Multiple solutions of the nonlinear reaction-diffusion model with fractional reaction

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Abstract

The purpose of this letter is to revisit the nonlinear reaction-diffusion model in porous catalysts when reaction term is fractional function of the concentration distribution of the reactant. This model, which originates also in fluid and solute transport in soft tissues and microvessels, has been recently given analytical solution in terms of Taylors series for different family of reaction terms. We apply the method so-called predictor homotopy analysis method (PHAM) which has been recently proposed to predict multiplicity of solutions of nonlinear BVPs. Consequently, it is indicated that the problem for some values of the parameter admits multiple solutions. Also, error analysis of these solutions are given graphically.

Key words: Predictor homotopy analysis method; Prescribed parameter; Reaction-diffusion model, multiple solutions.

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1 Introduction

The governing boundary value problem of the generalized one dimensional steady state reaction-diffusion model can be written in dimensional variables as

$$D\mathbf{U}'' - V\mathbf{U}' - r(\mathbf{U}) = 0, \quad 0 \le x \le L, \quad \mathbf{U}'(0) = 0, \quad \mathbf{U}(L) = \mathbf{U}_s, \quad (1.1)$$

where D is the diffusivity, V is the advective velocity and $r(\mathbf{U})$ denotes reaction process [1]. Now, by introducing nondimensional quantities $U(x) = \frac{\mathbf{U}(X)}{\mathbf{U}_s}$, $x = \frac{X}{L}$ and $R(U)$ as nondimensional reaction term and then substituting these nondimensional quantities into equation (1.1), we get

$$U'' - PU' - R(U) = 0, \quad 0 \le x \le 1, \quad U'(0) = 0, \quad U(1) = 1, \quad (1.2)$$

where $P = \frac{VL}{D}$ is so-called Péclet number. Without advective transport, we have $P = 0$ and in this case the model has been used to study porous catalyst pellets as the model of diffusion and reaction [2,3]. Furthermore, if we consider $R(U)$ as fractional function or so-called Michaelis-Menten reaction term then the model is converted to

$$U''(x) - \frac{\alpha U(x)}{\beta + U(x)} = 0, \quad 0 \le x \le 1, \quad (1.3)$$

with the boundary condition

$$U'(0) = 0, \quad U(1) = 1, \quad (1.4)$$

where α, characteristic reaction rate, and β is half saturation concentration. The problem (1.2) without advective transport ($P = 0$) and with reaction term $R(U) = \phi^2 U^n$ (ϕ is Thiele modulus) has been studied by Adomian decomposition method [4] and Homotopy analysis method [2,3]. Subsequently, S. Abbasbandy and E. Shivanian [5] have considered almost the same problem arising in heat transfer and have successfully obtained the exact analytical solution in the implicit form and proved the existence of dual solutions on some domain of x. The model (1.2) involves advective and diffusive transport with the Michaelis-Menten reaction model that is routinely used to represent biochemical processes [6–8].

This model encodes a number of important engineering processes including several applications in chemical engineering [9,10] and environmental engineering [7,8]. The boundary value problem (1.3)-(1.4) contains non-linear fractional term which makes it somewhat difficult to treat even by numerical methods. A.J. Ellery and M.J. Simpson [1] presented Taylor series solution for this model which truly is convergent on the condition that the Michaelis-Menten reaction term has bounded derivatives as they mentioned.

The aim of this paper is to go advance with this model by applying predictor homotopy analysis method (PHAM) [3,11,12] which is more general than HAM in some sense and can be applied to predict and calculate multiple solutions of BVPs simultaneously. The homotopy analysis method [13,14] has been successfully applied to several nonlinear problems such as the viscous flows of non- Newtonian fluids [15–21], the KdV-type equations [22], nano boundary layer flows [23], nonlinear heat transfer [24], finance problems [25], Riemann problems related to nonlinear shallow water equations [26], projectile motion [27], Glauert-jet flow [28], nonlinear water waves [29], ground water flows [30], Burgers-Huxley equation [31], time-dependent Emden Fowler type equations [32], differential difference equation [33], Laplace equation with Dirichlet and Neumann boundary conditions [34], thermalhydraulic networks [35] and also readers are referred to see [36–47]. It is not unknown to anyone familiar with the analytical methods that HAM series is general Taylor series [13,48] which uses the convergence-controller parameter to make convergence fast, so we use PHAM to get series solution more accurate than usual Taylor series solution. We consider nonlinear fractional term in equations (1.3)-(1.4) in some cases which have unbounded derivatives then it is revealed by PHAM that the problem admits multiple (dual) solutions in these cases, while the exact solution of this problem is unknown.

2   Application of the PHAM to the model

The predictor homotopy analysis method (PHAM) has been fully discussed by S. Abbasbandy and E. Shivanain in [11]. Let us rewrite the

Eqs. (1.3)-(1.4) as follows:

$$(\beta + U)\frac{\mathrm{d}^2 U}{\mathrm{d}x^2} - \alpha U = 0, \quad 0 \le x \le 1, \qquad (2.1)$$

or equivalently

$$\beta U'' + U U'' - \alpha U = 0, \quad 0 \le x \le 1. \qquad (2.2)$$

The boundary conditions by prescribed parameter γ, as it is straightforward in PHAM, become

$$U(0) = \gamma, \quad U'(0) = 0, \qquad (2.3)$$

with the additional forcing condition

$$U(1) = 1 \qquad (2.4)$$

which plays essential role in determining multiplicity of solutions as it is described in PHAM. Now, we apply predictor homotopy analysis method on Eqs. (2.2)-(2.3) where prescribed parameter γ, which is played important role to realize about multiplicity of solutions, will be obtained with the help of *rule of multiplicity of solutions.*
It is straightforward to use the set of base functions

$$\{x^n, \ n = 0, 1, 2, ...\}. \qquad (2.5)$$

Under the rule of solution expression and according to the initial conditions (2.3), it is easy to choose

$$u_0(x, \gamma) = \gamma + x^2, \qquad (2.6)$$

as initial guess of solution $u(x)$, $H(x) = 1$ as auxiliary function, and to choose auxiliary linear operator

$$\mathcal{L}\left[\phi(x, \gamma; p)\right] = \frac{\partial^2 \phi(x, \gamma; p)}{\partial x^2}, \qquad (2.7)$$

with the property

$$\mathcal{L}\left[c_1 + c_2 x\right] = 0. \qquad (2.8)$$

Therefore, after two subsequent integrations, the M-th order deformation Equation of PHAM yields for $M \geq 1$

$$u_m(x, \gamma) = \chi_m u_{m-1}(x, \gamma) + \hbar \int_0^x \int_0^\eta R_m(\vec{u}_{m-1}, \tau, \gamma) \mathrm{d}\tau \mathrm{d}\eta + c_1 + c_2 x, \quad (2.9)$$

where from (2.2)

$$R_m(\vec{u}_{m-1}, \tau, \gamma) = \beta u''_{m-1}(\tau, \gamma) + \sum_{j=0}^{m-1} u_j(\tau, \gamma) u''_{m-1-j}(\tau, \gamma) - \alpha u_{m-1}(\tau),$$
$$(2.10)$$

and integration constants c_1 and c_2 are obtained by the conditions

$$u_m(0, \gamma) = u'_m(0, \gamma) = 0. \quad (2.11)$$

In this way we obtain the functions $u_m(x, \gamma)$ for $m = 1, 2, 3, \ldots$ from Eq. (2.9) successively. Finally, we can obtain M-th order approximate solution

$$U_M(x, \gamma, \hbar) = \sum_{m=0}^M u_m(x, \gamma), \quad (2.12)$$

So, additional forcing condition (2.4), becomes

$$U_M(1, \gamma, \hbar) \approx 1. \quad (2.13)$$

3   Multiple solutions of the model

A.J. Ellery and M.J. Simpson [1] showed that Taylor series solution of the problem (1.3)-(1.4) is convergent when $\frac{\alpha u}{\beta + u}$ has bounded derivatives by applying the ratio test to this series. So, if we consider negative value for β then it is possible the Taylor series solution be divergent. In this section, not only we get convergent PHAM series solution but also we discover that the existence of multiple solutions are possible. To be specific, assume the case consist of ($\alpha = 0.4$ and $\beta = -0.3, -0.25, -0.2$) then according to the equation (2.13) γ as a function of convergence-controller parameter $\hbar$ has been plotted in the $\hbar$-range $[0, 0.5]$ implicitly in Figure 1. Two γ-plateaus can be identified in this Figure for each value of β, namely

$\gamma = 0.3812$ and $\gamma = 0.64983$ for $\beta = -0.3$, $\gamma = 0.2848$ and $\gamma = 0.6986$ for $\beta = -0.25$ and, $\gamma = 0.2069$ and $\gamma = 0.7298$ for $\beta = -0.2$. Consequently, we conclude that the PHAM furnishes dual solutions in each case. We remark here that both the first branch and second branch of solutions are calculated at the same time only by Eq. (2.12) with different γ and $\hbar$ which are specified from Figure 1. Furthermore, we emphasize that there is no need to use more than one initial approximation guess, one auxiliary linear operator, and one auxiliary function that is in a sharp contrast to all approximation methods which are used to converge to one solution.
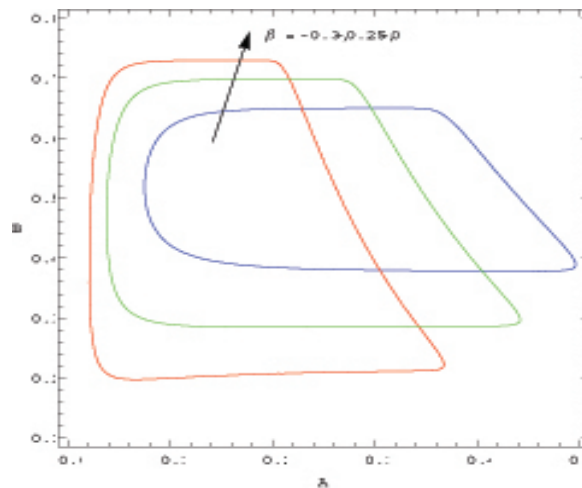

Fig. 1: Prescribed parameter γ via convergence-controller parameter $\hbar$ in according to (2.13) with $M = 30$.

In the plot shown in Figure 2, correspond to $\gamma = 0.3812$ and $\gamma = 0.64983$ the approximate dual PHAM solutions $U_{30}(x, 0.3812, 0.25)$ and $U_{30}(x, 0.64983, 0.25)$, correspond to $\gamma = 0.2848$ and $\gamma = 0.6986$ the approximate dual PHAM solutions $U_{30}(x, 0.2848, 0.2)$ and $U_{30}(x, 0.6986, 0.2)$ and correspond to $\gamma = 0.2069$ and $\gamma = 0.7298$ the approximate dual PHAM solutions $U_{30}(x, 0.2069, 0.2)$ and $U_{30}(x, 0.7298, 0.15)$ given by Eq. (2.12) have been plotted. To show the accuracy of these dual approximate solutions, we have shown the absolute residual error for these solution in Figure 3 and 4 in two different views.
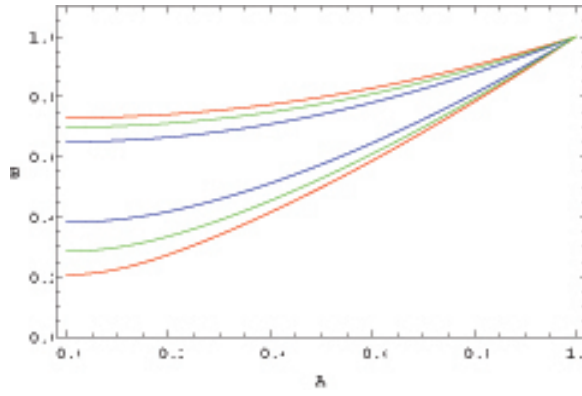
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Fig. 2: Dual approximate PHAM solutions with $M = 30$: Red color correspond to $\beta = -0.3$, green color correspond to $\beta = -0.25$ and blue color correspond to $\beta = -0.2$.
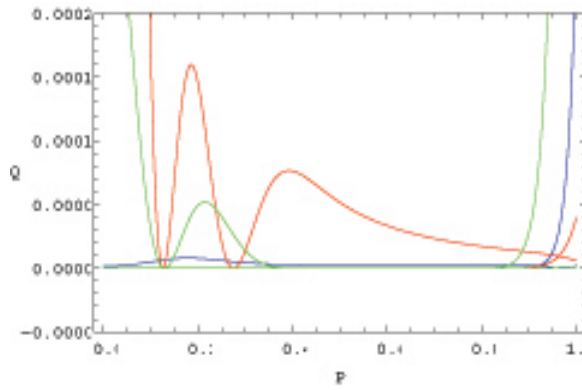

Fig. 3: The absolute residual error with $M = 30$; Red color correspond to $\beta = -0.3$, green color correspond to $\beta = -0.25$ and blue color correspond to $\beta = -0.2$.
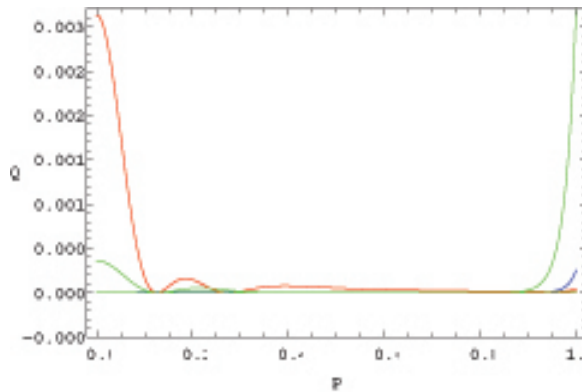

Fig. 4: The absolute residual error with $M = 30$; Red color correspond to $\beta = -0.3$, green color correspond to $\beta = -0.25$ and blue color correspond to $\beta = -0.2$.

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4 Conclusions

It is very important not to lose any solution of nonlinear differential equations with boundary conditions in engineering and physical sciences. In this regard, the present paper has revisited the nonlinear reaction diffusion equation with fractional reaction term via predictor homotopy analysis method (PHAM). It has been shown that not only we can get convergent series solution but also we can predict existence of multiple solutions.

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