Reduction of Cramer-Rao Bound in Arbitrary Pre-designed Arrays Using Altering an Element Position

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Received 6 July 2009; revised 20 February 2010; accepted 8 March 2010

Abstract

Simultaneous estimation of the range and the angle of close emitters usually requires a multidimensional search. This paper proposes an algorithm to improve the position of an element for arrays designed on the basis of some certain or random rules. In the proposed method, one element moves along the same previous direction, maintaining its vertical distance from each source, to reach a constellation with less Cramer-Rao Bound (CRB). The efficiency of this method has been demonstrated through simulation and a comparative study has been conducted, contrasting both the CRB and the determinant of the received signal's covariance matrix before and after applying our proposed scheme.

Keywords: Cramer-Rao Bound, Direction of Arrival, Range, Near-field.

1. Introduction

Direction of arrival (DOA) estimation is usually carried out under the assumption that signal sources are in the far reading field of the array, and hence the wavefront is planar across the array aperture. Supposing that the far field range is denoted by R_0, as the range of the largest departure of the wavefront from a plane wave across the array is $l\lambda$, where λ is the wavelength, it is straightforward to show that $R_0 \approx D^2/8l$, where D is the array aperture measured in wavelengths [1]. For arrays with small aperture, R_0 is rather small and the far field assumption holds very well. However, for arrays with large aperture, e.g., those used in sonar systems, sources are usually located in the near field.

Bearing estimation for near-field sources requires simultaneous estimation of the bearing and range since the curvature of wavefront cannot be ignored. This estimation usually requires a multidimensional search. Previous works on this estimation can be sought in [1-5]. Starer and Nehorai [4] developed an algorithm based on path-following. This algorithm is limited to the context of uniform linear arrays and to sources that are located in the Fresnel region; this region is taken to be between near-field case (with spherical wavefronts) and far-field case (where wavefronts can be estimated in plane form). Collins et al. [5] have proposed an analytic simulated algorithm to solve the problem of estimating the range and bearing. Their algorithm is also limited to uniform linear arrays. The works in [6-9] have also focused on transforming the problem of single dimensional search into polynomial rooting in different cases.

The aim of this paper is not to propose a constellation, but to change the position of one element designed on the basis of some certain or random rules to reach a constellation with less Cramer-Rao bound (CRB). In section 2, the received signal is modeled as a deterministic signal in Additive White Gaussian Noise (AWGN). Section 3 briefly discusses previously obtained results on CRB. Section 4 describes how to change an element's position to obtain less CRB. Section 5 evaluates our theoretical results using computer simulation. Section 6 concludes the article.

2. System Model

N sources of emitters are assumed to be observed by an arbitrary array of M sensors. The signal at the output of the mth sensor can be described by:

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$$x_m(t) = \sum_{n=1}^{N} s_n(t - \tau_{mn}) + v_m(t); \quad -T/2 \le t \le T/2, \quad m = 1,2,\ldots,M \tag{1}$$

where $\{s_n(t)\}_{n=1}^{N}$ are the radiated signals, $\{v_m(t)\}_{m=1}^{M}$ are waveforms of additive noise processes and T denotes the observation interval. The parameters τ_{mn} stand for the delays associated with the signal propagation from the nth source to the mth sensor. These parameters are significant for containing information about the position of source to arrays. Applying appropriate Fourier transform on (1) results in:

$$X_m(j) = \sum_{n=1}^{N} e^{-j\omega_{0n}\tau_{mn}} S_n(j) + V_m(j) \quad m = 1, 2, \ldots, M \tag{2}$$

where $S_n(j)$ and $V_m(j)$ are Fourier transforms of $s_n(t)$ and $v_m(t)$, respectively, and ω_{0n} denotes the center frequency of the radiated signal from the nth source. The index j represents the snapshot number. Using vector notations, formula (2) can be restated as follows:

$$\underline{X}(j) = \mathbf{A}\underline{S}(j) + \underline{V}(j) \tag{3}$$

where,

$$\begin{aligned} &\underline{X}(j) = [X_1(j), X_2(j), \ldots, X_M(j)]^T \\ &\underline{S}(j) = [S_1(j), S_2(j), \ldots, S_N(j)]^T \\ &\underline{V}(j) = [V_1(j), V_2(j), \ldots, V_M(j)]^T \\ &\mathbf{A}_{mn} = \left[e^{-j\omega_0\tau_{mn}}\right]_{M\times N} \quad ; m = 1, 2, \ldots, M, n = 1, 2, \ldots, N \end{aligned} \tag{4}$$

For the sake of simplicity, we assume that sensors and sources are located on the same common plane, so [10]:

$$\tau_{mn} = \frac{1}{c} r_n \left[1 + \left(\frac{\rho_m}{r_n}\right)^2 - 2\frac{\rho_m}{r_n}\cos(\theta_n - \phi_m)\right]^{1/2} \tag{5}$$

where c is the propagation velocity, r_n and θ_n are the range and bearing of the nth source, and p_m along with ϕ_m is the polar coordinate of the mth sensor. The problem is to estimate $\{\underline{r}_n, \underline{\theta}_n\}_{n=1}^{N}$ using data $\{\underline{X}(j)\}_{j=1}^{N_s}$, where N_s is the number of snapshots.

Before we obtain CRB, defining the following covariance matrices is necessary:

$$\mathbf{R}_s = E\{\underline{S}(j)\underline{S}^H(j)\} \tag{6}$$

$$\mathbf{R}_n = E\{\underline{V}(j)\underline{V}^H(j)\} = \eta\mathbf{I} \tag{7}$$

$$\mathbf{R}_x = E\{\underline{X}(j)\underline{X}^H(j)\} = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \mathbf{R}_n \tag{8}$$

We also adopt the following sample covariance matrix as an estimation of $\mathbf{R}_x$ [2]:

$$\hat{\mathbf{R}}_x = \frac{1}{N_s}\sum_{j=1}^{N_s} \underline{X}(j)\underline{X}^H(j). \tag{9}$$

3. Cramer Rao Bound

In this section, we briefly explain CRB for the estimation of the DOA of N observed sources with M elements. An array with M elements can at most separate $M-1$ sources. Therefore, $N < M$ is needed.

CRB gives a lower bound on the covariance matrix of any unbiased estimator. Let's assume that we have N_s independent samples of zero mean Gaussian process x which statistically depends on an arbitrary parameter vector $\underline{P}$. The Fisher Information Matrix (FIM) is, then, as follows [11]:

$$\mathbf{F}_{mn} = N_s.tr\left\{\mathbf{R}_x^{-1}\frac{\partial\mathbf{R}_x}{\partial p_m}\mathbf{R}_x^{-1}\frac{\partial\mathbf{R}_x}{\partial p_n}\right\}. \tag{10}$$

CRB is shown to be equal to the main diagonal elements of $\mathbf{F}$ inverse [11].

The following relation can be obtained from (8):

$$\mathbf{R}_x = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \eta\mathbf{I}. \tag{11}$$

We assume that the signals and the Gaussian noise are uncorrelated. However, the signals might be correlated or even coherent.

We define our parameter vector as follows:

$$\underline{P} = \left[\underline{\theta}^T, \underline{r}^T, \underline{\mu}^T, \nu\right]^T \tag{12}$$

where $\underline{\theta}$ is the DOA vector of N signals, $\underline{r}$ is the corresponding vector of N source ranges, $\underline{\mu}$ is a parameter vector that specifies the $\mathbf{R}_s$ entries, and ν denotes the noise variance. FIM can be partitioned into two blocks, each of which is linked to one or two parametric vector in $\underline{P}$. It is shown that these blocks are as follows [2]:

$$\begin{aligned} \mathbf{F}_{\underline{\alpha}\underline{\beta}} = 2\,\mathrm{Re}\Big\{&\left(\mathbf{R}_s\mathbf{A}^H\mathbf{R}_x^{-1}\mathbf{A}\mathbf{R}_s\right)\times\left(\dot{\mathbf{A}}_{\underline{\beta}}^H\mathbf{R}_x^{-1}\dot{\mathbf{A}}_{\underline{\alpha}}^H\right)^T \\ &+\left(\mathbf{R}_s\mathbf{A}^H\mathbf{R}_x^{-1}\dot{\mathbf{A}}_{\underline{\beta}}\right)\times\left(\mathbf{R}_s\mathbf{A}^H\mathbf{R}_x^{-1}\dot{\mathbf{A}}_{\underline{\alpha}}\right)^T\Big\} \end{aligned} \tag{13}$$

$$\mathbf{F}_{\underline{\alpha}\underline{\mu}} = \mathbf{Q}_4 = \left\{ \left(\mathbf{A}^H \mathbf{R}_x^{-1} \mathbf{A} \mathbf{R}_s \right)^T \otimes \left(\dot{\mathbf{A}}_{\underline{\alpha}}^H \mathbf{R}_x^{-1} \mathbf{A} \right) \right.$$
$$\left. = \left(\mathbf{A}^H \mathbf{R}_x^{-1} \dot{\mathbf{A}}_{\underline{\alpha}} \right)^T \times \left(\mathbf{R}_s \mathbf{A}^H \mathbf{R}_x^{-1} \mathbf{A} \right) \right\} \mathbf{Q}_t^H \tag{14}$$

$$\mathbf{F}_{\underline{\alpha}\nu} = 2\,\mathrm{Re}\left\{ diag\left(\mathbf{R}_s \mathbf{A}^H \mathbf{R}_x^{-2} \dot{\mathbf{A}}_{\underline{\alpha}} \right) \right\} \tag{15}$$

$$\mathbf{F}_{\underline{\mu}\underline{\mu}} = \mathbf{Q}_t \left[\left(\mathbf{A}^H \mathbf{R}_x^{-1} \mathbf{A} \right)^* \otimes \left(\mathbf{A}^H \mathbf{R}_x^{-1} \mathbf{A} \right) \right] \mathbf{Q}_t^H \tag{16}$$

$$\mathbf{F}_{\underline{\mu}\nu} = \mathbf{Q}_t \left[\left(\mathbf{R}_x^{-1} \mathbf{A} \right)^T \otimes \left(\mathbf{R}_x^{-1} \mathbf{A} \right)^H \right] \underline{I}(:) \tag{17}$$

$$F_{\nu\nu} = tr\left\{ \mathbf{R}_x^{-2} \right\} \tag{18}$$

where $\underline{\alpha}$ and $\underline{\beta}$ can be either one of the $\underline{\theta}$ and $\underline{r}$ parameter vectors. We have used $\times$ to denote the Hadamard product of two operands. Also, $\otimes$ signifies the Kronecker product [12]. $\underline{I}(:)$ denotes a vector consisting of the concatenation of the columns of the matrix and $diag\{B\}$ is a column vector containing the diagonal elements of the matrix B. The operators $(\,)^T$, $(\,)^*$ and $(\,)^H$ represent transposition, conjugation, and conjugate transposition, respectively. Further details concerning the above equations are saved for the appendix.

Equations 13 to 18 provide a set of closed form equations to compute the two dimensional CRB (2-D CRB).

4. Changing an Element's Position to Obtain Less CRB

The proposed scheme centers on the element with the strongest received signal. It is not intended to propose a constellation, but rather, to modify the position of an element in a pre-designed arbitrary array in a way that the element moves along the same direction (maintaining its vertical distance from each source) to reach a constellation with less CRB**.** It is known from [8] that:

$$\left| \det\left([\mathbf{A}]_{M \times M} \right) \right|_{abs} \le \left\{ \max \left| a_{ij} \right|_{abs} \right\}^M M^{M/2}; \qquad i, j = 1, 2, \ldots, M \tag{19}$$

Using the above formula, we can take the largest entry of $\mathbf{R}_x$ as the same kth element without loss of generality. In other words, it is possible to assign k to an element with the strongest received signal. We have:

$$\left| \det \mathbf{R}_x \right|_{abs} \le \left\{ E\left\{ \underline{X}_k(j) \underline{X}_k^*(j) \right\} \right\}^M M^{M/2}$$
$$= \left(\left| \underline{X}_k(j) \right|^2 \right)^M M^{M/2} \tag{20}$$

To reduce the determinant of $\mathbf{R}_x$, $\left| X_k(j) \right|^2$ should be reduced. Therefore:

$$\left| X_k(j) \right|^2 = \left| \sum_{n=1}^{N} S_n(j) \left(\cos \omega_{0n} \tau_{kn} + j \sin \omega_{0n} \tau_{kn} \right) \right|^2$$
$$= \left| \sum_{n=1}^{N} \left(S_n(j) \cos \omega_{0n} \tau_{kn} \right) + j \sum_{n=1}^{N} \left(S_n(j) \sin \omega_{0n} \tau_{kn} \right) \right|^2$$
$$= \left(\sum_{n=1}^{N} \left(S_n(j) \cos \omega_{0n} \tau_{kn} \right) \right)^2 + \left(\sum_{n=1}^{N} \left(S_n(j) \cos \omega_{0n} \tau_{kn} \right) \right)^2 \tag{21}$$

Since the sequence $S_n(j)$; $n = 1, 2, \ldots, N$, depends on the specification of the sources, it has no significant role in the minimization of the determinant. Supposing that $S_l(j)$ is the largest of the $S_n(j)$ sequence, we have:

$$\left| X_k(j) \right|^2 \le \left(S_l(j) \sum_{n=1}^{N} \cos \omega_{0n} \tau_{kn} \right)^2$$
$$+ \left(S_l(j) \sum_{n=1}^{N} \sin \omega_{0n} \tau_{kn} \right)^2 = S_l^2(j) \left\{ G + F \right\} \tag{22}$$

where,

$$G = \left(\sum_{n=1}^{N} \cos \omega_{0n} \tau_{kn} \right)^2 \quad ; n = 1, 2, \ldots, N \tag{23}$$

$$F = \left(\sum_{n=1}^{N} \cos \omega_{0n} \tau_{kn} \right)^2 \quad ; n = 1, 2, \ldots, N \tag{24}$$

Now some measure should be taken to reduce the $\{G + F\}$. Figure 1 shows an array with M elements and N sources in which H_{mn} and φ_{mn} denote the vertical distance and the angle of arrival between the mth element and the nth source. According to this figure, we have:

$$\sin \varphi_{kn} = \frac{H_{kn}}{C \tau_{kn}} \quad ; n = 1, 2, \ldots, N \tag{25}$$

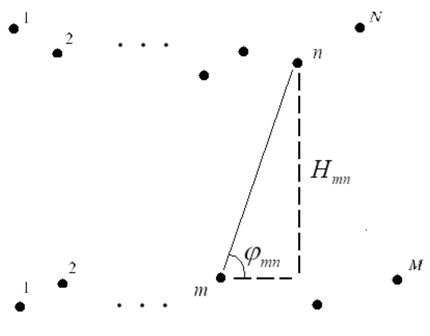

Fig. 1. Arbitrary constellation of sensors and sources.

$$\tau_{kn} = \frac{H_{kn}}{C \sin \varphi_{kn}} \quad ; n = 1, 2, \ldots, N \tag{26}$$

where C is the transmission velocity of the signals in the given environment. Then:

$$G + F = \left\{ \left(\sum_{n=1}^{N} \cos T_n \right)^2 + \left(\sum_{n=1}^{N} \sin T_n \right)^2 \right\} \tag{27}$$

where,

$$T_n = \frac{g}{\sin \varphi_{kn}} \quad ; n = 1, 2, \ldots, N \tag{28}$$

$$g = \frac{\omega_{0n} H_{kn}}{C} \quad ; n = 1, 2, \ldots, N \tag{29}$$

thus,

$$\begin{aligned} & G + F = \\ & (\cos T_1 + \cos T_2 + \cdots + \cos T_N)^2 + \\ & (\sin T_1 + \sin T_2 + \cdots + \sin T_N)^2 \\ & = \cos^2 T_1 + \cos^2 T_2 + \cdots + \cos^2 T_N \\ & + \sin^2 T_1 + \sin^2 T_2 + \cdots + \sin^2 T_N \\ & + 2\cos T_1 \cos T_2 + \cdots + 2\cos T_{N-1} \cos T_N \\ & + 2\sin T_1 \sin T_2 + \cdots + 2\sin T_{N-1} \sin T_N \\ & = N + 2\big(\cos T_1 \cos T_2 + \cdots + \cos T_{N-1} \cos T_N \\ & \qquad + \sin T_1 \sin T_2 + \cdots + \sin T_{N-1} \sin T_N \big) \end{aligned} \tag{30}$$

Equation (30) should be reduced.

It is known that the minimum of a phrase containing the summation of the cross-product of sinusoidal and cosine terms occurs when the angles are equally composed of 0's and $\pi/2$'s for an even number of angles, and when the angles are alternatively 0 and $\pi/2$ for an odd number of angles.

Hence, T_n; $n = 1, 2, \ldots, N$, should consist of an equal number of 0's and $\pi/2$'s. In other words, $\lfloor N/2 \rfloor$ of T_i's (where $\lfloor . \rfloor$ denotes the largest previous integer) should be equal to zero and the same number should be $\pi/2$. With an odd N, then, either the number of 0's or that of $\pi/2$'s is one unit larger, with either case having identical effects on final results. So:

$$T_n = 0 \quad , \quad T_{n+1} = \frac{\pi}{2} \qquad ; n = 1, 2, \ldots, N \tag{31}$$

$$\frac{\omega_{0n} H_{kn}}{C \sin \varphi_{kn}} = 0 \quad , \quad \frac{\omega_{0(n+1)} H_{k(n+1)}}{C \sin \varphi_{k(n+1)}} = \frac{\pi}{2} \tag{32}$$

Using a simple set of algebraic phrases and knowing that $\omega_{0n} = 2\pi f_{0n}$, we will obtain:

$$\varphi_{kn} = arc \sin \frac{2m f_{0n} H_{kn}}{C} \tag{33}$$

$$\varphi_{k(n+1)} = arc \sin \frac{4 f_{0(n+1)} H_{k(n+1)}}{C} \tag{34}$$

where the larger the parameter m is, the more accurate the estimation and determination of the position of the arrays will be. In fact, since T_i; $i = 1, 2, \ldots, N$, can not be zero, we can assume that it is a small number like π/m.

5. Numerical Results

5.1. Simulation Based on Three Sources

In this section, we evaluate our theoretical results using computer simulation. The simulation is based on three sources and four elements. With the index j representing the constant snapshot number, we assume baseband transmission signals of the form $S_1(j) = 2 + i2$, $S_2(j) = 1 + i3$, $S_3(j) = 5 + i3$, where $i = \sqrt{(-1)}$; also, the frequency and propagation velocity are taken as $f_{01} = 1.1787 \times 10^6\ Hz$, $f_{02} = 10^4\ Hz$, $f_{03} = 9.9298 \times 10^5\ Hz$, $C = 3 \times 10^8\ {}^{m}\!/_{s}$, respectively. Figure 2 demonstrates the DOAs of the array elements as well as the vertical ranges according to the values listed in Tables 1 and 2. The determinant of $\mathbf{R}_x$, CRB_θ and CRB_r have been computed

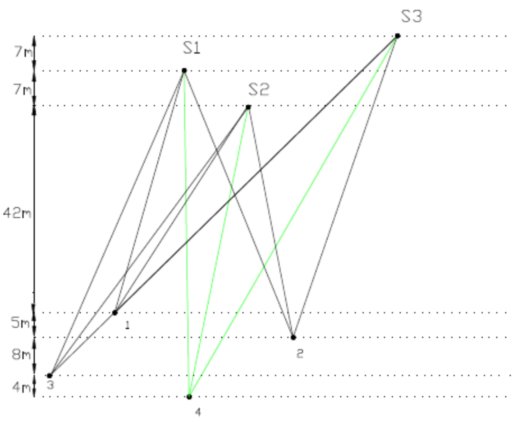

Fig. 2. Arbitrary designed array. The determinant of $\mathbf{R}_x$ is equal to 1.8112×10^{-42}.

Table 1

Angles between the elements and the sources (in degrees)

	Sources		
Elements	1	2	3
1	74	56	45
2	111	100	70
3	66	53	44
4	90	77	59

Table2

Vertical distances between the elements and the sources (in meters)

	Sources		
Elements	1	2	3
1	49	42	56
2	54	47	61
3	62	55	69
4	66	59	73

Table 3

Angles between the elements and the sources (in degrees)

	Sources	
Elements	1	2
1	74	56
2	111	100
3	66	53
4	90	77

Table 4

Vertical distances between the elements and the sources (in meters)

	Sources	
Elements	1	2
1	49	42
2	54	47
3	62	55
4	66	59

as 1.8112×10^{-42}, 1.0308×10^{-29}, 8.8505×10^{-26}, respectively. In order to evaluate the effect of changing an element's position, we have computed the power of received signals at each element and have identified the third element as the one bearing the strongest received signal. Once we change the array constellation according to the formalism discussed in section 4, the new angels are obtained as $\varphi_{31} = 103^\circ$, $\varphi_{32} = 90^\circ$, $\varphi_{33} = 66^\circ$, as depicted in Figure 3. Next, we compute the determinant of $\mathbf{R}_x$, CRB_θ and CRB_r for the proposed constellation, resulting in values equal to 3.4102×10^{-44}, 6.0716×10^{-31}, and 6.1691×10^{-27}, respectively.

B. Simulation Based on Two Sources

The simulation results discussed in this section are based on two sources and four elements. We assume transmission signals of the form $S_1(j) = 2 + i2$, $S_2(j) = 1 + i3$, with frequency and propagation velocity taken as $f_{01} = 1.5 \times 10^6, f_{02} = 9 \times 10^5, C = 3 \times 10^5$ respectively. Figure 4 depicts the DOAs of the array elements as well as the vertical ranges according to the values listed in Tables 3 and 4. We have computed the determinant of $\mathbf{R}_x$, CRB_θ and CRB_r as 2.3737×10^{-42}, 5.0207×10^{-30} and 4.3154×10^{-30}, respectively.

Once again, in order to evaluate the effect of changing an element's location, we have computed the power of the received signals at each element and have identified the third element as the one bearing the strongest received signal. Once we change the array constellation according to the formalism discussed in section 4, the new angels are obtained as $\varphi_{31} = 90^\circ$ and $\varphi_{32} = 41.3^\circ$. Next, we compute

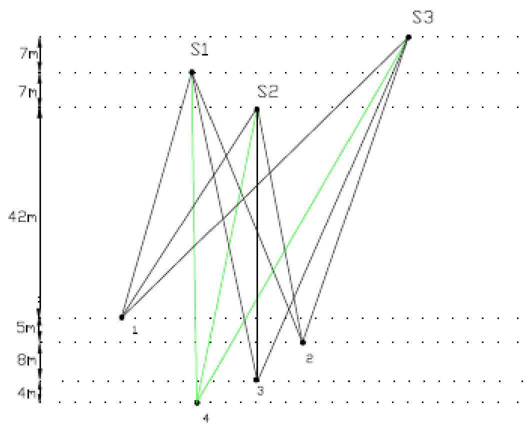

Fig. 3. Proposed constellation. The determinant of $\mathbf{R}_x$ is equal to 3.4102×10^{-44}.

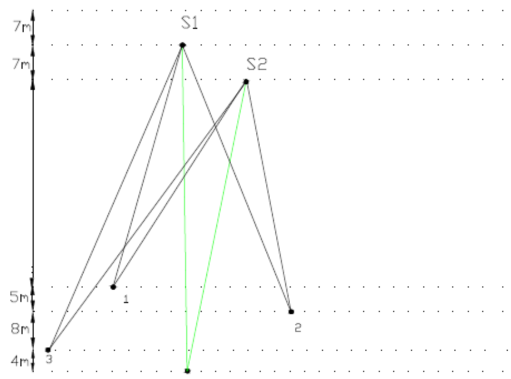

Fig.4. Arbitrary designed array using two sensors.

the determinant of $\mathbf{R}_x$, CRB_θ and CRB_r for the proposed constellation, resulting in values equal to 4.1649×10^{-43}, 1.1578×10^{-30}, and 2.1629×10^{-30}, respectively.

Tables 5 to 10 contrast the performance of the proposed algorithm against the primary constellation by altering the propagation velocity as well as the frequency of the first source. Tables 5 and 6 list the determinant of $\mathbf{R}_x$ for different frequencies and propagation velocities, respectively. Tables 7 and 8, on the other hand, report CRB_θ for different frequencies and propagation velocities. Finally, CRB_r for varying frequencies and propagation velocities is listed in Tables 9 and 10, respectively.

Table 5

The determinant of $\mathbf{R}_x$ versus frequency for an array of four elements (M=4)

Methods / Points (Hz)	Primary		An Element Changing	
	2sources	3sources	2sources	3sources
1000000	8.2468×10^{-42}	5.3085×10^{-42}	3.4098×10^{-42}	3.6145×10^{-42}
2000000	2.0544×10^{-42}	5.8325×10^{-43}	1.4084×10^{-42}	4.3982×10^{-43}
3000000	1.5749×10^{-43}	2.8194×10^{-42}	1.0932×10^{-43}	1.9487×10^{-42}
4000000	1.0692×10^{-43}	3.4566×10^{-42}	3.6470×10^{-46}	6.1592×10^{-43}
5000000	1.3054×10^{-43}	4.1266×10^{-41}	1.6300×10^{-43}	4.0036×10^{-41}
6000000	4.1237×10^{-43}	3.1241×10^{-42}	3.2253×10^{-43}	8.5009×10^{-43}
7000000	1.7775×10^{-42}	1.3006×10^{-42}	6.9780×10^{-45}	1.9980×10^{-43}
8000000	3.9428×10^{-43}	1.1554×10^{-43}	3.2493×10^{-45}	2.4330×10^{-44}
9000000	8.6702×10^{-44}	7.0083×10^{-42}	7.6557×10^{-46}	1.4468×10^{-42}
10000000	4.5189×10^{-44}	1.5868×10^{-41}	1.5247×10^{-45}	8.0344×10^{-42}

Table 6

The determinant $\mathbf{R}_x$ versus propagation velocity for an array of four elements (M=4)

Methods / Points (m/s)	Primary		An Element Changing	
	2sources	3sources	2sources	3sources
1000000	5.9711×10^{-43}	2.5484×10^{-41}	1.9427×10^{-43}	2.8985×10^{-42}
2000000	1.0951×10^{-41}	1.9735×10^{-41}	6.708×10^{-46}	4.1522×10^{-44}
3000000	1.8085×10^{-43}	6.9446×10^{-42}	3.9009×10^{-45}	1.8483×10^{-42}
4000000	3.7700×10^{-47}	7.7086×10^{-41}	3.1951×10^{-47}	8.2311×10^{-43}
5000000	3.5659×10^{-43}	3.1484×10^{-40}	3.3114×10^{-44}	4.4076×10^{-42}
6000000	7.7422×10^{-42}	1.4253×10^{-42}	1.0307×10^{-42}	2.0275×10^{-45}
7000000	1.1830×10^{-42}	5.5752×10^{-41}	6.6526×10^{-45}	1.2188×10^{-42}
8000000	1.3874×10^{-42}	5.9383×10^{-42}	6.0444×10^{-45}	3.0934×10^{-42}
9000000	2.4159×10^{-44}	1.2443×10^{-41}	2.1556×10^{-44}	5.3082×10^{-42}
10000000	8.7276×10^{-42}	3.2287×10^{-40}	5.5864×10^{-42}	1.5260×10^{-42}

Table 7

CRB_θ versus frequency for an array of four elements (M=4)

Methods / Points (Hz)	Primary		An Element Changing	
	2sources	3sources	2sources	3sources
1000000	2.2696×10^{-28}	3.0932×10^{-30}	3.5038×10^{-29}	2.1268×10^{-30}
2000000	2.7740×10^{-30}	1.1183×10^{-30}	1.0009×10^{-30}	7.8426×10^{-31}
3000000	5.3165×10^{-31}	2.4879×10^{-30}	2.5081×10^{-32}	1.4406×10^{-30}
4000000	1.4743×10^{-39}	4.9795×10^{-33}	2.2232×10^{-30}	2.2062×10^{-34}
5000000	4.4349×10^{-30}	2.7031×10^{-31}	4.2344×10^{-31}	1.4155×10^{-31}
6000000	1.0600×10^{-30}	6.7074×10^{-31}	1.2585×10^{-31}	6.1531×10^{-32}
7000000	5.8662×10^{-32}	6.5676×10^{-33}	4.6382×10^{-33}	5.5943×10^{-34}
8000000	3.4001×10^{-31}	1.1740×10^{-32}	8.3975×10^{-32}	1.0026×10^{-33}
9000000	2.0360×10^{-32}	6.2097×10^{-33}	1.4965×10^{-32}	1.2696×10^{-33}
10000000	1.4627×10^{-31}	3.5560×10^{-32}	9.9363×10^{-33}	1.6848×10^{-32}

Table 8

CRB_θ versus propagation velocity for an array of four elements (M=4)

Methods / Points (m/s)	Primary		An Element Changing	
	2sources	3sources	2sources	3sources
1000000	8.3219×10^{-35}	8.9179×10^{-36}	8.5501×10^{-36}	4.1315×10^{-37}
2000000	4.7536×10^{-36}	3.9642×10^{-34}	4.7522×10^{-36}	3.9588×10^{-35}
3000000	2.5533×10^{-35}	9.5407×10^{-35}	3.3845×10^{-36}	5.8610×10^{-37}
4000000	2.7540×10^{-33}	1.2440×10^{-33}	1.2274×10^{-33}	2.7070×10^{-35}
5000000	7.0811×10^{-34}	7.4506×10^{-35}	5.6050×10^{-34}	3.6407×10^{-36}
6000000	4.2372×10^{-33}	1.9872×10^{-32}	3.4724×10^{-33}	3.3847×10^{-33}
7000000	7.0252×10^{-34}	2.0089×10^{-32}	8.6511×10^{-35}	4.9795×10^{-33}
8000000	4.5400×10^{-35}	3.3976×10^{-32}	2.6775×10^{-35}	2.6244×10^{-32}
9000000	1.2252×10^{-33}	1.5149×10^{-32}	7.3698×10^{-34}	5.2732×10^{-34}
10000000	1.4381×10^{-33}	5.4370×10^{-33}	1.2839×10^{-33}	3.3276×10^{-33}

Table 9

CRB_r versus frequency for an array of four elements (M=4).

Methods / Points (Hz)	Primary		An Element Changing	
	2sources	3sources	2sources	3sources
1000000	2.4785×10^{-28}	1.5590×10^{-25}	4.8277×10^{-29}	6.1092×10^{-26}
2000000	8.5974×10^{-30}	8.2220×10^{-26}	2.5775×10^{-30}	3.8235×10^{-26}
3000000	4.0817×10^{-31}	5.4616×10^{-26}	8.7602×10^{-32}	6.1092×10^{-26}
4000000	2.4048×10^{-28}	3.0655×10^{-27}	2.3159×10^{-29}	1.4885×10^{-29}
5000000	2.2176×10^{-29}	9.4367×10^{-28}	2.1461×10^{-29}	1.1737×10^{-28}
6000000	1.4267×10^{-29}	1.6220×10^{-26}	7.7170×10^{-30}	3.3228×10^{-27}
7000000	1.0392×10^{-30}	1.0427×10^{-27}	3.4091×10^{-31}	1.001×10^{-27}
8000000	2.5689×10^{-29}	9.0454×10^{-27}	1.1824×10^{-30}	5.5565×10^{-28}
9000000	1.9810×10^{-30}	1.3019×10^{-25}	1.3549×10^{-30}	1.2179×10^{-25}
10000000	7.6638×10^{-30}	8.8748×10^{-26}	2.0034×10^{-30}	6.3073×10^{-26}

Table 10

CRB_r versus propagation velocity for an array of four elements (M=4).

Methods / Points (m/s)	Primary		An Element Changing	
	2sources	3sources	2sources	3sources
1000000	2.2184×10^{-34}	3.6944×10^{-32}	9.0762×10^{-35}	7.8264×10^{-33}
2000000	6.7851×10^{-35}	7.5931×10^{-29}	5.6365×10^{-35}	1.3934×10^{-30}
3000000	8.1053×10^{-35}	5.2576×10^{-30}	1.1763×10^{-35}	3.0189×10^{-31}
4000000	2.0674×10^{-33}	2.0001×10^{-29}	1.4828×10^{-33}	1.0399×10^{-30}
5000000	4.5562×10^{-33}	4.8737×10^{-30}	1.1500×10^{-33}	7.5032×10^{-31}
6000000	8.5730×10^{-34}	1.7672×10^{-28}	3.0725×10^{-34}	3.4891×10^{-29}
7000000	4.6894×10^{-34}	3.1980×10^{-29}	2.7233×10^{-35}	1.2080×10^{-29}
8000000	6.6242×10^{-36}	7.4334×10^{-28}	1.7407×10^{-36}	1.2640×10^{-28}
9000000	4.1619×10^{-32}	3.4597×10^{-28}	9.1560×10^{-33}	3.3314×10^{-28}
10000000	9.6997×10^{-34}	2.2495×10^{-28}	5.2477×10^{-34}	6.0149×10^{-29}

6. Conclusion

In this paper, we have proposed an algorithm to improve the position of an array element for arrays designed on the basis of some certain or random rules. In the proposed method, one element moves along the same previous direction, maintaining its vertical distance from each source, to reach a constellation with less CRB. The efficiency of the proffered method has been demonstrated through simulation; also, a comparative study has been conducted, contrasting both the CRB and the determinant of the received signal's covariance matrix before and after applying our proposed scheme. We have effectively shown that the new constellation formed by this element changing initiative is of superior performance and with less CRB compared to the primary constellation.

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Appendix: Further details on the CRB equations

In this appendix, we present further details concerning the equations (13)-(18) of the main text. It should be noted that $\mathbf{Q}_t$ and $\mathbf{Q}_4$ are constant matrices [2]:

$$\mathbf{Q}_t = \left[\mathbf{Q}^H \overline{\mathbf{Q}}^H\right]^H \tag{A.1}$$

$$\mathbf{Q}_4 = \mathbf{1}\left(\bar{\underline{J}}_4, \underline{J}_4\right) \tag{A.2}$$

where:

$$\mathbf{Q} = \mathbf{Q}_2\mathbf{Q}_1 \tag{A.3}$$

$$\overline{\mathbf{Q}} = -j\overline{\mathbf{Q}}_2\overline{\mathbf{Q}}_1 \tag{A.4}$$

$$\mathbf{Q}_1 = \mathbf{I} + \mathbf{1}\left(\underline{J}_1, \underline{J}_2\right) \tag{A.5}$$

where $\mathbf{I}$ is the Identity matrix of size equal to $N^2 \times N^2$ and $\mathbf{1}\left(\underline{J}_1, \underline{J}_2\right)$ represents a matrix of the same size, with "1" elements associated with the indices $\underline{J}_1$ and $\underline{J}_2$ and "0's" elsewhere. Also, we have:

$$\overline{\mathbf{Q}}_1 = \mathbf{I} - \mathbf{1}\left(\underline{J}_1, \underline{J}_2\right) \tag{A.6}$$

$$\mathbf{Q}_2 = \mathbf{1}\left(\bar{\underline{J}}_3, \underline{J}_3\right) \tag{A.7}$$

$$\overline{\mathbf{Q}}_2 = \mathbf{1}\left(\bar{\underline{J}}_1, \underline{J}_1\right) \tag{A.8}$$

$$\begin{aligned}\underline{J}_1 = [&2, 3, \ldots, N,\ N+3, N+4, \ldots, 2N, 2N+4 \\ &, 2N+5, \ldots, 3N, 3N+1, \ldots, (N+1)N]\end{aligned} \tag{A.9}$$

$$\bar{\underline{J}}_1 = [1, 2, \ldots, (N-1)N/2] \tag{A.10}$$

$$\begin{aligned}\underline{J}_2 = [&N+1, 2N+1, \ldots, (N-1)N+1,\ 2N+2, 3N+2, \\ &\ldots, (N-1)N+2,\ (N-1)N+3, \ldots, N^2-1]\end{aligned} \tag{A.11}$$

$$\begin{aligned}\underline{J}_3 = [&1, 2, \ldots, N,\ N+2, N+3 \\ &, \ldots, 2N, 2N+1, 2N+2, \ldots, N^2]\end{aligned} \tag{A.12}$$

$$\bar{\underline{J}}_3 = [1, 2, \ldots, N(N+1)/2] \tag{A.13}$$

$$\underline{J}_4 = \left[1,\ N+2,\ 2N+3,\ \ldots,\ N^2\right] \tag{A.14}$$

and $\bar{\underline{J}}_4$ is characterized simply by the indexes $[1, 2, \ldots, N]$, also:

$$\dot{\mathbf{A}}_{\underline{\alpha}} = \sum_{n=1}^{N} \frac{\partial \mathbf{A}}{\partial \underline{\alpha}_n} \tag{A.15}$$

The needed derivations in the formula (A.15) are calculated as follows:

$$\frac{\partial \mathbf{A}_{mk}}{\partial \underline{\theta}_n} = -\frac{\delta_{kn} j(\omega_0/c)\mathbf{A}_{mk}\rho_m \underline{r}_n \sin(\underline{\theta}_n - \phi_m)}{\left[\underline{r}^n + \rho_m^2 - 2\rho_m \underline{r}_n \cos(\underline{\theta}_n - \phi_m)\right]^{1/2}} \tag{A.16}$$

$$\frac{\partial \mathbf{A}_{mk}}{\partial \underline{r}_n} = -\frac{\partial \mathbf{A}_{mk}}{\partial \underline{\theta}_n}\frac{\left(\underline{r}_n - \rho_m \cos(\underline{\theta}_n - \phi_m)\right)}{\left(\rho_m \underline{r}_n \sin(\underline{\theta}_n - \phi_m)\right)} \tag{A.17}$$