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Optimization of the Black-Scholes Equation with the Numerical Method of Local Expansion to Minimize Risk Coverage

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ABSTRACT

In this paper, we present an efficient and accurate method for calculating the Black-Scholes differential equations and solve the Black-Scholes equations using Jacoby and Airfoil orthogonal bases, with the collocation method. The Black-Scholes equation is a partial differential equation, which describes the price of choice in terms of time and the collocation method is a method of deter-mining coefficients. Then we show the computational results and examine the performance of the method for the two options, the price of basic assets and its issues. These results show that the Jacoby method is more efficient in solving the Black Scholes equation, and the method error is less and the convergence rate is higher. In this paper, by applying numerical methods to the desired equation, nonlinear devices can be solved by nonlinear solution methods, such as Newton's iterative method. The existence, uniqueness of the solution, and convergence of the methods are examined, and we will show in an example that by repeating then $\frac{|u_{n+1}-u_n|}{|u_n|} < \varepsilon$ can be reached and this indicates the accuracy of the response to these methods.

1 Introduction

Black-Scholes equation is a mathematical model of a financial market comprising derivative investment instruments. From this model, the Black-Scholes formula can be obtained, which gives a theoretical estimate of the price of European style options. This formula created great prosperity in trading, and scientifically legitimized the activities of the Chicago franchise and other power markets around the world. The general pricing theory of bargaining offered by Fisher Black and Miron Schulz in 1973 [7], is a result of these two talented scholars from previous efforts because Kasow, Louis Bachelier, and... . After presenting the theory by Black and Schulz, the main focus of financial advisers was to extend the use of this theory in the field of financial science [10].
In 1973, Chicago Transaction Disclosure at the Chicago Chamber of Commerce, USA, was formed in order to bargain. Then American, Persian and Philadelphia exchanges followed the Chicago Stock Exchange, which began trading in the stock exchanges in 1977. Until the early 1980s, options for dealing with four hundred shares (in the US market) and other financial instruments were traded. The main idea behind making this formula was to use in Financial math and work in bonds (cash) and stocks. By publishing a paper by Black and Scholes on the pricing of purchase and sale papers, a new revolution was taking place on such securities. Solving this equation, which is a kind of heat equation, is important.

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The Black-Scholes equation is a partial differential equation, which describes the price of choice in terms of time. The key insight behind this equation is that one can thoroughly control the risk of buying and selling a fully-fledged asset, and thus risking it. This, in turn, implies that there is only one correct price available for the Black Scholes formula. However, many of the hypotheses that are presented in this pricing model are not apparently real in practical settings.

The assumptions that are considered in the Black-Scholes equation are: V, derivative price as a function of time and stock price, asset price S follows a Brownian geometric motion (GBM), a fixed drift parameter r, a constant instability rate σ, a lack of arbitrage opportunities (Lack of risk-free profits), a friction and competitive market, it should be noted that we prove the existence of a unique solution and convergence of methods for the above equation. One of the most famous models in the financial markets is Black-Scholes model. In the field of financial modeling, the Black-Scholes model plays an important role in determining the price of high-risk assets, and is the basis for many model pricing frameworks for options. The utility of this popular model has proven to be a theoretical basis in financial markets. However, it has been shown that this model is unable to predict the important characteristics observed in asset earnings and the implied fluctuations of the market. For this reason, many arguments have been made for the development of successor models and activities in this field. In the Black-Scholes model, the basic asset price is matched by the geometric Brownian motion process, in which the asset price fluctuations and movements are assumed to be constant, so for this reason it cannot predict or explain dynamic or random behavior in price changes. Anyone who follows the crisis will realize that the real economy of business and goods has been strengthened by the well-known financial instruments known as derivatives. In the Black-Scholes equation, the recommended price of other values can be directly measured:

Time, price and asset, according to which the interest rate is secured without risk.

The following assumptions are considered for extracting the Black-Scholes pricing formula:

There is no trading expense or tax for traders, the interest rate without short-term risk is clear and constant over time, there is no limitation on lending and borrowing at unrestricted interest rates, there is always the possibility of selling a loan. In the course of time, transactions are always ongoing, the base asset price changes are continuous, and there is no price swing; the buy option only is available on maturity (Optional European Purchase). As most engineers and researchers know, there are many issues in nature that can ultimately be modeled in the form of differential equations, and since, generally, issues appear in a non-linear form and there is no exact answer to them, or whether those answers are hardly obtained, so we have to look for methods that can solve these models at least with an approximate solution. One of these issues is the Black-Scholes differential equation, due to its extensive use of finance and stock markets, it will be expanded with a number of different methods, which we will examine in this study. Most of the phenomena around us have a nonlinear model and formulate with nonlinear equations. With the advent of advanced computers, it is easy and easier to solve linear problems, but it's hard to discuss the exact solution of a non-linear problem.

Although we currently have advanced computers, as well as software such as Maple, Mathematica, and similar, but in most cases it is difficult to obtain an analytical response to a nonlinear problem, so we need to look for ways to solve this models at least with an approximate solution. Repetitive methods and extension methods are methods that can be used to solve many functional equations. Integral-differential equations, differential equations and partial differential equations are equations that are important in many sciences, such as physics, chemistry, mechanics, economics. Functional equations with variable time have many applications in chemical engineering, meteorology, civil engineering, medical engineering, aerospace engineering, astronomy, marine science, automobile industry, shipbuilding and mechanical engineering, and its various trends. For example, the general pricing theory of bargaining offered by Fischer Black and Miron Scholes in 1973, is a result of these two talented scholars from previous efforts because Kasow, Louis Bachelier, and ... After presenting the theory by Black and Schulz, the main focus of financial advisers was to extend the use of this theory in the field of financial science. The Black-Scholes analysis results can be generalized to buy options and options for European sales on non-profit companies.

One method, is the use of the Black-Scholes formula, that way the current value of the projected gain over the life of the bargain option is less than the stock price, and the volatility equivalent to the volatility of the stock price is the net present value of the gain. Fischer Black offers an approximate method of using American buy valuation on non-profit companies. Black's approximate suggest, Putting the price equals the highest price options for a European deal. The first option of a European transaction expires at the same time as the American Delegation, and the second immediately expires before the date of payment of the interest. In Section 2, we introduce the local methodology. In this section, the Jacobi Method and the Airfoil Method are described. Then, in Section 2-3, the equation is solved in a local method. In Section 3, we will investigate the singularity and examine it with a numerical example of the error in the table. In Figures, two methods of Jacobi and Airfoil are compared and finally the result is examined.

2 Introducing the Collocation Method

A collocation l method in mathematics is a method of displacement for numerical solutions of ordinary differential equations, partial differential equations and integral equations. The idea of this scheme is based on the selection of a limited range of candidate solutions (usually polynomials to a certain degree) and a number of points in the domain (collocation points) to solve the given equation in coordination points. In fact, the collocation method is a method of determining coefficients. With the aid of the different orthogonal polynomials, the different solving methods are proposed, in which the Jacobi and Airfoil polynomials are used as polynomials in this method.

2.1 Jacobi Polynomial

We define the Jacobi polynomial as follows: [2]

$$u_n(x,t) = w(x)w(t)\sum_{i=0}^{n} a_i p_i^{\alpha,\beta}(x)p_i^{\alpha,\beta}(t), \quad \alpha,\beta > -1 \tag{1}$$

$$w(x) = \frac{(1-x)^{\alpha}}{(1+x)^{\beta}}, \quad w(t) = \frac{(1-t)^{\alpha}}{(1+t)^{\beta}}$$

$$p_i^{\alpha,\beta}(x) = \frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{(-2)^i i!}.\frac{d^i}{dx^i}\left[(1-x)^{i+\alpha}(1+x)^{i+\beta}\right] \tag{2}$$

$$\left(p_i^{\alpha,\beta}\right)'(x) = \frac{1}{2}(i+\alpha+\beta+1)p_{i-1}^{(\alpha+1,\beta+1)}(x).$$

So in general,

$$\left(p_i^{\alpha,\beta}\right)^{(m)}(x) = \frac{1}{m!}(i+\alpha+\beta+m)p_{i-m}^{(\alpha+m,\beta+m)}(x), i \geq m.$$

$$p_i^{\alpha,\beta}(t) = \frac{(1-t)^{-\alpha}(1+t)^{-\beta}}{(-2)^i i!} \cdot \frac{d^i}{dt^i}\left[(1-t)^{i+\alpha}(1+t)^{i+\beta}\right]$$
$$\left(p_i^{\alpha,\beta}\right)^{(m)}(t) = \frac{1}{m!}(i+\alpha+\beta+m)p_{i-m}^{(\alpha+m,\beta+m)}(t), i \geq m. \tag{3}$$

2.2 Airfoil Polynomial

$$u_n(x,t) = w(x)w(t)\sum_{i=0}^{n} a_i f_i(x) f_i(t),$$
$$w(x) = \sqrt{\frac{(1+x)}{(1-x)}} \ , \quad w(t) = \sqrt{\frac{(1+t)}{(1-t)}} \ ,$$
$$f_i(x) = \frac{\cos\left[\left(i+\frac{1}{2}\right) arc\cos x\right]}{\cos\left[\frac{1}{2} arc\cos x\right]},$$
$$f_i(t) = \frac{\cos\left[\left(i+\frac{1}{2}\right) arc\cos x\right]}{\cos\left[\frac{1}{2} arc\cos x\right]}, \tag{4}$$
$$u_i(x) = \frac{\sin\left[\left(i+\frac{1}{2}\right) arc\sin x\right]}{\cos\left[\frac{1}{2} arc\sin x\right]},$$
$$u_i(t) = \frac{\sin\left[\left(i+\frac{1}{2}\right) arc\sin t\right]}{\cos\left[\frac{1}{2} arc\sin t\right]},$$

$$(1+\mathrm{x})\mathrm{f_i}'(\mathrm{x}) = \left(\mathrm{i}+\tfrac{1}{2}\right)\mathrm{u_i}(\mathrm{x}) - \tfrac{1}{2}\mathrm{f_i}(\mathrm{x}),$$
$$(1+t)\mathrm{f_i}'(\mathrm{t}) = \left(\mathrm{i}+\tfrac{1}{2}\right)\mathrm{u_i}(\mathrm{t}) - \tfrac{1}{2}\mathrm{f_i}(\mathrm{t}) . \tag{5}$$

2.3 Introduction of the Black-Scholes Equation and Its Evolving with Jacobi Polynomial

$$V_t = -\frac{1}{2}\sigma^2 s^2 V_{ss} - rsV_s + rV. \tag{6}$$

where ***V*** is the price of the option as a function of stock price ***S*** and time ***t***, ***r*** is the risk-free interest rate, and ***σ*** is the volatility of the stock. To solve the equation, we have

$$V(s,t) = \int_a^t -\frac{1}{2}\sigma^2 s^2 V_{ss}\, dt - \int_a^t rsV_s\, dt + \int_a^t rV\, dt. \tag{7}$$

According to equation (1) and derivative of s:

$$V_s = w(t)\left[w'(s)\sum_{i=0}^{n} a_i p_i^{\alpha,\beta}(s) p_i^{\alpha,\beta}(t) \ + w(s)\sum_{i=0}^{n} a_i \left(p_i^{\alpha,\beta}(s)\right)' p_i^{\alpha,\beta}(t)\right], \tag{8}$$

$$V_{ss}=w(t)\left[w^{"}(s)\sum_{i=0}^{n}a_ip_i^{\alpha,\beta}(s)p_i^{\alpha,\beta}(t)\ +2w'(s)\ \sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s)\right)'p_i^{\alpha,\beta}(t)\right.$$
$$\left.+w(s)\sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s)\right)^{"}p_i^{\alpha,\beta}(t)\right],$$

$$V_s=w(t)\left[w'(s)\textstyle\sum_{i=0}^{n}a_ip_i^{\alpha,\beta}(s)p_i^{\alpha,\beta}(t)\ +w(s)\textstyle\sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s)\right)'p_i^{\alpha,\beta}(t)\right], \tag{9}$$
$$V_{ss}=w(t)\left[w^{"}(s)\sum_{i=0}^{n}a_ip_i^{\alpha,\beta}(s)p_i^{\alpha,\beta}(t)\ +2w'(s)\ \sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s)\right)'p_i^{\alpha,\beta}(t)\right.$$
$$\left.+w(s)\sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s)\right)^{"}p_i^{\alpha,\beta}(t)\right],$$

So, by inserting in (7) we will have:

$$w(s)w(t)\sum_{i=0}^{n}a_ip_i^{\alpha,\beta}(s_j)p_i^{\alpha,\beta}(t_j) \tag{10}$$
$$=\int_a^t-\frac{1}{2}\sigma^2s^2w(t)\left[w^{"}(s)\sum_{i=0}^{n}a_ip_i^{\alpha,\beta}(s_j)p_i^{\alpha,\beta}(t_j)\right.$$
$$+2w'(s)\ \sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s_j)\right)'p_i^{\alpha,\beta}(t_j)$$
$$\left.+w(s)\sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s_j)\right)^{"}p_i^{\alpha,\beta}(t_j)\right]dt$$
$$-\int_a^t rsw(t)\left[w'(s)\sum_{i=0}^{n}a_ip_i^{\alpha,\beta}(s_j)p_i^{\alpha,\beta}(t_j)\right.$$
$$\left.+w(s)\sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s_j)\right)'p_i^{\alpha,\beta}(t_j)\right]dt$$
$$+\int_a^t r\,w(s)w(t)\sum_{i=0}^{n}a_ip_i^{\alpha,\beta}(s_j)p_i^{\alpha,\beta}(t_j)\,dt$$
$$-\frac{1}{2}\sigma^2s^2(w^{"}(s)\sum_{i=0}^{n}a_ip_i^{\alpha,\beta}(s_j)+w'(s)\sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s_j)\right)'+w(s)\sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s_j)\right)^{"})$$
$$\times\left(\int_a^t w(t)\sum_{i=0}^{n}p_i(t_j)\,dt\right)-rs(w'(s)\sum_{i=0}^{n}a_ip_i^{\alpha,\beta}(s_j)+w(s)\sum_{i=0}^{n}a_i\left(p_i^{\alpha,\beta}(s_j)\right)')$$

$$\times\left(\int_a^t w(t)\sum_{i=0}^{n} p_i(t_j)\,dt\right) - (r - 1)w(s)\sum_{i=0}^{n} a_i p_i^{\alpha,\beta}(s_j) \times \left(\int_a^t w(t)\sum_{i=0}^{n} p_i(t_j)\,dt\right) = 0$$

Given the following assumptions, we arrive at a matrix:

$$w(s) = E, w'(s) = H, w''(s) = G \tag{11}$$

Which are obtained from formula (1).

$$\begin{aligned}
&\int_a^t w(t)\sum_{i=0}^{n} p_i(t_j)\,dt = D_{ij},\\
&\sum_{i=0}^{n} a_i p_i^{\alpha,\beta}(s_j) = a_i A_{ij},\\
&\sum_{i=0}^{n} a_i \left(p_i^{\alpha,\beta}(s_j)\right)' = a_i B_{ij},\\
&\sum_{i=0}^{n} a_i \left(p_i^{\alpha,\beta}(s_j)\right)'' = a_i C_{ij}
\end{aligned} \tag{12}$$

we have:

$$-\frac{1}{2}\sigma^2 s^2 (Ga_iA_{ij} + Ha_iB_{ij} + Ea_iC_{ij}) \times D_{ij} - rs(Ha_iA_{ij} + Ea_iB_{ij}) \times D_{ij} \times D_{ij} - rs(Ha_iA_{ij} + Ea_iB_{ij}) \times D_{ij} - (r-1)(Ea_iA_{ij} \times D_{ij}) = 0 \tag{13}$$

With factoring and summarizing:

$$\left(-\frac{1}{2}\sigma^2 s^2 G - rsH + (r-1)E\right) a_i A_{ij} D_{ij} + \left(-\frac{1}{2}\sigma^2 s^2 H - rsE\right) a_i B_{ij} D_{ij} + \left(-\frac{1}{2}\sigma^2 s^2 G\right) a_i C_{ij} D_{ij} = 0, \tag{14}$$

So

$$Ma_iA_{ij}D_{ij} + Na_iB_{ij}D_{ij} + Ka_iC_{ij}D_{ij} = 0. \tag{15}$$

where in

$$M = -\frac{1}{2}\sigma^2 s^2 G - rsH + (r-1)E, \qquad N = -\frac{1}{2}\sigma^2 s^2 H - rsE, \qquad K = -\frac{1}{2}\sigma^2 s^2 G. \tag{16}$$

2.4 Solving the Black-Scholes Equation with Airfoil Polynomial

$$V(s,t) = w(s)w(t)\sum_{i=0}^{n} a_i f_i(s) f_i(t), \tag{17}$$

$$\begin{aligned}
w(s)w(t)\sum_{i=0}^{n} a_i f_i(s_j) f_i(t_j) \\
&= \int_a^t -\frac{1}{2}\sigma^2 s^2 w(t)\Bigg[w''(s)\sum_{i=0}^{n} a_i f_i(s_j) f_i(t_j) \\
&+ 2w'(s)\sum_{i=0}^{n} a_i \left(f_i(s_j)\right)' f_i(t_j) + w(s)\sum_{i=0}^{n} a_i \left(f_i(s_j)\right)'' f_i(t_j)\Bigg] dt \\
&- \int_a^t rsw(t)\Bigg[w'(s)\sum_{i=0}^{n} a_i f_i(s_j) f_i(t_j) \\
&+ w(s)\sum_{i=0}^{n} a_i \left(f_i(s_j)\right)' f_i(t_j)\Bigg] dt \\
&+ \int_a^t r\, w(s)w(t)\sum_{i=0}^{n} a_i f_i(s_j) f_i(t_j)\, dt
\end{aligned} \tag{18}$$

$$\begin{aligned}
&-\frac{1}{2}\sigma^2 s^2 (w''(s)\sum_{i=0}^{n} a_i f_i(s_j) + w'(s)\sum_{i=0}^{n} a_i \left(f_i(s_j)\right)' + w(s)\sum_{i=0}^{n} a_i \left(f_i(s_j)\right)'') \\
&\times\left(\int_a^t w(t)\sum_{i=0}^{n} f_i(t_j)\, dt\right) - rs(w'(s)\sum_{i=0}^{n} a_i f_i(s_j) + w(s)\sum_{i=0}^{n} a_i \left(f_i(s_j)\right)') \\
&\times\left(\int_a^t w(t)\sum_{i=0}^{n} f_i(t_j)\, dt\right) - (r-1)w(s)\sum_{i=0}^{n} a_i f_i(s_j) \times \left(\int_a^t w(t)\sum_{i=0}^{n} f_i(t_j)\, dt\right) = 0
\end{aligned}$$

Given the following assumptions, we have a matrix:

$$w(s) = E, \qquad w'(s) = H, \qquad w''(s) = G \tag{19}$$

Which are obtained from formula (1).

$$\begin{aligned}
&\int_a^t w(t)\sum_{i=0}^{n} f_i(t_j)\, dt = D_{ij}, \\
&\sum_{i=0}^{n} a_i f_i(s_j) = a_i A_{ij}, \\
&\sum_{i=0}^{n} a_i \left(f_i(s_j)\right)' = a_i B_{ij}, \\
&\sum_{i=0}^{n} a_i \left(f_i(s_j)\right)'' = a_i C_{ij}.
\end{aligned} \tag{20}$$

we have:

$$-\frac{1}{2}\sigma^2 s^2\big(Ga_iA_{ij} + Ha_iB_{ij} + Ea_iC_{ij}\big) \times D_{ij} - rs\big(Ha_iA_{ij} + Ea_iB_{ij}\big) \times D_{ij} \\ -(r-1)(Ea_iA_{ij} \times D_{ij}) = 0. \tag{21}$$

With factoring and summarizing:

$$\left(-\frac{1}{2}\sigma^2 s^2 G - rsH + (r-1)E\right) a_iA_{ij}D_{ij} + \left(-\frac{1}{2}\sigma^2 s^2 H - rsE\right) a_iB_{ij}D_{ij} \\ + \left(-\frac{1}{2}\sigma^2 s^2 G\right) a_iC_{ij}D_{ij} = 0, \tag{22}$$

So

$$Ma_iA_{ij}D_{ij} + Na_iB_{ij}D_{ij} + Ka_iC_{ij}D_{ij} = 0. \tag{23}$$

where in

$$M = -\frac{1}{2}\sigma^2 s^2 G - rsH + (r-1)E, \qquad N = -\frac{1}{2}\sigma^2 s^2 H - rsE, \qquad K = -\frac{1}{2}\sigma^2 s^2 G. \tag{24}$$

3 Uniqueness of Solution

The problem is unique when $0 < \alpha_1 < 1$ and the value of $\alpha_1 = \lambda MLT$. Suppose $f\big(y(t)\big) = [y(t)]^p$, $0 \le t \le T$, $\forall x \in J = [0, T]$ and $y(t)$ a nonlinear function that satisfy in the Lipschitz condition and have:

$$\left|\frac{k(x,t)}{(x-t)^\beta}\right| \le M, \\ |f(y) - f(z)| \le L|y - z| \tag{25}$$

Proof: We assume that the problem is not unique solution and y, y* be the solution of the problem, then:

$$|y - y^*| = \Bigg| I^\alpha g(x) + \lambda I^\alpha \int_a^x k(x,t)\frac{1}{(x-t)^\beta} f(y)dt - I^\alpha g(x) + \lambda I^\alpha \int_a^x k(x,t)\frac{1}{(x-t)^\beta} f(y^*)dt \Bigg| \\ \le |\lambda| \left| I^\alpha \int_a^x \frac{k(x,t)}{(x-t)^\beta}\big(f(y) - f(y^*)\big)dt \right| \\ \le |\lambda| ML|y - y^*|T \le \lambda MLT|y - y^*| = \alpha_1|y - y^*|. \tag{26}$$

So $(1-\alpha)|y - y^*| = 0$ and consequently $y = y^*$.

4 Numerical Examples

Example 1:

$$u_t(x,t) + x^2 u_{xx}(x,t) + 0.5x u_x(x,t) - u(x,t) = 0. \quad (27)$$

With initial condition:

$$u(x,0) = x^2 \;, \quad \varepsilon = 10^{-4}.$$
$$\alpha = 0.5\,, \beta = 0.2.$$
$$\alpha_1 = 0.78206.$$

Table 1: Approximate Solution for Exam1

(x,t)	Error		Approximate Solution	
	Jacobi error n=8	Airfoil error n=9	Jacobi	Airfoil
(0.1,0.13)	0.000816514	0.000788329	0.1249657	0.1180117
(0.2,0.18)	0.000823319	0.000777431	0.2312379	0.2166221
(0.3,0.27)	0.000824924	0.000762147	0.3256407	0.3068415
(0.4,0.32)	0.000836708	0.000743761	0.4272314	0.4123908
(0.5,0.38)	0.000845247	0.000734473	0.5358607	0.5141528
(0.7,0.43)	0.000851352	0.000729884	0.6479815	0.6227609

The accuracy of the solution and the stopping condition $\frac{|u_{n+1}(x,t) - u_n(x,t)|}{|u_n(x,t)|} < \varepsilon$.

Example 2:

$$u_t(x,t) + 0.08(2 + sinx)^2 + x^2 u_{xx}(x,t) + 0.06x u_x(x,t) - 0.06u(x,t) = 0. \quad (28)$$

With boundary condition:

$$u(0,t) = 0,$$
$$u(x,T) = \max((x - 25)^{-0.06}\, e, 0),$$
$$, \quad \varepsilon = 10^{-5}.$$

Table 2: Approximate Solution for Exam2

(x,t)	Error		Approximate Solution	
	Jacobi error n=8	Airfoil error n=9	Jacobi	Airfoil
(0.1,0.12)	0.00004271	0.00003987	0.1455417	0.1217315
(0.2,0.22)	0.00004625	0.00004117	0.2526704	0.2439417
(0.3,0.32)	0.00005116	0.00004822	0.3435129	0.3276638
(0.4,0.38)	0.00005479	0.00005128	0.4581609	0.4352127
(0.5,0.42)	0.00006013	0.00005516	0.5637428	0.5481379
(0.6,0.47)	0.00006339	0.00006146	0.6726085	0.6553478
(0.7,0.52)	0.00006625	0.00006144	0.7523482	0.7643251

$$\alpha = 0.6\,, \beta = 0.4.$$
$$\alpha_1 = 0.56702.$$

The accuracy of the solution and the stopping condition $\frac{|u_{n+1}(x,t) - u_n(x,t)|}{|u_n(x,t)|} < \varepsilon$.

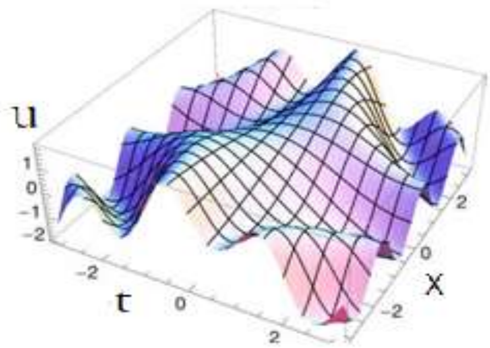

Fig 1: Jacobi polynomial

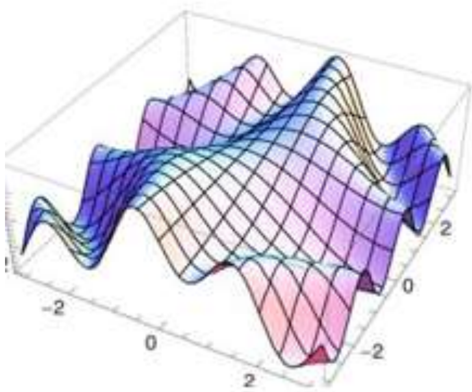

Fig 2: Airfoil Polynomial

5 Conclusion

In this paper, we implement the collocation methods with Jacobi and Airfoil bases on the Black-Scholes nonlinear differential equation and obtained the approximate solution. So far, Not much work has been done on this category of equations, which is one of the most used and complex differential equations in financial mathematics. In this article, we examined the collocation extension method. One can also use the collocation method with other bases in future work on integral-differential equations as follows.

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