

Fuzzy Multivariate Process Capability Index for Measuring Process Capability

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Abstract. In the case of process capability index several methods are identified. In this paper ,a new process capability index using fuzzy number and fuzzy probability concept, in order to remove the weakness of other famous method is suggested. After introduction of fuzzy index in univariate case, fuzzy multivariate process capability index is investigated. Finally, this new method is compared to three well-known methods in literature review, with numerical example.

Keywords: Component, Process Capability Index, Fuzzy Number, Multivariate Normal Distribution.

1. Introduction

In the statistical quality control field, it is needed to know process capability in order to applying control tools such as control charts. The object of process capability analysis consists of analyzing the process capability of production in order to assess whether or not the process meets given specification. It means that we should know how many of products satisfy technical specifications. There may not exist an exact definition of the term of "process capability" but there is an agreement to consider a process as capable if with high probability the quality characteristics X of the products items lies between some lower and upper specification limits. Therefore the idea of process capability implies that the fraction p of product non-conforming items should be small if the process is said to be capable. Three factors in the case of process capability are considered as mean process value, target value as a goal in tolerance region and process variation. Less different between mean process with target and less variation result in product near to target value and reduction in defect goods. If process mean is far from target and variation increase, defect rate will be go up and cause to have not capable process.

A well know method to study process capability use process capability index. There is different index that proposed in univariate and multivariate domain. In univariate case assume that quality characteristic X and

corresponding random sample (X_1, X_2, \dots, X_n) are normal, in fact $X \sim N(\mu, \sigma^2)$. let LSL denoted the lower and USL the upper specification limit , $m = (USL + LSL) / 2$ the midpoint of tolerance interval (LSL, USL) , T the target for μ for which we assume that $t=m$. The simplest index is the process potential index defined by

$$C_p = \frac{USL - LSL}{6\sigma} \quad (1)$$

That suggested by Kane. This index is ratio of tolerance region to process region. It is clear that this method consider the variation of process.

In order to reflect departures from the target value as well as change in the process variation several order indexes have been proposed C_{pk} (Sullivan) and C_{pm} (Chan, Chen and spring) like below:

$$C_{pk} = \frac{\min \{USL - \mu, \mu - LSL\}}{3\sigma} \tag{2}$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - t)^2}} \tag{3}$$

The last method is more efficient because of target value and process mean consideration. The shortage seems in Cpk is that these methods don't mentioned target value, and main constraint for all of this method is that normal distribution assumed for probability distribution. On the other approach using of non -conforming ratio (NC) as an index for capability process are identify by Carr(1991) at first time:

$$NC = \Pr \{x \notin [L, U]\} = 1 - \left\{ \phi \left(\frac{U - \mu}{\sigma} \right) - \phi \left(\frac{L - \mu}{\sigma} \right) \right\} \tag{4}$$

This approach based on the stepwise loss function as below:

$$\text{Loss} = \begin{cases} 0 & L \leq x \leq U \\ 1 & \text{otherwise} \end{cases} \tag{5}$$

In the next section, we identify fuzzy process capability index and investigate it to multivariate domain in order to remove discussed weakness. Numerical example for comparison in multivariate case to other method is available.

2. Method

As we see in pervious section process capability is equal to below probability value:

$$p = \int_L^U f(x)dx \tag{6}$$

That $f(x)$ is the probability distribution of quality characteristic. In fact this relation means discrete probability for $[LSL, USL]$ interval. If we use indicator function to this definition we could have below equation as:

$$p = \int_{-\infty}^{\infty} \mu(x)f(x)dx \quad (7)$$

$$\mu(x) = \begin{cases} 1 & x \in [L, U] \\ 0 & x \notin [L, U] \end{cases} \quad (8)$$

This concept is using of stepwise loss function that is defined up to tolerance region. Figure 1 provide fuzzy quality, when $X = t$, $\mu(t) = 1$, which shows products are completely up-to-standard, when $X = LSL$ or USL , $\mu(LSL) = \mu(USL) = 0$, it shown product are totally below standards with both sides symmetrical. The curve shown in Figure 1 is worked out through the fuzzy statistical, processing of users suitability appraisals. However, in actual calculation, it can sometimes be simplified: using a linear line segment rather than a curve.

The process capability index can be defined as the probability of fuzzy up-to-standard products turn out in the process of production it is labeled as \tilde{C}_p . when the quality index of process products is a continues random variable,

$$\tilde{C}_p = \int_{-\infty}^{\infty} \mu(x)f(x)dx \quad (9)$$

In the formula, $f(x)$ is the probability density function of x , $\mu(x)$ is the membership function of fuzzy up-to-standard products. When the quality characteristic of process is a discrete random variable X and the value of X is x_i ($i = 1, \dots, n$).

$$\tilde{C}_p = \sum_{i=1}^n \mu(x_i)f(x_i) \quad (10)$$

In this formula $p_x(x_i)$ is the probability when $X = x_i$, $\mu(x_i)$ is the membership function of fuzzy up-to-standard product. Obviously we have to know the value of the probability density of quality characteristic and the membership function of fuzzy up to standard products, before we can work out \tilde{C}_p

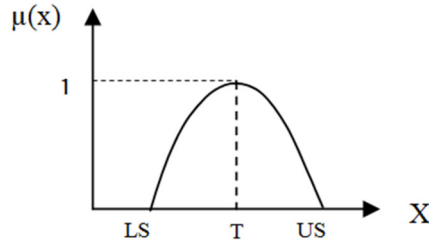


Figure 1. Fuzzy tolerance interval for up-to-standard product.

The membership function that we define for up to standard product, in fact, is a fuzzy interval .Since then we use LR fuzzy number especially normal and triangular type in order to have simple and reasonable calculation. Figure 2 and Figure 3 shows triangular and normal membership function type, respectively.

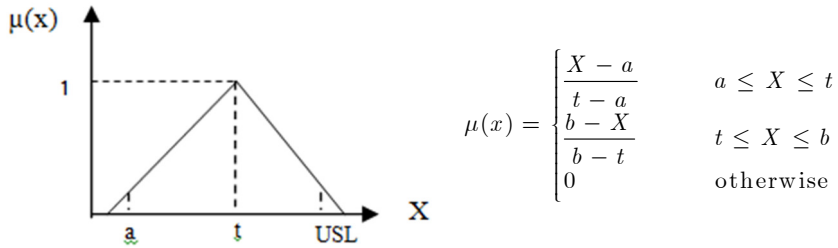


Figure 2. Triangular membership function and its function.

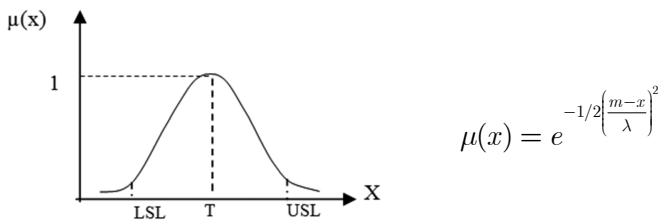


Figure 3. Normal membership function and its function.

As an example if we have normal membership function and normal distribution, we could calculate fuzzy index as below integral:

$$\tilde{C}_p = \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2\lambda^2}} \frac{1}{\sqrt{2\pi\sigma}} e^{-1/2 \left(\frac{x-\mu}{\sigma} \right)^2} dx \tag{11}$$

If we solve it we have below answer:

$$\tilde{C}_p = \left(\frac{\lambda}{\sqrt{\sigma^2 + \lambda^2}} \right) e^{\frac{-(\mu-t)^2}{2(\sigma^2 + \lambda^2)}} \quad (12)$$

For index estimation if we don't have distribution parameters we would use estimation of parameter in it like S^2 for σ^2 and \bar{X} for μ . this method for other membership function and other distribution function is applicable and final solution will be available using analytical or numerical methods.

3. Findings

The important topic that we face to it in process capability analysis is multivariate PCIs. In the univariate case tolerance interval defined such as $\{LSL \leq x \leq USL\}$, but in multivariate case we should consider tolerance zone, that in some case it will be a hyper cube as below:

$$\bigcap_{i=1}^v (L_i \leq x_i \leq U_i) \quad (13)$$

A graphical approach has been presented for monitoring whether or not our process is capable by using the process capability plots for two families of the one-sided PCIs based on fuzzy data in Sadeghpour Gildeh& Moradi (2012) study. Tsai-Wai Chen et al. (2012), proposes a fuzzy inference method to select the best among the competing suppliers based on an estimated capability index of C calculated from sampled data. Cengiz Kahraman, İhsan Kaya,(2010) in study, the fuzzy set theory is used to add more sensitiveness to PCA including more information and flexibility Ming-Hung Shu, Hsien-Chung Wu, (2012) propose a constructive methodology for obtaining the fuzzy estimate of Cpm using fuzzy data, which is based on "resolution identity" in fuzzy sets theory. In Moradi& Sadeghpour Gildeh study, (2012), a graphical approach has been presented for monitoring whether or not our process is capable by using the process capability plots for two families of the one-sided PCIs based on fuzzy data. Zİhsan Kaya, (2010) in study, the fuzzy set theory is used to add more sensitiveness to PCA including more information and flexibility. Zeinab Ramezania(2011) presented four

approximate $100 (1 - \alpha)\%$ fuzzy confidence regions for the fuzzy PCI \tilde{C}_{pm} . In the multivariate process capability index, Shahriari and et.all (1995), suggest a multivariate capability vector. The proposed vector consists of three components [CPM, PV, and LI]. Two components use the assumption that the process data is from a multivariate normal distribution with elliptical counters defining probability region. The first component of the vector is the ratio of volume of engineering tolerance region to volume of modified process region. The second component of the vector is based on the assumption that the center of the engineering specification is considered to be the true underlying mean of the process. Probability value (PV) is defined as a significant level of the observed values. It will never exceed 1; values close to zero indicate that the center of the process is "far" from the engineering target values. The third component has a comparison of the location of the process region and tolerance region. It has a value of 1 if the entire modified process regions contained within the tolerance region and otherwise a value of 0. another method suggested by Taam & et.all (1993) that is a fraction which the numerator of it is ratio of modified tolerance region to process region that is suitable for process variation analysis .and denominator of it is statistical distance between process mean and target value, this index compare to 1 .if final result of it is more than 1 process will be capable. In 1994 Chen was proposed different method base on non-conforming product fraction that it consider dispersion of distribution, and it has weakness rather than other method to change of target or mean vectors. An efficient argument about calculation of three expressed methods and its applications was existed in reference. After quick review on famous multivariate methods we investigate fuzzy index to multivariate case.

In order to this get aim we should investigate fuzzy membership function and use multivariate distribution function .if we use vector and matrix we could have a multivariate normal membership function as below:

$$\mu(X) = e^{-1/2(X-t)'\Lambda^{-2}(x-t)} = e^{-1/2\left(\frac{(x_1-t_1)^2}{\lambda_1^2} + \frac{(x_2-t_2)^2}{\lambda_2^2} + \dots + \frac{(x_n-t_n)^2}{\lambda_n^2}\right)} \quad (14)$$

That

$$\Lambda_{n \times n} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad X_{n \times 1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad t_{n \times 1} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} \quad (15)$$

$\Lambda_{n \times n}$ Is the parameter matrix of normal membership function and T is target vector important result in is that investigated membership function is equal to multiple of it in univariate case .on the other word it will be as below

$$\mu(X) = \mu(x_1) \times \mu(x_2) \times \dots \mu(x_n) \quad (16)$$

We could use above equation to investigate every membership function.

Totally if $f(X)$ is multivariate probability distribution and $\mu(x)$ is investigated membership function, Multivariate process capability index will be defined as below:

$$M\tilde{C}_p = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(X) f(X) dX \quad \text{For Continues PDF} \quad (17)$$

$$M\tilde{C}_p = \sum \sum \dots \sum \mu(X) f(X) \quad \text{For Discrete PDF} \quad (18)$$

In general case assume that $f(X)$ has p variable normal distribution as below

$$f(X) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-1/2(X-\mu)'\Sigma^{-1}(X-\mu)} \quad (19)$$

That μ is mean vector and Σ is covariance matrix; and $\mu(x)$ is an investigated normal membership function.

Therefore multivariate fuzzy process capability index will calculate as:

$$M\tilde{C}_p = \int \int \dots \int e^{-1/2(X-t)'\Lambda^{-2}(x-\mu)} \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} e^{-1/2(X-\mu)'\Sigma^{-1}(X-\mu)} dX$$

Especially in bivariate case and use of bivariate normal distribution this index can show as:

$$M\tilde{C}_p = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-1/2 \left[\frac{(x_1-t_1)^2}{\lambda_1^2} + \frac{(x_2-t_2)^2}{\lambda_2^2} \right]} \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12})^2}} \quad (20)$$

$$\exp \left[\frac{1}{-2(1-\rho_{12}^2)} \left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}} \right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}} \right)^2 - 2\rho_{12} \left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}} \right) \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}} \right) \right] \right] dx_1 dx_2 \quad (21)$$

That σ_{11} , σ_{22} is variance of x_1 , x_2 and ρ_{12} is correlation between x_1 , x_2 and T1 and T2 is target values for it.

There are two simple methods in order to estimation of the fuzzy index. In first method we directly use the definition of index; so that by substitution of estimated distribution parameter and membership function in main clause and solve computed sigma or integral we will have a final value for fuzzy index:

$$M\hat{C}_p = \int \int \dots \int e^{-1/2(X-T_0)' \Lambda_0^{-2}(X-T_0)} \frac{1}{(2\pi)^{p/2} |S|^{1/2}} e^{-1/2(X-\bar{X})' S^{-1}(X-\bar{X})} d(X) \quad (22)$$

That \bar{X} is sample mean vector and S is sample covariance matrix. In second method, relation between index definition and expected value concepts and properties is used as:

$$M\hat{C}_p = \int \int \dots \int \mu(X) f(X) d(X) = E(\mu(X)) \quad (23)$$

If we use for general model that express it in previous section, estimating of fuzzy multivariate PCI will be as:

$$M\hat{C}_p = E(\mu(X)) = E \left[e^{-1/2(X-t_0)' \Lambda_0^{-2}(X-t_0)} \right] = \frac{1}{N} \sum_{i=1}^N e^{-1/2(X-t_0)' \Lambda_0^{-2}(X-t_0)} \quad (24)$$

It means that average of membership values for every sample that calculates regarding its membership function (normal membership function in last equation); will be a good estimation for fuzzy index. When a random sample of distribution is available will be applicable.

In order to have better insight and a comparison of suggested index to other methods two example will solve in this section.

Example 1. Assume that data of presented example in Shahriari and et.all (1995) is considered in this section. The specifications were defined as an interval for each variety rang (235,295) and (440,500) and together form a rectangular tolerance region. The center of the specifications was $t_0' = [265,470]$ and was assumed to be target of the process. The sample mean vector was

$$\bar{X}' = [264.32, 471.48]$$

And the sample covariance matrix was:

$$S = \begin{bmatrix} 102.65 & 68.87 \\ 68.87 & 107.96 \end{bmatrix}.$$

For fuzzy multivariate process capability index we should solve below integral that use first estimation method:

$$M\tilde{C}_p = \int_{400}^{500} \int_{200}^{300} e^{-1/2 \left[\frac{(x_1-250)^2}{14.7^2} + \frac{(x_2-460)^2}{14.7^2} \right]} \frac{1}{2\pi\sqrt{102.5 \times 107.96 \times (1 - 0.654)^2}} \quad (25)$$

$$\exp \left\{ \frac{1}{-2(1 - 0.654^2)} \left[\left(\frac{x_1 - 264.32}{\sqrt{102.65}} \right)^2 + \left(\frac{x_2 - 471.48}{\sqrt{107.96}} \right)^2 - 2 \times 0.654 \left(\frac{x_1 - 264.32}{\sqrt{102.65}} \right) \left(\frac{x_2 - 471.48}{\sqrt{107.96}} \right) \right] \right\} dx_1 dx_2 = 0.46358 \quad (26)$$

Three multivariate process capability indexes were calculated in Shahriari and et.al (1995) and now table 5.1 shows the result for this method and output for fuzzy method.

Table 1. Output Of Multivariate Process Capability Indexes
(Target Near Mean)

Method	Method 1: Vector three component [CPM, PV, LI]	Method 2: process capability ratio MCPM	Method 3: NC based Method MCP	Fuzzy Multivariate Index
Suggested by Output	Shahriari [0.85,0.1,0]	Taam, Subbiah 0.96/1.03=0.93	Chen 0.91	- 0.7

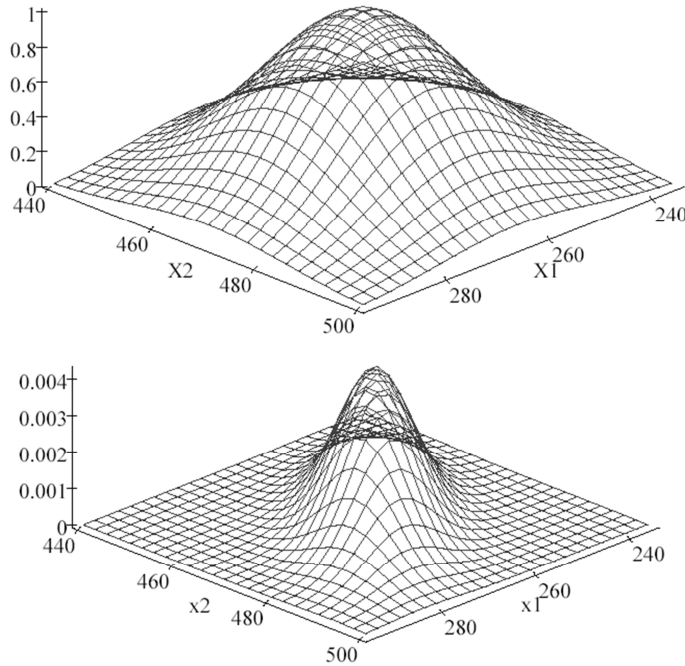


Figure 4. Membership function to example 1 with centered tolerance target vector Bivariate Normal Surface to example 1

The first components of method 1, as well as the index of Methods 2 and 3, has value less than 1, implying that variability of the process is more than allowed by engineering drawing (probability level is 99.73%). However the second component of Method 1 and denominator of method 2 indicate that the process is not on target value. About fuzzy index result shows that some reason lead to reduction in it and related to decision maker point view ,this process will be rather capable .

Example 2. With keep assumption in example 1 and just change in target vector to $T_0' = [250,460]$ in order to sensitivity analysis if methods to target vector, indexes was calculated that shown in table II.

Result shows that second component of method 1, method 2 and fuzzy index is sensitive to difference between mean vector and target vector. As it is clear that Method 3 is not sensitive to target and mean change.

In order to analyzing the main factor on fuzzy index we calculate this index with effect of three

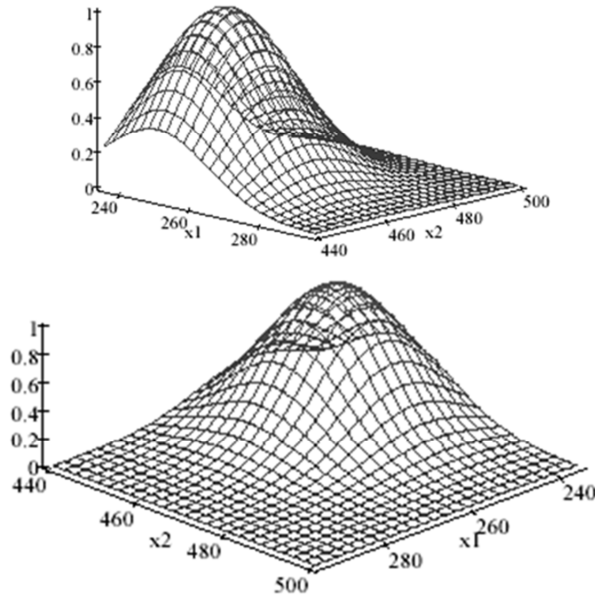


Figure 5. Membership function for Example 2 with uncentered tolerance target vector

Factor mean, target, variability separately, result shown in tables below:

Table 2. Output of multivariate Process capability indexes (Target far from mean)

Method	Method 1: Vector three component [CPM, PV, LI]	Method 2: process capability ratio MCPM	Method 3: NC based Method MCP	Fuzzy Multivariate Index
Suggested by Output	Shahriari [0.85,0,0]	Taam, Subbiah 0.32/1.74=0.18	Chen 0.91	- 0.56

Table 3. Fuzzy multivariate Process capability indexes (Target change analysis)

Raw	Target vector	Position	Fuzzy MCp
1	[265,470]	Default value	0.7
2	[230,420]	Far from mean(increasing)	0.18
3	[290,480]	Far from mean(decreasing)	0.49

Table 4. Fuzzy multivariate Process capability indexes (mean change analysis)

Raw	Mean vector	Position	Fuzzy MCp
1	[265,470]	Default value	0.7
2	[250,450]	Far from target(increasing)	0.32
3	[290,490]	Far from target(decreasing)	0.15

Table 5. Fuzzy multivariate Process capability indexes (variation change analysis)

Raw	Target vector	Position	Fuzzy MCp
1	S11=90	Reduction value S11 in S matrix	0.72
2	S11=110	Addition value S11 in S matrix	0.68
3	S22=95	Reduction value S22in S matrix	0.71
4	S22=120	Addition n value S22in S matrix	0.69
5	S12=30	Reduction value S12in S matrix	0.69
6	S12=90	Addition value S12in S matrix	0.72

Table 6. Fuzzy multivariate Process capability indexes (symmetric membership function spread change analysis)

Raw	Landa Vector	Position	Fuzzy MCp
1	[14.7,14.7]	Default value	0.7
2	[9.6,9.6]	reduction	0.5
3	[20,20]	Addition	0.8

As that is obvious that the multivariate fuzzy process capability index is sensitive to change of mean vector, target value and process variation. Addition in deference between mean vector and target value and any addition in variance value will cause to reduction to this index.

4. Conclusion

In this paper fuzzy multivariate PCIs using fuzzy concepts was presented. An attention to this method and two way of its estimation, some distinguish rather than other methods will be seen as below:

- a. This method is sensitive to target value, process variation and consideration loss functions, especially in multivariate case that other method haven't attention to loss function.
- b. This method is not restrict to normal distribution function and applicable for all of PDFs.
- c. Estimation of other methods in multivariate case is complicated but in fuzzy based method using second estimation procedure, just we should calculate an average.
- d. Just by suggested method we can have a PCI when the PDF of distribution is completely unknown via second method estimation because it needs just a random sample to calculation. This property cause to practical usage of it.
- e. Final output of this method is a number that will be so useful to compare two or more process defined to specify tolerance region and selection the best of them.

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