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A Method to Reduce Effects of Packet Loss in Video Streaming Using Multiple Description Coding

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Abstract

Multiple description (MD) coding has evolved as a promising technique for promoting error resiliency of multimedia system in real-time application programs over error-prone communicational channels. Although multiple description lattice vector quantization (MDCLVQ) is an efficient method for transmitting reliable data in the context of potential error channels, this method doesn't consider discreteness of network so that losing all descriptions is highly possible. It means all videos may be removed. In this study, we have implemented scheme of MDCLVQ in real-time environment of network, in a method that, raw video (i.e. video with no standard encoding (like MPEG)) is transmitted through independent packets inside of network. This technique leads in low or close to zero loss of all packets. Our purpose is to increase error resiliency and reliable data transmission in error-prone channels. The technique has been tested on some videos sources of Akiyo, Carphone, Foreman and Miss-America. The experimental results indicate that quality of reconstructed videos are substantially improved in terms of central and side PSNR.

Keywords: *Multiple Description Coding, Real-Time Video, Lattice, Sub-lattice hexagonal, Quantization.*

1. Introduction

In recent years, concerning widespread use of smart-phones, tablet and personal computers, the demand for multimedia communication applications has been increased [1]. Multimedia services have widely applied in wired and wireless networks [2]. One common kind of multimedia services is video stream, being classified into two categories: real-time and on demand [3]. Today, real-time video stream transmission over wireless network has been widely accepted and made people to rely on it ([1],[4]). Therefore, video streaming traffic is rising drastically. On the other hand, transmitting real-time services are highly sensitive. Thus supporting high quality video on wireless devices is very important [5].

The transmission of uncompressed (raw) services requires a very large bandwidth. Then, in many of cases, video compression technique is applied. Through removing redundant data, this technique reduces necessary data for transmission. Moreover, transmitting compressed data in network is so sensitive ([1],[6]).

The transmission of real-time services like recorded videos by drone-hat is done through uncompressed data,

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because no strong processor is available in drone-hat and advanced compression, like MPEG, is totally impossible. Therefore, drone-hat transmit videos with no compression and in raw form. Thus, transmitting real-time services with highest quality has become a challenging issue for researchers.

Transmitting multimedia services in noise networks confronts with many challenges. The most critical one is loss of packet. Three methods have been proposed to impede loss of packets in the course of transmitting multimedia services, including ARQ, FEC, and ERC [7].

Automatic Repeat Request (ARQ) method retransmits lost packets in order to receive message of acknowledgment packet using feedback channel. Then, it sends all other packets in the same way. This method is mainly applicable for distributing videos in YouTube and on demand services. Due to the time it takes to ask for retransmission and then to receive it, this method is not suitable for real-time communications [8].

Forward Error Correction (FEC) transmits redundant information in combination with the main data for receiver. Receiver utilizes redundant information to reconstruct corrupted or lost data. Instead of retransmitting packet, FEC uses carrying redundant information, resulting in reduced delay in transmission time. This would be highly suitable for real-time video streaming. Therefore, carrying redundancy packets requires more bandwidth, and not applicable for cases with burst loss of packets ([8],[9]).

The third category is Error Resilient Coding (ERC), a type of which is Layered Coding (LC). In this method, multimedia service is encoded into a base layer and one or more enhancement layers. The base layer provides the basic quality of multimedia service, and enhancement layers improve the quality of the base layer. In other words, this is a hierarchical method, implying that a given layer cannot be decoded unless all of its lower layers have been received correctly. Therefore, this technique is efficient for situations with low rates of loss, and doesn't present acceptable error resiliency [10].

MDC is a sample of the third category. This encodes a single video stream into two or more equally important sub-streams, called descriptions, each of which is packed individually. They are transmitted with a mutual physical channel or separately. With low possibility, both descriptions are simultaneously damaged. Getting every description, receiver delivers the minimum quality and is capable of reconstructing an acceptable video. Moreover, through getting more descriptions (or all descriptions), a video with better quality is produced. This method fits with real-time applications, with high error-rate and no need to retransmit ([7],[11][12][13],[14],[15]).

Multiple description lattice vector quantization (MDLVQ) is a very common model of MDC ([1],[11]). It is a combination of a lattice vector quantizer and a labelling function [16]. The lattice vector quantization is a well-known technique for data compression, which offers less computational complexity due to the regular structure of the lattice [17]. MDCLVQ method was first described in [18]. In this method, the input stream is encoded by a lattice vector quantizer (LVQ). Then, the quantized stream is transformed into two descriptions using the labelling function. Different schemes for MDLVQ have been presented so far, The difference between these schemes of MDLVQ is in labelling function ([16],[18]).

There are an odd number of decoders in an m-channel MD coding schemes. That is, this schemes generating K descriptions has $2^K - 1$ decoders [1]. Storage space requirement in encoder depends on the number K of descriptions. Therefore, time complexity and space complexity of the MDC schemes depend on the encoding and decoding operations. MDLVQ generic scheme do not need to store any codebook for encoding and decoding operations, then time complexity and space complexity has been less than other models MDC. However, the side codebooks are not optimized [19].

The proposed scheme of MDCLVQ-A2 in [1] is a sample of MDLVQ. It includes a new labelling function based on the coinciding similar sub-lattices of the root lattices. This scheme utilizes two different sub-lattices rather than a single sub-lattice. Because of partitioning, presence of two sub-lattices and a new labelling function, MDCLVQ is different from other proposed schemes. Also, it reduces entropy of side videos drastically. On the other hand, this scheme is not relying on specific standard properties of video encoding, implying not dependency on properties of video encoding. While in [13] standard of H.264/AVC and JPEG-2000 has considered for scheme. In these two standards, due to parallel running of encoders, more computational sources are required than main encoders. Because entropies of side videos are decreased, the total computation time of multi description encoders would be lowered. Thus, the required time for encoding each standard video is reduced.

The schema proposed in [1] is a promising method to promote error resiliency and reliable transmission of data. However, it didn't consider discreteness of network, leading in high possibility of loss for all descriptions. It means this scheme whether receives the whole video or nothing. The result is partial loss in video quality. The aim of this study is not only implementing MDCLVQ-A¬2 scheme in real-time network environment but also evaluate efficiency of this method in real-time conditions. Network is designed in a method that packets are transmitted and lost independently, and the possibility of losing all packets be close to zero.

The rest of this study is organized as follows. In Section 2, the definitions of the elementary of lattices are presented. The proposed scheme are presented in Section 3. The experimental results are included in Section 4. Finally section 5 concludes the paper.

2. Elementary of Lattices

A lattice is considered as a subset of points in the Euclidean space that share a common property [1]. The lattice points are generated using a generator matrix. The generator matrix is composed of the basis vectors of the lattice [17], The generator matrix is described in [1]. Therefore, the lattice a can be defined by equation (1).

$$A_n = \{(x_0, x_1, \dots, x_n) \in Z^{n+1} : x_0 + x_1 + \dots + x_n = 0\} \quad (1)$$

In equation (1), the lattice A_n is a subset of points with n+1 coordinates, such that the sum of these coordinates is zero. For example, the A2 lattice is a subset of the complex space ℂ and at unit scale it is generated by the basis vectors {1, ω}⊂ ℂ, where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ [18]. Therefore, the A_2 lattice at unit scale is generated by equation (2).

$$G_{2\times2} = \begin{pmatrix} \mathrm{Re}(1) & \mathrm{Im}(1) \\ \mathrm{Re}(\omega) & \mathrm{Im}(\omega) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \quad (2)$$

In an n-dimensional lattice Λ, the Voronoi region of a lattice point is defined as the union of all non-lattice points within $\mathbb{R}^n$ that are closer to this particular lattice point than any other lattice point. Thus, the Voronoi region of λ ∈ Λ is defined by equation (3) [18].

$$V(\lambda) \triangleq \{\chi \in \mathbb{R}^n : \|\chi - \lambda\| \le \chi - \lambda'\| , \forall\, \lambda' \in \Lambda\} \quad (3)$$

As a consequence, all the points within V(λ) must be quantized to λ. The Voronoi regions of the points in the A2 are hexagons; therefore, it is called the hexagonal lattice. The Voronoi region of a sub-lattice point λ' is the set of all lattice points that are closer to λ' than any other sub-lattice points. Thus, the Voronoi region of λ' ∈ Λ' is defined as equation (4) [18].

$$V(\lambda') \triangleq \{ \lambda \in \Lambda : \|\lambda - \lambda'\| \le \|\lambda - \lambda''\|, \forall\, \lambda'' \in \Lambda' \} \quad (4)$$

Assume that Λ is an L-dimensional lattice with the generator matrix G. A sub-lattice Λ' ⊂ Λ with generator matrix G' is said to be geometrically similar to Λ if and only if G'=*c*UGB, for nonzero scalar *c*, an integer matrix U with det U = ±1, and a real orthogonal matrix B (with $BB^t = I$) [17].

The index *N* is defined as the ratio of the fundamental volume of the sub-lattice Λ' to the fundamental volume of the lattice Λ. The fundamental volume of the lattice (vol) is equal to the determinant of the generator. Thus, *N* is calculated by equation [16].

$$N = \frac{vol'}{vol} = \sqrt{\frac{\det \Lambda'}{\det \Lambda}} = \frac{\det G'}{\det G} \quad (5)$$

For the hexagonal lattice, Λ' is similar to Λ if N is of the form equation [18].

$$N = \alpha^2 - \alpha\beta + \beta^2 \text{ For } \alpha,\beta \in \mathbb{Z} \quad (6)$$

Sub-lattice $\Lambda' \subset \Lambda$ is considered as a clean sub-lattice if all the points of the Λ reside only inside the Voronoi region of the sub-lattice points rather than on the boundary of the Voronoi region. It means that the lattice points are not shared between the Voronoi regions of adjacent sub-lattice points [20].

The sub-lattices of A_2 are clean, if and only if, α and β are relatively primes. It follows that A2 has a clean similar sub-lattice of index N if and only if N is a product of primes congruent to 1 (mod 6). The commonly used values of N are 7, 13, 19, and 37. In other words, α and β are selected such that the value of N satisfies these conditions and hence a clean similar sub-lattice of the hexagonal is generated [20].

In order to find the suitable values of α and β, all twofold combinations of integers [-10, -10] …[+10, +10] have been examined in [13] and only the combinations that generate clean sub-lattices of the hexagonal lattice with the desired indices are selected. The values of α and β that generate clean sub-lattices with N= 7, 13, 19, and 37 combinations are provided in [1].

As shown in Table 1, there are 12 combinations of integer values that satisfy these conditions for N =7, 13 ([1], [13]). Thus, 12 generator matrices are calculated using different values. The generator matrices corresponding to N=7 are provided in [13].

Table. 1. Different values of α and β for different values of N (7, 13) ([1],[13])

i	N	α	β	N	α	β
1	7	-3	-2	13	-4	-3
2		-3	-1		-4	-1
3		-2	-3		-3	-4
4		-2	1		-3	1
5		-1	-3		-1	-4
6		-1	2		-1	3
7		1	-2		1	-3
8		1	3		1	4
9		2	-1		3	-1
10		2	3		3	4
11		3	1		4	1
12		3	2		4	3

The coinciding similar sub-lattices are defined as geometrically similar sub-lattices of a root lattice with the same index N but generated by different generator matrices. In other words, the similar sub-lattices are coinciding with each other in a regular manner. Thus, these sub-lattices are called coinciding similar sub-lattices of the hexagonal lattice [1].

It is seen in Fig. 1, that there are only two distinct sub-lattices for index N=7. This occurs because the sub-lattices form two groups, each group consists of 6 sub-lattices which are coinciding with each other. The same scenario happens for other values of N, there are only two distinct coinciding sub-lattices for every value of N. the two coinciding sub-lattices overlap with each other in a regular pattern. That is, they overlap with each other every N lattice points in every direction. These points are shown by big red circles with black border. The lattice, the first group of sub-lattices coincide and the second group coincide and are shown with triangles, blue squares and red circles respectively ([1],[13]).

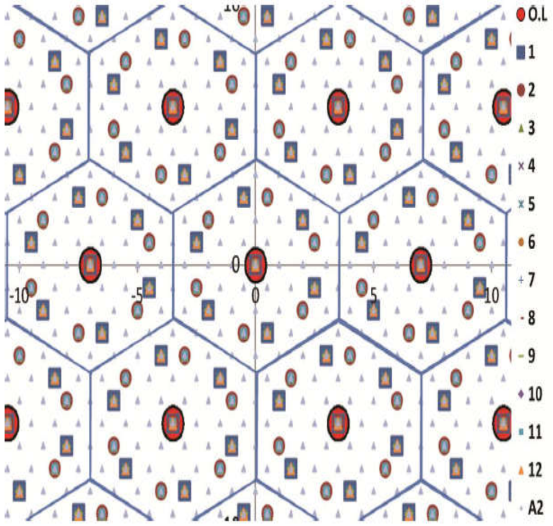

Fig. 1. The coinciding similar sub-lattices of hexagonal lattice with index N=7 but in a limited range [1].

The Super-Voronoi set of an overlapping point is the set of all lattice points that are closer to this point than any other overlapping point. The Super-Voronoi regions of the overlapping sub-lattice points with N=7 is shown with blue hexagon in Fig. 1. This symmetry is used in construction of the partitions. Partitions are used to define the new equivalence class, and the new shift property that can be used to simplify the labeling function. The new equivalence class and the new labeling function are presented in ([1],[13]).

3. Proposed Method

The proposed method of this study, like [11], are used A_2 points as the codebooks of the quantizer, and the coinciding similar sub-lattice points as the labels. Codebooks are optimized and time complexity per sample is been O(K). However, our scheme doesn't utilize any standard encoding method. It means our scheme is implemented on raw video. Proposed scheme diagram is shown in Fig. 2, Therefore, the proposed scheme consists of some modules (Wavelet decomposition module, Lattice Vector Quantizer (LVQ) module, Coinciding Labelling Function module, Inverse Wavelet Transforms module, Video Decoders module), being explained in ([1],[13]).

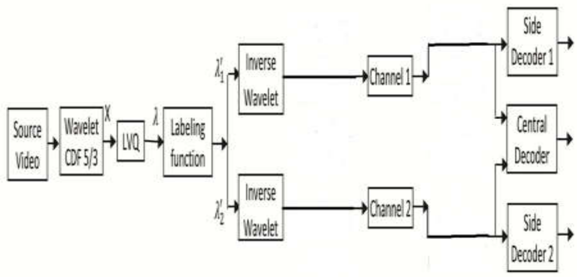

Fig. 2. MDCLVQ-A_2 video coding scheme applied to raw video [13].

First, the input video with the QCIF format (144×176 pixels) is read to save data. Then, using the wavelet module, every single video is decomposed into four videos (72×88 pixels). At end, decomposed video is placed into two-dimensional vectors with 144×176÷2 row. Then, it enters in LVQ module.

In the second stage, the two-dimensional input vectors are mapped into 3D vectors by a transformation matrix. Then, the 3D vectors are mapped to the lattice points by simple arithmetic manipulations. The procedure of LVQ module is described comprehensively in [1].

In the third stage, the labeling function maps each lattice point into two coinciding sub-lattice points that form a label. At the end of this stage, two independent descriptions of the video stream are produced [1]. The next stage, the inverse wavelet transform on the streams generated by the labeling function is done. The original data (144×176 pixels) has been restored.

The proposed scheme, MDCLVQ-A2, is implemented in real environment of network. The network environment consists of two transmitters, A FIFO queue, one receiver, and two identical channels. It is assumed that transmitter (like drone-hat) has a weak processor and no possibility for encoding is available, implying no encoding for raw video. Also, there is no limitation for bit rate, and a bit rate of 10 Mb/s is chosen. Error bit rate was set between 10^0 to 10^ -10, the result of which is independent transmission and loss of packets. In addition, the FIFO queue is used as an essential part of a discrete event simulator, in order to put packets in proper queue.

4. Experimental Results

In this section, the experimental results of the proposed scheme, MDCLVQ-A_2, being implemented in the real network environment for standard videos of Akiyo and Foreman, with format of (144×176 pixels) are presented. We are intended to evaluate efficiency of our proposed method comparing to MDCLVQ-A_2 scheme in [1] in terms of both PSNR and error resiliency.

The MDCLVQ-A_2 scheme is a multivariable system, because it is controlled by several factors, such as the wavelet threshold, the fundamental area of the hexagonal lattice and the index of the coinciding similar sub-lattices. Fundamental area of the hexagonal lattice is changed by sigma. The variable sigma is used by the fast quantizing algorithm to affect the generator matrix of the lattice being used. The fundamental area of the hexagonal lattice is changed by sigma = 0.1, 0.2…, 1, while maintaining the shape of the hexagonal lattice. Thus, the granularity of the quantization is adapted [1]. In this experiment the labeling function with index N=7 has been used constantly and the wavelet threshold is constantly zero. Therefore, the analyses provided below are based on different values of sigma and Variable bit error rate. The performance of the MD coding scheme is evaluated based on the average required bit rate and reconstruction fidelity in the decoders.

A common measure to evaluate the quality of the video encoding is to compare the reconstructed video with the original video through the average peak signal to noise ratio (PSNR) of the luminance (Y) components ([1],[13]). In the following results, the PSNR is measured for the luminance component. The average PSNR values of the central decoder and the side decoders of the proposed scheme are provided in Table 2. The average PSNR values of the central decoder and side decoders are improved as sigma is increased. In other words, the qualities of the reconstructed videos are directly proportional to the value of sigma.

As an example, PSNR values of the central decoder for the all videos corresponding to sigma=0.2 and Variable error bit rate are shown in Figure 3.

The experimental results for the "Foreman" video sequences are shown in Fig. 4, that average PSNR values of the central decoder and the side decoders of the proposed scheme are higher than the corresponding values of the MDCLVQ-A_2 in [1]. Thus, this is because of better reconstruction quality of the proposed scheme as compared with the MDCLVQ-A_2 in [1]. However, the general pattern is the same and all the PSNR values increase as the sigma is increased because reconstruction fidelity is more affected by the performance of the MDCLVQ-A_2.

According to Fig.3. and the reconstruction performance of the proposed scheme in terms of the central PSNR and side PSNR increase, the sigma is increased. In order to demonstrate the visual quality of the reconstructed videos, the frames of the central videos of Foreman sequence with sigma values that range from 0.1 to 1 are provided in Fig.5. The corresponding value of sigma and the PSNR of the central videos are also provided, as shown in Fig.5. The quality of the reconstructed videos of the Foreman increase as the value of sigma is increased. In other words, the proposed scheme can offer acceptable side and central reconstruction qualities and error resiliency without extra bit rate.

Table. 2. The average PSNR (dB) of the central and the side decoders for the selected video sequences.

Video/Sigma		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
	Side1	31.50	31.86	32.93	33.86	34.56	35.43	37.04	37.50	38.40	38.45
Miss-America	Side2	31.50	31.90	32.93	33.96	34.43	35.43	37.04	37.50	38.40	38.45
	Central	31.50	31.86	32.93	33.86	34.56	35.43	37.04	37.50	38.40	38.45
	Side1	37.61	37.67	38.15	39.45	38.81	39.80	40.77	42.51	42.51	43.88
Foreman	Side2	37.61	37.52	38.15	39.53	39.81	40.24	41.77	42.51	43.88	43.88
	Central	37.61	37.67	38.15	39.35	39.73	40.24	41.77	42.51	42.80	43.88
	Side1	34.04	34.54	35.20	36.21	38.02	38.82	36.49	35.75	38.82	41.56
Akiyo	Side2	33.96	34.54	35.13	36.35	38.38	38.82	36.49	35.75	38.82	41.56
	Central	34.00	34.48	35.20	36.21	38.02	38.82	36.49	35.75	38.82	41.56
	Side1	31.50	31.73	32.44	33.42	34.71	37.60	35.93	35.93	39.77	39.77
Carphone	Side2	31.50	31.72	32.44	33.42	34.71	38.41	35.93	39.77	39.77	38.40
	Central	31.50	31.72	32.44	33.42	34.71	37.60	35.93	38.67	39.77	38.40

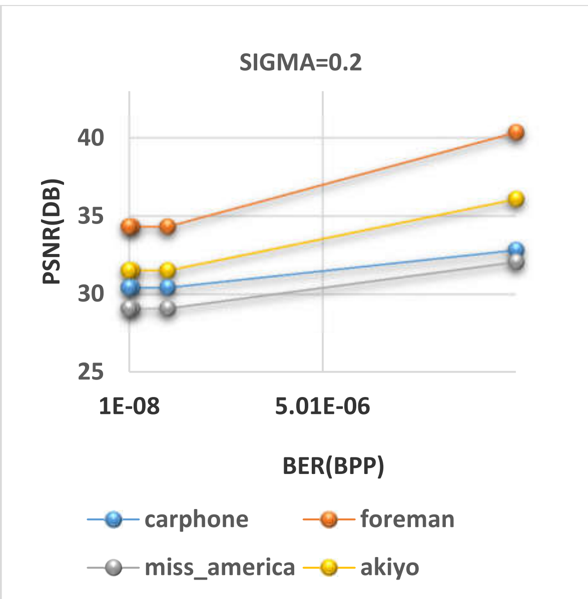

Fig. 3. PSNR values of the central decoder for videos tested with sigma=0.2 and error bit rate.

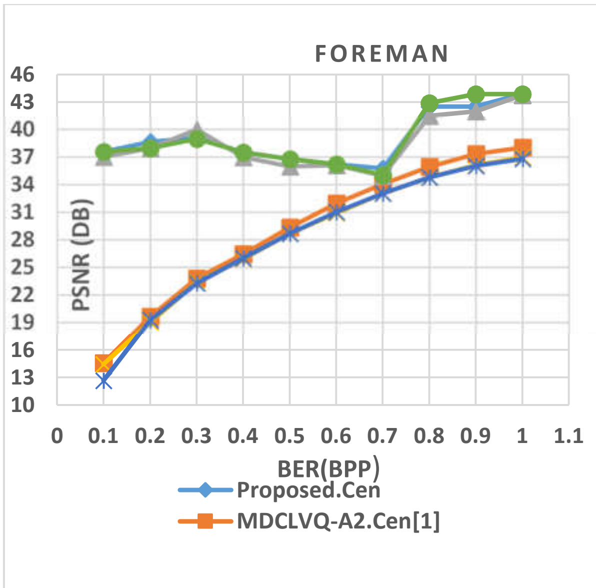

Fig. 4. PSNR values for the Foreman video by the proposed MDCLVQ-A_2 and referenced algorithms.

5. Conclusion

MDCLVQ-A_2 scheme is an efficient technique for real-time application programs in error-prone communicational channels. This is particularly suitable in situations with no possibility of retransmission. The main advantage of this scheme is promoting robustness of video transmission over error-prone communication channels while conserving the bandwidth.

In this study, we have implemented scheme of MDCLVQ-A_2 for real-time environment of network, in a method that, discreteness of network was considered, videos were transmitted through independent packets, and possible loss of all packets were close to zero. Also, scheme was evaluated in real environment of network. To evaluate, the proposed scheme was implemented over some standard video streams. Results indicated that quality of all video streams was substantially improved in terms of central and side PSNR. Therefore, both resistance for video transmission and error resiliency have been improved comparing previous scheme.

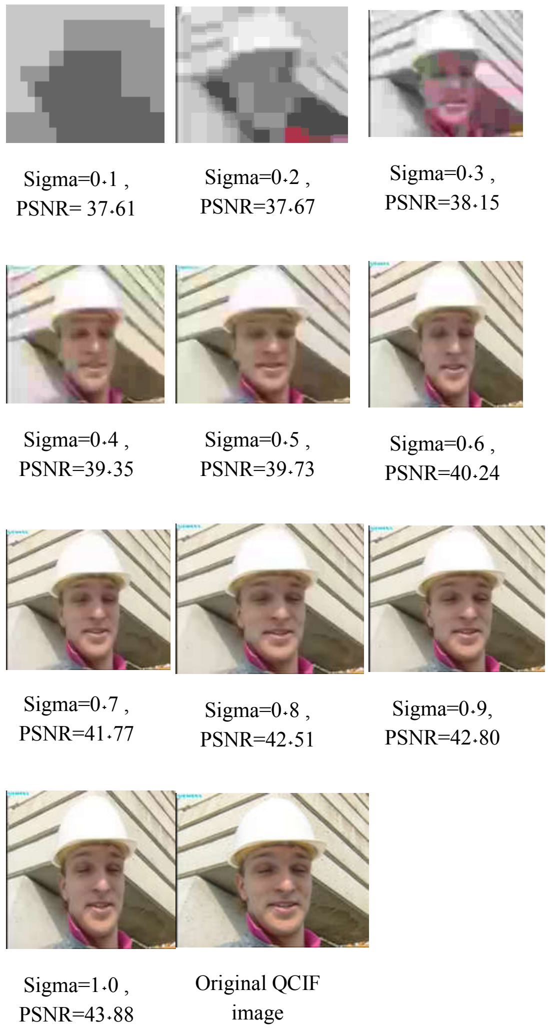

Fig. 5. The 1st frames of the QCIF Foreman sequence reconstructed by the central decoder corresponding to sigma = 0.1 to 1.

References

[1] Akhtarkavan, E.; Salleh, M.F.M., "Multiple descriptions coinciding lattice vector quantizer for wavelet image coding". IEEE Transaction on Image Processing, vol. 21, no. 2, pp.653-661 (2012).

[2] Kusetogullari, H., "Dynamic multiple description wavelet based image coding using enhanced particle swarm optimization". IEEE Journal of Selected Topics in Signal Processing, vol. 9, no. 1, pp. 125-138 (2015).

[3] Sheshjavani, A.G.; Akbari, B., "A study on repeat-request chunks in pull-based peer-to-peer video-on-demand streaming". IEEE 22nd Iranian Conference on Electrical Engineering (ICEE). pp. 891-896 (2014).

[4] Immich, R., et al. "Adaptive motion-aware fec-based mechanism to ensure video transmission". 2014 IEEE Symposium in Computers and Communication (ISCC), pp.1-6 (2014).

[5] Sgardoni, V.; Nix, A.R., "Raptor code-aware link adaptation for spectrally efficient unicast video streaming over mobile broadband networks". IEEE Transactions on Mobile Computing, vol. 14, no. 2, pp. 401-415 (2015).

[6] Chuang, K.P.; Lee, W.T.; Wei, H.W., "Dynamically Adjust Packet Size and Block Size to Achieve Maximum Effectiveness of Forward Error Correction". 2014 Tenth International Conference on Intelligent Information Hiding and Multimedia Signal Processing (IIH-MSP). IEEE. pp. 658-661 (2014).

[7] Kazemi, M.; Shirmohammadi, S.; Sadeghi, K.H., "A review of multiple description coding techniques for error-resilient video delivery". Multimedia Systems, vol. 20, no. 3, pp. 283-309 (2014).

[8] Ali, F.A.; et al., "Bandwidth efficient adaptive forward error correction mechanism with feedback channel". Journal of communications and networks, vol. 16, no.3, pp. 322-334 (2014).

[9] Wu, T.-Y.; et al., "An enhanced structure of layered forward error correction and interleaving for scalable video coding in wireless video delivery". IEEE Wireless Communications, vol. 20, no. 4, pp.146-152 (2013).

[10] Ghahremani, S.; Ghanbari, M., "Error resilient video transmission in ad hoc networks using layered and multiple description coding". Multimedia Tools and Applications, vol. 76, no.6, pp. 9033-9049 (2016).

[11] Akhtarkavan, E.; Salleh, M.F.M., "Multiple descriptions coding for H.264/AVC using coinciding A2 lattice vector quantizer". IEEE 2011

International Conference on Signal and Image Processing Applications (ICSIPA 2011). Kuala Lumpur, Malaysia, pp. 505-510 (2011).

[12] Sun, Y.-C.; Tsai, W.-J., "Rate-distortion optimized mode selection method for multiple description video coding". Multimedia tools and applications, vol. 72, no. 2, pp. 1411-1439 (2014).

[13] Akhtarkavan, E.; Salleh, M.; Sidek, O., "Multiple Descriptions Video Coding Using Coinciding Lattice Vector Quantizer for H. 264/AVC and Motion JPEG2000". World Applied Sciences Journal, vol. 21, no. 2, pp. 157-169 (2013).

[14] Goyal, V.K.; Kovacevic, J., "Generalized multiple description coding with correlating transforms". IEEE Transactions on Information Theory, vol. 47, no. 6, pp. 2199-2224 (2001).

[15] Goyal, V.K., "Multiple description coding: Compression meets the network". IEEE Signal Processing Magazine, vol. 18, no. 5, pp. 74-92 (2001).

[16] Servetto, S.D.; Vaishampayan, V.A.; Sloane, N., "Multiple description lattice vector quantization". Data Compression Conference (DCC), Snowbird, UT, pp.13-22, IEEE (1999).

[17] Conway, J.; Sloane, N.J.A., "Sphere Packings, Lattices and Groups". New York: Springer-Verlag (1998).

[18] Vaishampayan, V.A.; Sloane, N.; Servetto, S., "Multiple description vector quantization with lattice codebooks: Design and analysis". IEEE Trans. on Information Theory, vol. 47, no. 5, pp. 1718-1734 (2001).

[19] Wu, H.; Zheng, T.; Dumitrescu, S., "On the Design of Symmetric Entropy-Constrained Multiple Description Scalar Quantizer with Linear Joint Decoders". IEEE Trans. on communications, vol. 65, no. 8, pp. 3453-3466 (2017).

[20] Conway, J.H.; Rains, E.M.; Sloane, N.J.A., "On the existence of similar sublattices". Canad. J. Math, vol. 51, no.6, pp. 1300-1306 (1999).