n} \bar{\lambda}_j y_{rj}^t \ge \beta_{ro}^t;\ 1 \le r \le s$$
$$\bar{\lambda}^t \in \gimel^t$$

This suggest $(\bar{\lambda}^t, \theta_o^t)$ $(\bar{\lambda}^t, \theta_o^t)$ as a feasible solution to the criterion model of (5) $\forall t \in \{1,2,\dots,\bar{t}\}$. This implies that $\theta_o^{tN} \le \theta_o^t{:}\ \forall t \in \{1,2,\dots,\bar{t}\}$. Now if $\exists t' \in \{1,2,\dots,\bar{t}\}$ such that $\theta_o^{t'N} < \theta_o^{t'}$ then $\theta_o^{t'N} = k\theta_o^{t'}, 0 < k < 1$ and then we have

$$\sum_{j=1}^{n} \lambda_j^{*t'} x_{ij}^{t'} \le \theta_o^{t'N} \bar{\alpha}_i^{t'} = \theta_o^{t'} \overline{(k\alpha_i^{t'}});\ 1 \le i \le m$$
$$\sum_{j=1}^{n} \lambda_j^{*t'} y_{rj}^{t'} \ge \beta_{ro}^{t'};\ 1 \le r \le s$$
$$\lambda^{*t'} \in \gimel^{t'}$$

, where $(\lambda^{*t'}, \theta_o^{t'N})$ is the optimal solution of the criterion model (5) at the time period $t' \in \{1,2,\dots,\bar{t}\}$. Now setting $\tilde{\lambda}^t = \begin{cases} \lambda^{*t} & t = t' \\ \bar{\lambda}^t & t \ne t' \end{cases}$ and $\tilde{\alpha}^t = \begin{cases} k\alpha_i^{t'} & t = t' \\ \alpha_i^t & t \ne t' \end{cases}$ we see that $(\tilde{\lambda}, \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_m)$ is a feasible solution of MOLP model of (6) and this contradicts with weak efficiency of $(\bar{\lambda}, \bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_m)$ since $1^T(\tilde{\lambda}, \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_m) < 1^T(\bar{\lambda}, \bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_m)$.□

One of the main factors in the potential of the production is the production size. This issue is dealt with by the returns to scale concept in the DEA and inverse DEA modelling. There is a high potential for production in the increasing returns to scale area, and the manager should suggest scaling up. The constant returns to scale area is a vital and important area since it is some sort of boundary between increasing and decreasing returns to scale areas. In the area of decreasing returns to scale wise managers should be careful of the production scale of the firm, and shrinking the scale of production may be beneficial for the firm. The proposed models (3)-(6) provide the possibility of considering any type of returns to scale in the case of the input-output estimation, based on the characteristics of the production technology in each period. This helps decision makers in using the potential of increasing production and saving resources, considering the scale opportunities in each period. MOLP models are well-known for having alternative efficient solutions, and different methods are presented for solving these models in the literature, see e.g., (Steuer, 1986) for more details on the solving method. One may use the weighted sum method for solving MOLP model (6), which yields the following model:

$$Min = \sum_{t=1}^{\bar{t}} \sum_{i=1}^{m} \alpha_i^t = \sum_{t=1}^{\bar{t}} \sum_{i=1}^{m} x_{io}^t + \Delta x_i^t$$
$$\sum_{j=1}^{n} \lambda_j x_{ij}^t \le \theta_o^t (x_{io}^t + \Delta x_i^t);\ 1 \le i \le m, t = 1,2,\dots,\bar{t} \qquad (7)$$
$$\sum_{j=1}^{n} \lambda_j y_{rj}^t \ge y_{ro}^t + \Delta y_{ro}^t = \beta_{ro}^t;\ 1 \le r \le s, t = 1,2,\dots,\bar{t}$$
$$\lambda^t \in \lambda^t$$

Theorem3. The MOLP model (6) requires less input compared with the MOLP model of (3) over time period of $t \in \{1,2,\dots,\bar{t}\}$ for producing a given total output level and preserving the efficiency score of DMU₀ over a given period.

Proof. Assume a given expected output level of $\beta_{ro}^t = y_{ro}^t + \Delta y_{ro}^t$ that needs to be produced with current efficiency scores over time. The required input level using the MOLP model (3) is $\sum_{t=1}^{\bar{t}} \alpha_i^t = \sum_{t=1}^{\bar{t}} x_{io}^t + \Delta x_i^t$. Note that this model needs to be solve $\bar{t}$ times regarding to $t = 1,2,\dots,\bar{t}$ to find the required input level. Now let $(\bar{\lambda}^t, \bar{\alpha}^t), = 1,2,\dots,\bar{t}$ is the weak efficient solution of MOLP (3) at each time period of$= 1,2,\dots,\bar{t}$. This suggests $(\tilde{\lambda}, \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_m)$ as a feasible solution of MOLP model (6), where $\tilde{\lambda}_j^t = \bar{\lambda}_j^t$ and $\tilde{\alpha}_j^t = \bar{\alpha}_j^t$ are getting from the weak efficient solution of MOLP (3) for each $t = 1,2,\dots,\bar{t}$. This implies $(\tilde{\lambda}, \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_m) <\neq (\hat{\lambda}, \hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_m)$,where $(\hat{\lambda}, \hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_m)$ is weak efficient solution of MOLP model (6).

For long-run policy making and planning, if the decision maker is able, then he may pool all perturbed inputs of a specific DMU over all time periods and then use the following model to decide about the required input level, producible output level, and also the optimal time period for each input consumption and output production.

$$Min\ \mathbf{1}^T(\alpha_1^t, \alpha_2^t,, \dots, \alpha_m^t) = \mathbf{1}^T(x_{1o}^t + \Delta x_{1o}^t, x_{2o}^t + \Delta x_{2o}^t, \dots, x_{mo}^t + \Delta x_{mo}^t,) = (\sum_{t=1}^{\bar{t}} x_{io}^1 + \Delta x_i^1, \sum_{t=1}^{\bar{t}} x_{io}^2 + \Delta x_i^2, \dots \sum_{t=1}^{\bar{t}} x_{io}^{\bar{t}} + \Delta x_i^{\bar{t}})$$
$$\sum_{j=1}^{n} \lambda_j x_{ij}^t \le \theta_o^t (x_{io}^t + \Delta x_i^t);\ 1 \le i \le m \qquad (8)$$
$$\sum_{j=1}^{n} \lambda_j y_{rj}^t \ge y_{ro}^t + \Delta y_{ro}^t = \beta_{ro}^t;\ 1 \le r \le s$$
$$\sum_{t=1}^{\bar{t}} \beta^t = \sum_{t=1}^{\bar{t}} \beta_o^t$$
$$\lambda^t \in \lambda^t$$

In fact, the above model provides the possibility of consuming some inputs from one period of time for another period of time. This makes more flexibility of input consumption for producing the expected output. However, model (8) does not necessarily guarantee the expected output of each time period, but the total produced output is guaranteed with minimum input consumption. It is also important to point out that, like the MOLP model (6), the current efficiency of all units is guaranteed to be unchanged. The proof of this statement is straightforward and is ignored. If we consider an inter-temporal structure for the problem, we get the input-oriented InDEA model of (Jahanshahloo et al., 2015) as a special case of our model. This means considering a general initial stuck input and a terminal stuck input and adding relative constraints to our model as follows:

$\sum_{j\epsilon J} \lambda_j Z_j^{\tau+T} \ge Z_o^{\tau+T}$
$\sum_{j\epsilon J} \lambda_j Z_j^{\tau-1} \le \mathcal{L}_o^{\tau-1}$
$\mathcal{L}_o^{\tau-1} =\ge Z_o^{\tau+T} + \sum_{t=\tau}^{\tau+T} \rho_o^t$
$\rho_o^t \ge Z_o^t$

where $Z_j^{\tau-1}$is the initial stuck and $Z_j^{\tau+T}$ is the terminal stuck of *j*-th DMU. Another important setting that makes the model of (Jahanshahloo et al., 2015) a special case of our model is that we need to consider a special case of $\lambda_{rj}^t = \lambda_j$ for the intensity variable our model to the model of (Jahanshahloo et al., 2015). In fact, a unique set of intensity variables for *all* time periods is considered by (Jahanshahloo et al., 2015). This means a unique peer analysis, a unique target setting, etc., for *all* periods, which seems a bit strict. However, we assume a separate intensity variable for each time period which makes our model a generalized model in this sense.

4. Output-oriented multi-period InDEA

In the literature of the InDEA, one may ask different questions, such as if the input level of a specific DMU perturbs to a certain level, then how much output should we expect to be produced at the current efficiency level. This may be called an output-oriented InDEA problem. This section aims to deal with this question over time. Moreover, this section finds when we should expect the produced output too. Assume DMU$_o$, for instance, that perturbs its input level from x_{io}^t to $\alpha_{io}^t = x_{io}^t + \Delta x_{io}^t$ x_{io}^t at time period *t*.

Now the question is how much output should be expected for the perturbed inputs at the current efficiency level. The following model aims to achieve this for us.

$$\begin{aligned} & Max\ (\beta_1^t, \beta_2^t, , \dots, \beta_s^t) \\ & \sum_{j=1}^{n} \lambda_j x_{ij}^t \leq \left(x_{io}^t + \Delta x_{io}^t\right);\ 1 \leq i \leq m \\ & \sum_{j=1}^{n} \lambda_j y_{rj}^t \geq \varphi_o^t (y_{ro}^t + \Delta y_r^t) = \varphi_o^t \beta_r^t;\ 1 \leq r \leq s \\ & \lambda \in ƛ^t \end{aligned} \tag{9}$$

, where φ_o^t is the optimal value of the following models, that is, the output efficiency score of DMU$_o$ at the period of *t*.

$$\begin{aligned} & Max\ \varphi \\ & \sum_{j=1}^{n} \lambda_j x_{ij}^t \leq x_{io}^t;\ \ 1 \leq i \leq m \\ & \sum_{j=1}^{n} \lambda_j y_{rj}^t \geq \varphi y_{ro}^t;\ 1 \leq r \leq s \\ & \lambda \in ƛ^t \end{aligned} \tag{10}$$

Model (9) finds the maximum producible output using perturbed input in the current efficiency level.

Theorem4. If $(\lambda^t, \bar{\beta}^t)$ is a weak efficient solution of MOLP model (9), then the new input-output level of $(\alpha_o^t, \bar{\beta}^t)$ preserve the relative efficiency of DMU$_o$ in the period of $t = 1,2, \dots, \bar{t}$.

Proof. It is straightforward to follow the proof of Theorem 1.

Now, considering the whole time period, the following MOLP model guarantees an unchanged efficiency score of all DMUs over all periods for a perturbed input level of$\alpha_{io}^t = x_{io}^t + \Delta x_{io}^t$:

$$\begin{aligned} & Max\ \mathbf{1}^T (\beta_1^t, \beta_2^t, , \dots, \beta_s^t) \\ & \quad = \mathbf{1}^T (y_{1o}^t + \Delta y_{1o}^t, y_{2o}^t + \Delta y_{2o}^t, \dots, y_{so}^t + \Delta y_{so}^t,) \\ & \quad = (\sum_{t=1}^{\bar{t}} y_{ro}^1 + \Delta y_r^1, \sum_{t=1}^{\bar{t}} y_{ro}^2 + \Delta y_r^2, \dots, \sum_{t=1}^{\bar{t}} y_{ro}^{\bar{t}} + \Delta y_r^{\bar{t}}) \\ & \sum_{j=1}^{n} \lambda_j x_{ij}^t \leq \left(x_{io}^t + \Delta x_{io}^t\right);\ 1 \leq i \leq m \\ & \sum_{j=1}^{n} \lambda_j y_{rj}^t \geq \varphi_o^t (y_{ro}^t + \Delta y_r^t) = \varphi_o^t \beta_r^t;\ 1 \leq r \leq s \\ & \lambda \in ƛ^t \end{aligned} \tag{11}$$

The above model not only finds the required input and producible output but also determines the optimal time of input consumption and output production.

Theorem5. Weak efficient solutions of MOLP (11) preserve the efficiency score DMU$_o$ for *all* periods of $t = 1,2, \dots, \bar{t}$.

Proof. It is straightforward, following the proof of Theorem 2.

The associated linear programming of MOLP model (11) is also as follows:

$$\begin{aligned} & Max \sum_{t=1}^{\bar{t}} \sum_{r=1}^{s} \beta_r^t = \sum_{t=1}^{\bar{t}} \sum_{i=1}^{s} y_{ro}^t + \Delta y_i^t \\ & \sum_{j=1}^{n} \lambda_j x_{ij}^t \leq \left(x_{io}^t + \Delta x_{io}^t\right);\ 1 \leq i \leq m, t = 1,2, \dots, \bar{t} \\ & \sum_{j=1}^{n} \lambda_j y_{rj}^t \geq \varphi_o^t (y_{ro}^t + \Delta y_r^t) = \varphi_o^t \beta_r^t;\ 1 \leq r \leq s, t = 1,2, \dots, \bar{t} \\ & \lambda \in ƛ^t \end{aligned} \tag{12}$$

Theorem6. The total produced *r*-th input level for using the new input level is $\sum_{t=1}^{\bar{t}} \bar{\beta}_{ro}^t = \sum_{t=1}^{\bar{t}} y_{ro}^t + \bar{\Delta} y_{ro}^t$ and the produced output using the MOLP model (11) is greater than the produced output level using the MOLP model of (9) over period of $t = 1,2, \dots, \bar{t}$, using a perturbed input level and preserving the efficiency score of DMU$_o$ over the given period.

Proof. It is straightforward, following the proof of Theorem 3.

Considering a more flexible situation where the total expected output is targeted instead of the individual output level at each period, one may use the following model:

$$Max\ \mathbf{1}^T(\beta_1^t, \beta_2^t, , \dots, \beta_s^t) = \mathbf{1}^T(y_{1o}^t + \Delta y_{1o}^t, y_{2o}^t + \Delta y_{2o}^t, \dots, y_{so}^t + \Delta y_{so}^t,) = (\sum_{t=1}^{\bar{t}} y_{ro}^1 + \Delta y_r^1, \sum_{t=1}^{\bar{t}} y_{ro}^2 + \Delta y_r^2, \dots, \sum_{t=1}^{\bar{t}} y_{ro}^{\bar{t}} + \Delta y_r^{\bar{t}})$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^t \leq (x_{io}^t + \Delta x_{io}^t);\ 1 \leq i \leq m, t = 1,2,\dots,\bar{t} \quad (13)$$

$$\sum_{j=1}^{n} \lambda_j y_{rj}^t \geq \varphi_o^t(y_{ro}^t + \Delta y_r^t) = \varphi_o^t \beta_r^t;\ 1 \leq r \leq s, t = 1,2,\dots,\bar{t}$$

$$\textstyle\sum_{t=1}^{\bar{t}} \alpha_i^t = \sum_{t=1}^{\bar{t}} \alpha_{io}^t,\ 1 \leq i \leq m$$

$$\lambda \in \lambda^t, \quad t = 1,2,\dots,\bar{t}$$

Just to remind you that the above model also preserves the efficiency scores of all DMUs over all periods. The MOLP model (5.4) focuses on the *total* input consumption (output production) in contrast with the MOLP model (11) that cares about separate input consumption (output production) of each period of time. However, both models produce the maximum output level within their own framework.

5. An Application of Efficiency and Sensitivity Analysis of Selected Iranian Economic Sectors

This section analyzes the efficiency status of five selected Iranian economic sectors (agriculture, oil, industry, transportation and domestic, commercial, and public) as the basic research units during the period of 2003-2013. Deeper investigation is done using the proposed models to have a sensitivity analysis and more structural efficiency insight of the aforementioned economic sectors within the period of study. Labor and capital stock are assumed as two inputs in the analysis, and the value added is considered a single output. Employed labor is considered the first input that is found from the Iran Statistical Yearbook (1997-2014). Capital and net capital stock of each sector are selected as a proxy of input, which is reported as billion IRR. The added value of each sector that is output is extracted from the Statistical Center of Iran. Table 1 reports the statistical description of the data set.

Table 1
Summary statistics of data

Variable name	Mean	Standard error	Min	Max
Labor	2623270	1525832	118270	4404110
Capital	1606084.109	2044746.575	148612	12217851
GDP	519960.5	372673	80516.2	1589633

The first step of efficiency analysis is done and reported in table 2 over the time period of the study, assuming constant returns to scale. As can be seen, the oil industry is the efficient sector during the study period, and the agriculture sector is efficient in some years. The last sector, which is the domestic, commercial, and public sector, has the lowest efficiency within sectors during the study period.

Table 2
The efficiency score of sectors over period of study

	Agricultural	Oil	Industry	Transportation	Domestic…
Year 2003	1	1	0.4632	0.5092	0.1871
Year 2004	1	1	0.4535	0.468	0.1831
Year 2005	1	1	0.4382	0.4495	0.1817
Year 2006	1	1	0.4312	0.4249	0.1711
Year 2007	0.9826	1	0.4163	0.3955	0.1467
Year 2008	0.8188	1	0.3862	0.3479	0.1278
Year 2009	0.8654	1	0.4069	0.3613	0.1373
Year 2010	0.8071	1	0.3896	0.3349	0.1308
Year 2011	0.7191	1	0.3842	0.308	0.1152
Year 2012	1	1	0.5	0.4574	0.1572
Year 2013	1	1	0.4715	0.4924	0.1553

In the next part of the analysis, let us consider two sectors: one efficient and one inefficient, for instance oil sector (S2) and the domestic, commercial, and public sector (S5). Assuming the current efficiency level for each sector, let us consider an increment for the output of each sector and find the required input level. This analysis is performed considering a steady increment from 5 to 100 percent, and the results are reported in Table 3a-3c. Columns report the required inputs (first input shown by Dx1 and second input shown by Dx2) associated with the expected output. Rows are associated with the S2 and S5 during the period from 2003 to 2013.

S5 is found inefficient, meaning it does not use its current inputs efficiently; thus, it does not require any extra input to produce up to 30 percent of the output increment in 2003. It does not need even extra inputs for producing up to 100 percent output increment in some years, like 2004. It is expected, given the efficiency score of this sector, which is 0.1831 in 2004, that means 81 percent waste of input on producing output. This analysis suggests consuming wasted input for new perturbed output levels. The story for the second sector (S2), which is the oil sector, and found to be efficient, is different. It is found relatively efficient, meaning there is no waste of inputs. This implies extra input for producing new, increased outputs. This can be seen in Table 3, which shows the requirement for extra inputs in almost all periods of study. In the next analysis, we consider the above investigation for the whole years of the study period and find interesting results that is reported in Table 4a-4c. Then provide a comparison between the classical inverse DEA that considers each period separately, with the results of the proposed time-flexible inverse DEA. For the first scenario, that is, a five percent extension of output level, if we consider a time-separate framework, then we need 326939 units of the first input and 69420 units of the second input, and 395659 units of the inputs generally. In this scenario, if we consider the time-flexible framework, then we see that we need 1286.323 units of the first input and no extra amount of the second input.

Observe that the aforementioned expected output level in the previous analysis needs a lower input level compared with the previous study, as proved in Theorem 3. It is important to point out that this fact does not hold for individual DMUs; for instance, the sector S2 does not require much input in the previous analysis, but it demands some inputs using a generalized model over time. This result is summarized in Table 5. However, the total required inputs using the generalized InDEA over time do not exceed the required input level using separate years' analysis. This is due to the flexibility of the proposed models that provides the possibility of consuming inputs in all years and then finds the optimal time and optimal amount of consuming inputs for producing the targeted output. Remember that the optimal time and optimal amount are associated with the minimum level of inputs for producing the targeted output. The first and second rows of Table 5 are associated with the sum of the required input considering separate years, and the third and fourth rows are associated with the total required input using the generalized model. Observe that the latter is not greater than the former. This means we need fewer inputs using a generalized model compared with the separate models for producing the expected output level. In short, we observe that the proposed models provide a more flexible framework for producing a targeted output. This framework provides the possibility of identifying and preventing any excess of inputs from the beginning of the period in each year, and consequently prevents any waste of inputs that may occur in each year and may transfer to the next year. It requires fewer inputs compared with the separate year investigation and thus yields savings in inputs that are used for producing desired outputs.

Regarding the nature of the inverse DEA methodology, it can be used as a proxy for the sensitivity analysis in the efficiency analysis of production units. It helps decision makers in finding how sensitive the resources (inputs) are for the production lines and how sensitive the production is to consuming resources. This provides vital information on the prioritization of inputs and outputs for decision makers. If a resource is sensitive in case of its effects on the performance of production units, then it should be given high priority of attention by decision makers. If an output is sensitive and plays an important role for the firm, then it needs more consideration to prevent any possible potential loss of the firm in the market. Beyond that, inverse DEA methods can be used as a complementary tool for performance measurement by managers and policy makers. It helps managers design the optimal production lines. If a manager is interested in assigning available resources to production units, then the inverse DEA plays a core role. This role becomes more important when the resource assignment should be done over time, and the proposed models in the current paper help decision makers and managers with this task. On the output side, if a manager desires to produce a specific amount of output, then the inverse DEA models are a powerful method for such planning, optimally. The proposed models in the current paper help managers in designing an optimal production plan over a period of time. They help decision-makers in reaching targets by consuming the minimum amount of resources.

The proposed models are based on the available crisp data. However, in many real-world problems, there is uncertainty or incomplete information on the data. Thus, one of the potential areas for future study is working on the uncertainty issue in the inverse DEA problem over time. We consider proportional and radial measurements in all processes of input-output estimation in the current paper. One potential future line for further investigation is dealing with non-radial measurements within the concept. However, the proposed models are generalized in terms of the returns to scale, and any type of returns to scale may assumed for the input-output estimation process. The underlying production technology for the proposed models in the current paper is a straight production technology line. In case of any production and technological complexity, the proposed models need modification, which is left for future research.

Table 3a
InDEA results for separate years

	5 percent increment		10 percent increment	
	Dx1	Dx2	Dx1	Dx2
S2(2003)	11291.1	5913.5	22582.2	11827
S5(2003)	0	0	0	0
S2(2004)	0	0	26314.2	11899.5
S5(2004)	0	0	0	0
S2(2005)	15215.05	6105.05	30430.1	12210.1
S5(2005)	0	0	0	0
S2(2006)	16814.5	6227	33629	12454
S5(2006)	0	0	0	0
S2(2007)	18665.75	6289.4	37331.5	12578.8
S5(2007)	0	0	0	0
S2(2008)	21270.25	6347.3	42540.5	12694.6
S5(2008)	0	0	0	0
S2(2009)	23854.25	6383.95	47708.5	12767.9
S5(2009)	0	0	45483.98	0
S2(2010)	25408.05	6697.7	50816.1	13395.4
S5(2010)	0	0	0	0
S2(2011)	28838.7	7543.9	57677.4	15087.8
S5(2011)	0	0	0	0
S2(2012)	61599.15	8449.15	123198.3	16898.3
S5(2012)	8182.716	0	381522.7	0
S2(2013)	95099.5	9463.05	190199	18926.1
S5(2013)	0	0	0	0

Table 3b.
InDEA results for separate years

	15 percent increment		30 percent increment	
	Dx1	Dx2	Dx1	Dx2
S2(2003)	33873.3	17740.5	67746.6	35481
S5(2003)	0	0	37490.26	0
S2(2004)	39471.3	17849.25	78942.6	35698.5
S5(2004)	0	0	0	0
S2(2005)	45645.15	18315.15	91290.3	36630.3
S5(2005)	0	0	0	0
S2(2006)	50443.5	18681	100887	37362
S5(2006)	0	0	0	0
S2(2007)	55997.25	18868.2	111994.5	37736.4
S5(2007)	0	0	0	0
S2(2008)	63810.75	19041.9	127621.5	38083.8
S5(2008)	0	0	0	0
S2(2009)	71562.75	19151.85	143125.5	38303.7

S5(2009)	205475.9	0	685451.6	0
S2(2010)	76224.15	20093.1	152448.3	40186.2
S5(2010)	0	0	444450.5	0
S2(2011)	86516.1	22631.7	173032.2	45263.4
S5(2011)	0	0	0	0
S2(2012)	184797.5	25347.45	369594.9	50694.9
S5(2012)	754862.6	0	1874882	0
S2(2013)	285298.5	28389.15	570597	56778.3
S5(2013)	120017	0	1729304	0

Table 3c
InDEA results for separate years

	50 percent increment		75 percent increment		100 percent increment	
	Dx1	Dx2	Dx1	Dx2	Dx1	Dx2
S2(2003)	11291.1	5913.5	22582.2	11827	33873.3	17740.5
S5(2003)	0	0	0	0	0	0
S2(2004)	0	0	26314.2	11899.5	39471.3	17849.25
S5(2004)	0	0	0	0	0	0
S2(2005)	15215.05	6105.05	30430.1	12210.1	45645.15	18315.15
S5(2005)	0	0	0	0	0	0
S2(2006)	16814.5	6227	33629	12454	50443.5	18681
S5(2006)	0	0	0	0	0	0
S2(2007)	18665.75	6289.4	37331.5	12578.8	55997.25	18868.2
S5(2007)	0	0	0	0	0	0
S2(2008)	21270.25	6347.3	42540.5	12694.6	63810.75	19041.9
S5(2008)	0	0	0	0	0	0
S2(2009)	23854.25	6383.95	47708.5	12767.9	71562.75	19151.85
S5(2009)	0	0	45483.98	0	205475.9	0
S2(2010)	25408.05	6697.7	50816.1	13395.4	76224.15	20093.1
S5(2010)	0	0	0	0	0	0
S2(2011)	28838.7	7543.9	57677.4	15087.8	86516.1	22631.7
S5(2011)	0	0	0	0	0	0
S2(2012)	61599.15	8449.15	123198.3	16898.3	184797.5	25347.45
S5(2012)	8182.716	0	381522.7	0	754862.6	0
S2(2013)	95099.5	9463.05	190199	18926.1	285298.5	28389.15
S5(2013)	3875020	0	6557166	0	9239311	0

Table 4a.
InDEA results using generalized model over time

	5 percent increment		10 percent increment	
	Dx1	Dx2	Dx1	Dx2
S2(2003)	0	0	22582.2	11827
S5(2003)	0	0	0	0
S2(2004)	0	0	26314.2	11899.5
S5(2004)	0	0	0	0
S2(2005)	0	0	30430.1	12210.1
S5(2005)	0	0	0	0
S2(2006)	0	0	33629	12454
S5(2006)	0	0	0	0
S2(2007)	0	0	37331.5	12578.8
S5(2007)	0	0	0	0
S2(2008)	0	0	42540.5	12694.6
S5(2008)	0	0	0	0
S2(2009)	0	0	47708.5	12767.9
S5(2009)	0	0	33377.61	0
S2(2010)	0	0	50816.1	13395.4
S5(2010)	0	0	0	0
S2(2011)	0	0	57677.4	15087.8
S5(2011)	0	0	0	0
S2(2012)	0	0	123198.3	16898.3
S5(2012)	1286.32	0	381522.7	0
S2(2013)	0	0	190199	18926.1
S5(2013)	0	0	0	0

Table 4b.
InDEA results using generalized model over time

	15 percent increment		30 percent increment	
	Dx1	Dx2	Dx1	Dx2
S2(2003)	22582.2	11827	22582.2	11827
S5(2003)	0	0	0	0
S2(2004)	26314.2	11899.5	26314.2	11899.5
S5(2004)	0	0	0	0
S2(2005)	30430.1	12210.1	30430.1	12210.1
S5(2005)	0	0	0	0
S2(2006)	33629	12454	33629	12454
S5(2006)	0	0	0	0
S2(2007)	55997.25	18868.2	111994.5	37736.4
S5(2007)	0	0	0	0
S2(2008)	63810.75	19041.9	127621.5	38083.8
S5(2008)	0	0	0	0
S2(2009)	71562.75	19151.85	143125.5	38303.7
S5(2009)	0	0	503006.4	0
S2(2010)	76224.15	20093.1	152448.3	40186.2
S5(2010)	0	0	444450.5	0
S2(2011)	86516.1	22631.7	173032.2	45263.4
S5(2011)	0	0	0	0
S2(2012)	184797.5	25347.45	369594.9	50694.9
S5(2012)	1286.323	0	1874882	0
S2(2013)	285298.5	28389.15	570597	56778.3
S5(2013)	0	0	1729304	0

Table 4c.
InDEA results using generalized model over time

	50 percent increment		75 percent increment		100 percent increment	
	Dx1	Dx2	Dx1	Dx2	Dx1	Dx2
S2(2003)	112911	59135	169366.	88702.5	225822	118270
S5(2003)	217661.2	0	442874	0	668088	0
S2(2004)	131571	59497.5	197356.5	89246.25	263142	118995
S5(2004)	178526.1	0	435860	0	693193.9	0
S2(2005)	152150.5	61050.5	228225.7	91575.75	304301	122101
S5(2005)	18491.16	0	280912.4	0	543333.5	0
S2(2006)	168145	62270	252217.5	93405	336290	124540
S5(2006)	32650.48	0	361623.6	0	690596.6	0
S2(2007)	186657.5	62894	279986.3	94341	373315	125788
S5(2007)	0	0	106481.6	0	485267.5	0
S2(2008)	212702.5	63473	319053.7	95209.5	425405	126946
S5(2008)	0	0	501068.7	0	1048217	0
S2(2009)	238542.5	63839.5	357813.8	95759.25	477085	127679
S5(2009)	972635.3	0	1559671	0	2146707	0

S2(2010)	254080.5	66977	381120.7	100465.5	508161	133954
S5(2010)	1110435	0	1942917	0	2775398	0
S2(2011)	288387	75439	432580.5	113158.5	576774	150878
S5(2011)	0	0	203102.9	0	947571.9	0
S2(2012)	615991.5	84491.5	923987.3	126737.3	1231983	168983
S5(2012)	3368242	0	5234942	0	7101642	0
S2(2013)	950995	94630.5	1426493	141945.8	1901990	189261
S5(2013)	3875020	0	6557166	0	9239311	0

Table 5
Comparison InDEA for separate years and the generalized InDEA over time

	5 percent increment		10 percent increment	
	Dx1	Dx2	Dx1	Dx2
Sum of Dx1 of S2	318056.3	69420	662426.8	150739.5
Sum of Dx1 of S5	8182.716	0	427006.6	0
S2(2004)	0	0	662426.8	150739.5
S5(2004)	1286.323	0	414900.3	0
	15 percent increment		**30 percent increment**	
Sum of Dx1 of S2	993640.2	226109.2	1987280	452218.5
Sum of Dx1 of S5	1080356	0	4771579	0
S2(2004)	937162.4	201913.9	1761369	355437.3
S5(2004)	1286.323	0	4551644	0
	50 percent increment		**75 percent increment**	
Sum of Dx1 of S2	3312134	753697.5	4968201	1130546
Sum of Dx1 of S5	10126446	0	18192327	0
S2(2004)	3312134	753697.5	4968201	1130546
S5(2004)	9773662	0	17626620	0
	100 percent increment			
Sum of Dx1 of S2	6624268	1507395		
Sum of Dx1 of S5	27117958	0		
S2(2004)	6624268	1507395		
S5(2004)	26339327	0		

6. Conclusion

InDEA models like classical DEA models have been developed and widely used over the last two decades. Dealing with the data over time and analyzing the efficiency is an important concept that needs more theoretical attention. The current paper deals with the time-dependent InDEA and proposes a generalized model that is capable of not only the input-output estimation but also optimal time determination in terms of input consumption and output production. As mentioned before, the proposed model has a generalized structure, and different theoretical extensions of InDEA models that are proposed so far can be considered in its generalized over-time framework. Although we had the data set in as crisp form but there are many real-world problems involved with uncertainty that should be considered in the process of input or output estimation. It is a well-known fact that the source of uncertainty may be different from case to case. As long as we have information about the source of uncertainty, we should consider it in the performance assessment process and input-output estimation.

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