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GENERALIZED INVERSES OF ANTI-TRIANGULAR BLOCK OPERATOR MATRICES

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Abstract

We introduce a new class of generalized inverse which is called π −Hirano inverse. In this paper some elementary properties of the π −Hirano inverse are obtained. We investigate the existence of the π −Hirano inverse for the anti-triangular operator matrix $N = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}$ with $DCB = 0$ and et al. Certain multiplicative and additive results for the π −Hirano inverse in a Banach algebra are presented. We then apply some conditions under which a 2×2 block operator matrix has π −Hirano inverse over Banach spaces.

1- Introduction

Let $\mathcal{A}$ be a Banach algebra with an identity. We first recall the definitions of some generalized inverses. As is well known, in 1958, Drazin [6] defined, an element $a \in \mathcal{A}$ has Drazin inverse if there is the element $a \in \mathcal{A}$ which satisfies

$$ax = xa, xax = x \text{ and } a - a^2x \in N(\mathcal{A})$$

or

$$ax = xa, xax = x \text{ and } a^k = a^{k+1}x. \qquad (1)$$

Here $N(\mathcal{A})$ is the set of all nilpotent elements in $\mathcal{A}$. The element x above is uniqe if it exists and is denoted by a^d and called the Drazin inverse of a. The smallest such nonnegetive integer k is called the Drazin index of a, denoted $ind(a)$. Drazin proved that a has Drazin inverse if and only if a is strongly π−regular, that is $a^m \in a^{m+1}\mathcal{A} \cap \mathcal{A}a^{m+1}$ for some $m \in \mathbb{N}$ [6]. Here, $\mathbb{N}$ stands for the set of all natural numbers. Recently, several subclasses of the Drazin inverse have been studied. In 2017, Wang [12] gave the notion of the strongly Drazin inverse in a ring. An element $a \in \mathcal{A}$ has strongly Drazin inverse if there is a uniqe common solution to the equations

$$ax = xa, xax = x \text{ and } a - ax \in N(\mathcal{A})$$

and we denoted by a^{sd}. We know that in a Banach algebra $\mathcal{A}$, $a \in \mathcal{A}^{sd}$ if and only if it is the sum of an idempotent and a nilpotent that commute, if and only if $a - a^2 \in N(\mathcal{A})$ [2]. Here, $q \in \mathcal{A}$ is an idempotent if $q^2 = q$.

In same year, Chen and Sheibani [3] definded, the Hirano inverse of $a \in \mathcal{A}$ is the uniqe element $x \in \mathcal{A}$ satisfying

$$ax = xa, xax = x \text{ and } a^2 - ax \in N(\mathcal{A})$$

and we denoted by a^h. They characterized the Hirano inverse by tripotents. Here, $q \in \mathcal{A}$ is a tripotent if $q^3 = q$. Also, they showed that, $a \in \mathcal{A}^H$ if and only if

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$a - a^3 \in N(\mathcal{A})$ [3], if and only if it is the sum of a tripotenet and a nilpotent that commute [1]. In addition, they obtained $\mathcal{A}^{sD} \subsetneq \mathcal{A}^{H} \subsetneq \mathcal{A}^{D}$, where $\mathcal{A}^{sD}$, $\mathcal{A}^{H}$ and $\mathcal{A}^{D}$ mean the sets of all strongly Drazin invertible, Hirano invertible and Derazin invertible elements in $\mathcal{A}$, respectively. In 2019, Mosic [10] gave the notion of the n-strongly Drazin inverse, which is a new class of Drazin inverse and the n-strongly Drazin inverse $a \in \mathcal{A}$ is the uniqe element $x \in \mathcal{A}$ if such element exists, and if it satisfies

$$ax = xa, xax = x \text{ and } a^n - ax \in N(\mathcal{A}) \qquad (2)$$

for some $n \in \mathbb{N}$ and we denoted by a^{nsd}. Clearly, the n-strongly Drazin inverse covers the strongly Drazin inverse and Hirano inverse, that is, $a^{1\mathrm{sd}} = a^{\mathrm{sd}}$ and $a^{1\mathrm{sd}} = a^{\mathrm{h}}$. In [16], Zou and Mosic et al. investigate the structure of a ring in which every element satisfies the condition $a - a^{n+1} \in N(\mathcal{A})$ is nilpotent for a fixed n. This inspires us to introduce and study a new class of generalized inverse, that it forms a subclass of Drazin inverses in a Banach algebra. We say that an element $a \in \mathcal{A}$ has π −Hirano inverse if there exists $x \in \mathcal{A}$ such that

$$ax = xa, xax = x \text{ and } a - a^{n+2}x \in N(\mathcal{A}) \qquad (3)$$

for some $n \in \mathbb{N}$ and we denoted by $a^{\pi h}$. The preceding x shall be unique, if such element exists. We observed these inverses form a subclass of Drazin inverses which is related to periodic elements in a Banach algebra $\mathcal{A}$. We denote the set of all π−Hirano invertible elements in $\mathcal{A}$ by $\mathcal{A}^{\pi H}$. It is proved that an element $a \in \mathcal{A}$ has π −Hirano inverse if and only if $a - a^{n+1} \in N(\mathcal{A})$ for some $n \in \mathbb{N}$. The invertibility of the sum of two π −Hirano invertible elements in a Banach algebra under some conditions will be presented.

2- Additive results

In this section we are concern on additive property of the π −Hirano inverse of the sum in a Banach algebra $\mathcal{A}$.

Lemma 1.2. [16] Let $a, b \in \mathcal{A}$ with $ab = ba$. Then

(i) If $a \in N(\mathcal{A})$ or $b \in N(\mathcal{A})$ then $ab \in N(\mathcal{A})$.

(ii) If $a, b \in N(\mathcal{A})$, then $a + b \in N(\mathcal{A})$.

Lemma 2.2. [14] Let $a \in \mathcal{A}$. If $a - a^2 \in N(\mathcal{A})$, then there exists a monic polynomial $f(a) \in Z[a]$ such that $f(a) = f(a)^2$ and $a - f(a) \in N(\mathcal{A})$.

Lemma 3.2. [16] Let $a, b \in \mathcal{A}$ be such that $ab = 0$. Then

$$a, b \in N(\mathcal{A}) \Leftrightarrow a + b \in N(\mathcal{A}).$$

Theorem 4.2. [7] Let $n \in \mathbb{N}$. Then $a \in \mathcal{A}^{\pi H}$ if and only if $a - a^{n+1} \in N(\mathcal{A})$.

Corollary 5.2. Every Hirano invertible element in a Banach algebra is π−Hirano invertible element.

Proof. It is obvious by Theorem 2.4. □

In the next example we show that the converse Corollary 5.2 is not true.

Example 6.2. Let $\mathcal{A} = M_2(Z_2)$ and $a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathcal{A}$. Then a has π−Hirano inverse but it is not Hirano invertible. Because, it is obvious that $a = a^4$ and so $a - a^4 \in N(\mathcal{A})$. Then by Theorem 2.4, a has π−Hirano inverse. If a has Hirano inverse, it follows by [3],

$$a^h = a^d = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

But $a^2 - aa^h = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ is not nilpotent. This gives a contradiction.

Corollary 7.2. Every π−Hirano invertible element in a Banach algebra is Deazin invertible element.

Proof. Let $a \in \mathcal{A}$ has π−Hirano inverse. By Theorem 4.2, $a - a^{n+1} \in N(\mathcal{A})$. Then there exists some $m \in \mathbb{N}$ such that $(a - a^{n+1})m = 0$. Hence, by Lemma 2.2, we can find some polynomial $f(a)$ such that $a^n = a^{n+1}f(a)$and so a is strongly π−regular which is Drazin invertible element [6]. □

Theorem 8.2. [7] Let $a, b, c \in \mathcal{A}$. If $aba = aca$, then ac has π −Hirano inverse if and only if ba has π −Hirano inverse.

Corollary 9.2. Let $a, b \in \mathcal{A}$. If ab has π−Hirano inverse, then so has ba.

Corollary 10.2. Let $a \in \mathcal{A}$ and $m \in \mathbb{N}$. Then $a \in \mathcal{A}^{\pi H}$ if and only if $a^m \in \mathcal{A}^{\pi H}$.

Lemma 11.2. Let $a, b \in \mathcal{A}$ and $ab = 0$. Then $a, b \in \mathcal{A}^{\pi H}$ if and only if $a + b \in \mathcal{A}^{\pi H}$.

Proof. Since $a + b \in \mathcal{A}^{\pi H}$, there exist $m, k \in \mathbb{N}$ such that

$$a - a^{m+1}, \ b - b^{k+1}$$

are nilpotent. Let $n = mk$. Then $a - a^{n+1}$ and $b - b^{n+1}$ are nilpotent. Becuase,

$$\begin{aligned} a - a^{n+1} &= a - a^{mk+1} \\ &= (a - a^{m+1}) + (a^{m+1} - a^{2m+1}) + \cdots + (a^{(k-1)m+1} - a^{km+1}) \\ &= (a - a^{m+1})\left(1 + a^m + \cdots + a^{(k-1)m}\right) \in N(\mathcal{A}) \end{aligned}$$

Likewise, we have $b - b^{n+1} \in N(\mathcal{A})$. By hypothesis $ab = 0$, we have

$$\begin{aligned} a + b - (a+b)^{n+1} &= (a - a^{n+1}) + (b - b^{n+1}) + (ba^n + b^2a^{n-1} + \cdots + b^n a) \\ &= x + y + z. \end{aligned}$$

It is clear that x, y and z are nilpotent, $x(y + z) = 0$ and $zy = 0$. In view of Lemma 3.2, we get $x + y + z$ is nilpotent and by Theorem 4.2, $a + b$ is π−Hirano invertible. On the contrary, let $a + b \in \mathcal{A}^{\pi H}$. Then $x(y + z) = 0$ and $zy = 0$. By Lemma 3.2 and Theorem 4.2, we obtain $x, y, z \in N(\mathcal{A})$ and $a + b \in \mathcal{A}^{\pi H}$. □

Theorem 12.2. [11] Let $a, b \in \mathcal{A}^{\pi H}$. If $a^3b = 0$, $ba^2b = 0$, $abab = 0$ and $b^2ab = 0$, then $a + b \in \mathcal{A}^{\pi H}$.

Corollary 13.2. Let $a, b \in \mathcal{A}^{\pi H}$. If $a^2b = 0$, $aba = 0$ and $bab = 0$, then $a + b \in \mathcal{A}^{\pi H}$.

Corollary 14.2. [11] Let $a, b \in \mathcal{A}^{\pi H}$. If $a^2b = 0$ and $bab = 0$, then $a + b \in \mathcal{A}^{\pi H}$.

We are now ready to prove:

Theorem 15.2. Let $a, b \in \mathcal{A}^{\pi H}$. If $a^2b = 0$, $bab^2 = 0$ and $(ab)^2 = 0$ then $a + b \in \mathcal{A}^{\pi H}$.

Proof. Let $p = a^2 + ab$ and $q = b^2 + ba$. Since $(ab)^2 = 0$ and we have $ab - (ab)^2 \in \mathcal{A}$ is nilpotent. Hence, by using Theorem 4.2, we see that $ab \in \mathcal{A}^{\pi H}$. By using Corollary 9.2, $ba \in \mathcal{A}^{\pi H}$. In view of Corollary 10.2, $a^2, b^2 \in \mathcal{A}^{\pi H}$. Since $a^2(ab) = 0$, it follows by Lemma 11.2, that $p \in \mathcal{A}^{\pi H}$. As $(ba)b^2 = 0$, we see that $q \in \mathcal{A}^{\pi H}$. Clearly,

$$p^2q = (a^2 + ab)(ab^3 + ab^2a) = 0,$$

$$qpq = (b^2 + ba)(ab^3 + ab^2a) = 0.$$

According to Corollary 14.2, $(a + b)^2 = p + q \in \mathcal{A}^{\pi H}$. Therefore $a + b \in \mathcal{A}^{\pi H}$, by Corollary 10.2.

Clearly, $(ab)a^2 = 0$ and $(ba)b^2 = 0$. It follows by Lemma 11.2, that $p, q \in \mathcal{A}^{\pi H}$. Furthermore, we check that $pq = 0$ and then $(a + b)^2 = p + q \in \mathcal{A}^{\pi H}$ and $a + b \in \mathcal{A}^{\pi H}$, as required. □

Lemma 16.2. Let $a, b \in \mathcal{A}^{\pi H}$. If $ab^2 = 0$ and $aba = 0$, then $a + b \in \mathcal{A}^{\pi H}$.

Proof. Let $p = a^2 + ab$ and $q = b^2 + ba$. Since $(ab)^2 = 0$, then $ab, ba \in \mathcal{A}^{\pi H}$. Clearly, $(ab)a^2 = 0$ and $(ba)b^2 = 0$. It follows by Lemma 11.2, that $p, q \in \mathcal{A}^{\pi H}$. Furthermore, we check that $pq = 0$ and then $(a + b)^2 = p + q \in \mathcal{A}^{\pi H}$ and $a + b \in \mathcal{A}^{\pi H}$, as required. □

Theorem 17.2. Let $a, b \in \mathcal{A}^{\pi H}$. If $ab^2 = 0$, $aba^2 = 0$ and $(ab)^2 = 0$, then $a + b \in \mathcal{A}^{\pi H}$.

Proof. Let $p = a^2 + ab$ and $q = b^2 + ba$. Clearly, $ab, ba \in \mathcal{A}^{\pi H}$. Since $(ab)a^2 = 0$ and $(ba)b^2 = 0$ it follows by Lemma 11.2, that $p, q \in \mathcal{A}^{\pi H}$. Clearly, $pq^2 = 0$ and $pqp = 0$. According to Lemma 16.2., $(a + b)^2 = p + q \in \mathcal{A}^{\pi H}$. Therefore $a + b \in \mathcal{A}^{\pi H}$. □

Proposition 18.2. Let $a, b \in \mathcal{A}^{\pi H}$. If $ab^2 = 0$, $a^2ba = 0$ and $(ab)^2 = 0$, then $a + b \in \mathcal{A}^{\pi H}$.

Proof. Let $p = a^2 + ba$ and $q = b^2 + ab$. As in the proof in Theorem 17.2, we see that $a + b \in \mathcal{A}^{\pi H}$. □

3- OPERATOR MATRICES

To illustrate the preceding results, we are concerned with the π−Hirano inverse for an operator matrix. Throughout this section, the operator matrix

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad (4)$$

Where $A\in\mathcal{L}(X)^{\pi H}$, $B\in\mathcal{L}(X,Y)$, $C\in\mathcal{L}(Y,X)$ and $D\in\mathcal{L}(Y)^{\pi H}$. Using different splitting approach and Theorem 17.2, we will obtain various conditions for the π−Hirano inverse of M.

Lemma 1.3. [7] Let $A\in\mathcal{L}(X)^{\pi H}$ and $D\in\mathcal{L}(Y)^{\pi H}$. Then

$$K=\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, L=\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \text{ and } L=\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

have π−Hirano inverse.

Lemma 2.3. [4] Let $B\in\mathcal{L}(X,Y)^{\pi H}$ and $C\in\mathcal{L}(Y,X)^{\pi H}$. If $CB\in\mathcal{L}(Y)^{\pi H}$, then

$$L=\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$$

has π −Hirano inverse.

Lemma 3.3. Let $D, CB\in\mathcal{L}(Y)^{\pi H}$. If $DCB=0$, then

$$N=\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}$$

has π −Hirano inverse.

Proof. Consider the splitting of,

$$N=\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}+\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}=P+Q$$

By Lemmas 1.3 and 2.3, P and Q have π−Hirano inverse. According to the assumptions, we have,

$$PQ^2=\begin{bmatrix} 0 & 0 \\ 0 & DCB \end{bmatrix}, PQP^2=\begin{bmatrix} 0 & 0 \\ DCBC & 0 \end{bmatrix}$$

$$\text{and } (PQ)^2=\begin{bmatrix} 0 & 0 \\ DC & 0 \end{bmatrix}^2.$$

Therefore $PQ^2=0$, $PQP^2=0$ and $(PQ)^2=0$. Applying Theorem 17.2, $M=P+Q\in\mathcal{L}(X\oplus Y)^{\pi H}$, as asserted. ☐

Theorem 4.3. Let $A\in\mathcal{L}(X)^{\pi H}$ and $D, CB\in\mathcal{L}(Y)^{\pi H}$. If $DCB=0$, $ABC=0$ and $ABD=0$, then

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

has π −Hirano inverse.

Proof. Write

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}=P+Q.$$

By Lemma 3.1 and Lemma 3.3, P and Q have π−Hirano inverse. We check that $PQ^2=0$, $PQP^2=0$ and $(PQ)^2=0$. Then by Theorem 17.2, we complete the proof and $M=P+Q\in\mathcal{L}(X\oplus Y)^{\pi H}$. ☐

Corollary 5.3. Let $A\in\mathcal{L}(X)^{\pi H}$ and $D, CB\in\mathcal{L}(Y)^{\pi H}$. If $DCB=0$, and $AB=0$, then $M\in\mathcal{L}(X\oplus Y)^{\pi H}$.

Proposition 6.3. Let $A\in\mathcal{L}(X)^{\pi H}$ and $D, CB\in\mathcal{L}(Y)^{\pi H}$. If $DCB=0$, $CAB=0$ and $CA^2=0$ then $M\in\mathcal{L}(X\oplus Y)^{\pi H}$.

Proof. Write

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\begin{bmatrix} 0 & B \\ C & D \end{bmatrix}+\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}=P+Q.$$

Similary Theorem 4.3, we have $PQ^2=0$, $PQP^2=0$ and $(PQ)^2=0$. By Theorem 17.2, we complete the proof. ☐

Corollary 7.3. Let $A\in\mathcal{L}(X)^{\pi H}$ and $D, CB\in\mathcal{L}(Y)^{\pi H}$. If $DCB=0$ and $CA=0$ then $M\in\mathcal{L}(X\oplus Y)^{\pi H}$.

Lemma 8.3. Let $D, CB\in\mathcal{L}(Y)^{\pi H}$. If $BDC=0$ and $BD^2=0$, then

$$N=\begin{bmatrix} 0 & B \\ C & D \end{bmatrix} \qquad (5)$$

has π−Hirano inverse.

Proof. Consider the splitting of,

$$N=\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}=P+Q$$

Similary Lemma 3.3, we have $PQ^2=0$, $PQP^2=0$ and $(PQ)^2=0$. Applying Theorem 17.2, $N\in\mathcal{L}(X\oplus Y)^{\pi H}$, as asserted. ☐

Theorem 9.3. Let $A\in\mathcal{L}(X)^{\pi H}$ and $D, CB\in\mathcal{L}(Y)^{\pi H}$. If $BDC=0$, $BD^2=0$, $CA^2=0$, and $CAB=0$, then

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

has π −Hirano inverse.

Proof. Write

$$M = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} = P + Q.$$

By Lemma 1.3 and Lemma 8.3, P and Q have π−Hirano inverse. We have $PQ^2 = 0$, $PQP^2 = 0$ and $(PQ)^2 = 0$. Then by Theorem 17.2, we complete the proof. □

Corollary 10.3. Let $A \in \mathcal{L}(X)^{\pi H}$ and $D, CB \in \mathcal{L}(Y)^{\pi H}$. If $BD = 0$ and $CA = 0$, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Theorem 11.3. Let $A \in \mathcal{L}(X)^{\pi H}$ and $D, CB \in \mathcal{L}(Y)^{\pi H}$. If $DCB = 0$, $CAB = 0$ and $CA^2 = 0$, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Proof. Write

$$M = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = P + Q.$$

We get the result by Lemma 3.3 and Theorem 9.3.

Corollary 12.3. Let $A \in \mathcal{L}(X)^{\pi H}$ and $D, CB \in \mathcal{L}(Y)^{\pi H}$. If $DCB = 0$ and $CA = 0$, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

4- PERTURBATIONS

Let M be an operator matrix. It is of interest to consider the π −Hirano inverse of M under generalized Schur condition $D = CA^{\pi h}B$ [9]. Let

$$W = AA^{\pi h} + A^{\pi h}BCA^{\pi h}. \qquad (6)$$

We now derive

Theorem 1.4. Let $A \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $AA^{\pi}BC = 0$, $A^{\pi}BCA^{\pi} = 0$, $ABCA^{\pi} = 0$, $D = CA^{\pi h}B$ and AW has π −Hirano inverse, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Proof. We easily see that

$$M = \begin{bmatrix} A & B \\ C & CA^{\pi h}B \end{bmatrix} = \begin{bmatrix} 0 & A^{\pi}B \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A & AA^{\pi h}B \\ C & CA^{\pi h}B \end{bmatrix}$$
$$= P + Q.$$

It is not hard to see that $P^3Q = 0$, $QP^2Q = 0$, $PQPQ = 0$ and $Q^2PQ = 0$. In view of Theorem 12.2, P is nilpotent and it has π−Hirano inverse. Moreover, we have $Q = Q_1 + Q_2$,

$$Q_1 = \begin{bmatrix} AA^{\pi} & 0 \\ CA^{\pi} & 0 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} A^2A^{\pi h} & AA^{\pi h}B \\ CAA^{\pi h} & CA^{\pi h}B \end{bmatrix}$$

that $Q_2Q_1 = 0$ and Q_1 is nilpotent. Easily check that

$$Q_2 = \begin{bmatrix} AA^{\pi h} \\ CA^{\pi h} \end{bmatrix} \begin{bmatrix} A & AA^{\pi h}B \end{bmatrix}.$$

By hypothesis, we see that

$$\begin{bmatrix} A & AA^{\pi h}B \end{bmatrix} \begin{bmatrix} AA^{\pi h} \\ CA^{\pi h} \end{bmatrix} = A^2A^{\pi h} + AA^{\pi h}BCA^{\pi h} = AW$$

has π−Hirano inverse. Obviously, Q_2 has π−Hirano inverse. Therefore Q has π−Hirano inverse. According to Theorem 2.12, $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Corollary 2.4. Let $A \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $ABC = 0$, $A^{\pi}BC = 0$, $D = CA^{\pi h}B$ and AW has π −Hirano inverse, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Theorem 3.4. Let $A \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $BCAA^{\pi} = 0$, $A^{\pi}BCA = 0$ and $D = CA^{\pi h}B$. If AW has π −Hirano inverse, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Proof: Clearly, we have

$$M = \begin{bmatrix} A & B \\ C & CA^{\pi h}B \end{bmatrix} = P + Q$$

Where

$$P = \begin{bmatrix} A^2A^{\pi h} & B \\ C & CA^{\pi h}B \end{bmatrix} \text{ and } Q = \begin{bmatrix} AA^{\pi} & 0 \\ 0 & 0 \end{bmatrix}.$$

By assumption, we verify that $P^2Q = 0$, $QPQ^2 = 0$ and $(PQ)^2 = 0$. In view of in Theorem 15.2, Q is nilpotent, and then it has π−Hirano inverse. Moreover, we have $P = P_1 + P_2$,

$$P_1 = \begin{bmatrix} 0 & A^{\pi}B \\ CA^{\pi} & 0 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} A^2A^{\pi h} & AA^{\pi h}B \\ CAA^{\pi h} & CA^{\pi h}B \end{bmatrix}$$

and $P_2P_1 = 0$. Since ${P_1}^3 = 0$, therefore P_1 has π−Hirano inverse. Moreover, we have

$$P_2 = \begin{bmatrix} AA^{\pi h} \\ CA^{\pi h} \end{bmatrix} \begin{bmatrix} A & AA^{\pi h}B \end{bmatrix}.$$

By hypothesis, we see that

$$\begin{bmatrix} A & AA^{\pi h}B \end{bmatrix} \begin{bmatrix} AA^{\pi h} \\ CA^{\pi h} \end{bmatrix} = A^2A^{\pi h} + AA^{\pi h}BCA^{\pi h} = AW$$

is π−Hirano invertible. Therefore P_2 has π−Hirano inverse. By virtue of Theorem 15.2, $M \in \mathcal{L}(X \oplus Y)^{\pi H}$, as required.

□

Corollary 4.4. Let $A \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $BCA = 0$, $D = CA^{\pi h}B$ and AW has π −Hirano inverse, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Theorem 5.4. Let $A \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $BCAA^{\pi} = 0$, $BCA^{\pi}BC = 0$, $A^{\pi}BCA = 0$, $D = CA^{\pi h}B$ adn AW has π −Hirano inverse, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Proof. Clearly, we have

$$M = \begin{bmatrix} A & B \\ C & CA^{\pi h}B \end{bmatrix} = P + Q$$

Where

$$P = \begin{bmatrix} A & B \\ CA^{\pi h}A & CA^{\pi h}B \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 \\ CA^{\pi h} & 0 \end{bmatrix}.$$

By assumption, we verify that

$$PQ^2 = 0, PQP^2 = 0 \text{ and } (PQ)^2 = 0.$$

In view of Theorem 17.2, Q is nilpotent, and then it has π−Hirano inverse. We see that $P = P_1 + P_2$,

$$P_1 = \begin{bmatrix} AA^{\pi h} & A^{\pi}B \\ 0 & 0 \end{bmatrix} \text{ and } P_2 = \begin{bmatrix} A^2A^{\pi h} & AA^{\pi h}B \\ CAA^{\pi h} & CA^{\pi h}B \end{bmatrix}$$

that $P_1P_2 = 0$ and P_1 is nilpotent. Moreover, we have

$$P_2 = \begin{bmatrix} AA^{\pi h} \\ CA^{\pi h} \end{bmatrix} \begin{bmatrix} A & AA^{\pi h}B \end{bmatrix}.$$

By hypothesis, we see that

$$\begin{bmatrix} A & AA^{\pi h}B \end{bmatrix} \begin{bmatrix} AA^{\pi h} \\ CA^{\pi h} \end{bmatrix} = A^2A^{\pi h} + AA^{\pi h}BCA^{\pi h} = AW$$

has π−Hirano inverse. Therefore P2 has π−Hirano inverse. By virtue of Theorem 17.2, $M \in \mathcal{L}(X \oplus Y)^{\pi H}$, as required.

Corollary 6.4. Let $A \in \mathcal{L}(X)^{\pi H}$ and $D \in \mathcal{L}(Y)^{\pi H}$. If $BCA = 0$, $BCA^{\pi} = 0$, $D = CA^{\pi h}B$ and AW has π −Hirano inverse, then $M \in \mathcal{L}(X \oplus Y)^{\pi H}$.

Regarding a complex matrix as the operator matrix on $\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}$, we now present a numerical example to demonstrate Theorem 5.4.

Example 7.4. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -1 & -1 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

be complex matrices and set

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

then

$$A^{\pi h} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A^{\pi} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We easily check that $BCAA^{\pi} = 0$, $BCA^{\pi}BC = 0$, $A^{\pi}BCA = 0$ and $D = CA^{\pi}B$. In this case, A and D have π −Hirano inverses.

Conclusion

We introduce a new class of generalized inverse which is called π −Hirano inverse. In this paper some elementary properties of the π −Hirano inverse are obtained. We investigate the existence of the π −Hirano inverse for the anti-triangular operator matrix $N = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}$ with $DCB = 0$ and et al. Certain multiplicative and additive results for the π −Hirano inverse in a Banach algebra are presented. We then apply some conditions under which a 2×2 block operator matrix has π −Hirano inverse over Banach spaces.

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Author Contributions

Tahereh Haddadi, Samaneh Bahramian collected the data, carried out the data analysis and interpreted the results and wrote the manuscript.

Conflict of Interest

The author declares that there is no conflict of interests regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancy have been completely observed by the authors.