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Convergence Theorems for α-Nonexpansive Mappings in $CAT(0)$ Spaces

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Abstract

In this paper we derive convergence theorems for an α-nonexpansive mapping of a nonempty closed and convex subset of a complete $CAT(0)$ space for SP-iterative process and Thianwan's iterative process.

Key words: $CAT(0)$ spaces, α-Nonexpansive mapping, Δ-convergence, SP-iteration, Thianwan's iteration.
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1 Introduction

The purpose of this paper is to study fixed point theorems for α-non-expansive mappings in $CAT(0)$ spaces. A metric space X is a $CAT(0)$

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space (see Bridson and Haefliger [2]) if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. Our approach is to prove Δ-convergence theorems for SP-iteration and Thianwan's iterations for α-nonexpansive mappings in $CAT(0)$ spaces.

Here are the details. Let X be a $CAT(0)$ space and let C be a nonempty subset of X and $T : C \to X$ be a mapping. Denote $F(T)$ by the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$.

Definition 1.1 *T is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$ and that T is quasi-nonexpansive if $F(T) \neq \phi$ and $d(Tx, y) \leq d(x, y)$ for all $x \in C$ and $y \in F(T)$.*

In 2011, Aoyama and Kohsaka [1] defined α-non-expansive mappings in Banach spaces. We introduce the notion of this mapping in $CAT(0)$ spaces.

Definition 1.2 *A mapping $T : C \to X$ is said to be an α-non-expansive for some real number $\alpha < 1$ if $d(Tx, Ty)^2 \leq \alpha d(Tx, y)^2 + \alpha d(Ty, x)^2 +$ $+(1 - 2\alpha)d(x, y)^2$ for all $x, y \in C$.*

Clearly, 0-non-expansive maps are exactly non-expansive maps.

Definition 1.3 ([6]) *Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.*

Remark 1.4 *$A(\{x_n\})$ consists of exactly one point in $CAT(0)$ spaces (see, e.g., [5, Proposition 7].*

Definition 1.5 ([6]) *A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim x_n = x$ and call x the Δ-limit of $\{x_n\}$.*

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset R$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_2)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = (x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic space is said to be a $CAT(0)$ space if all geodesic triangles satisfy the following comparison axiom:

Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \leq d_{\mathbb{E}^2}(\bar{x}, \bar{y})$.

If x, y_1, y_2 are points in a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \qquad \text{(CN)}$$

This is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfy (CN) inequality.

Now we collect some results which are used in our main results:

Lemma 1.6 ([6]) *Let (X, d) be a $CAT(0)$ space. Then for $x, y \in X$ and for each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y). \qquad (1.1)$$

Lemma 1.7 ([6]) *Let X be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z) \tag{1.2}$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Lemma 1.8 ([6]) *Let (X,d) be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)d(x,z)^2 + td(y,z)^2 - t(1-t)d(x,y)^2 \tag{1.3}$$

for all $t \in [0,1]$ and $x, y, z \in X$.

Lemma 1.9 ([7]) *Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence.*

Lemma 1.10 ([4]) *If C is a closed convex subset of a complete $CAT(0)$ space and if $\{x_n\}$ is a bounded sequence in C then the asymptotic center of $\{x_n\}$ is in C.*

In 2009, Thianwan [12] introduced an iteration named by his name, i.e., Thianwan's iteration which is defined by $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= \alpha_n T y_n + (1-\alpha_n) y_n, \\ y_n &= \beta_n T x_n + (1-\beta_n) x_n, \quad \text{for all } n \geq 1, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta\}$ are sequences in $[0,1]$.

In 2011, Phuengrattana and Suantai [10] defined the SP-iteration as follows:

$$\begin{aligned} x_{n+1} &= \alpha_n T y_n + (1-\alpha_n) y_n, \\ y_n &= \beta_n T z_n + (1-\beta_n) z_n, \\ z_n &= \gamma_n T x_n + (1-\gamma_n) x_n, \end{aligned} \tag{1.5}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$. They showed that the Mann, Ishikawa and SP-iterations are equivalent and the SP-iteration converges better than the others for the class of continuous and non-decreasing functions. Clearly Thianwan's iteration and Mann iterations are special cases of the SP-iteration.

Example 1.11 *Here we provide an example which proves that the SP-iteration converges much faster than the other iterations, for the increasing function* $f(x) = 2x^3 - 7x^2 + 8x - 2$*, the Ishikawa iteration converges in 22nd iterations while the SP-iteration converges in 2nd iteration with* $x_0 = 0.8$.

Recall that SP-iteration and Thianwan's iteration in $CAT(0)$ *spaces are described as follows:*

For any initial point x_1 *in* C*, we define the iterates* $\{x_n\}$ *by*

$$\begin{cases} x_{n+1} = \alpha_n T y_n \oplus (1-\alpha_n) y_n \\ y_n \quad\;\; = \beta_n T x_n \oplus (1-\beta_n) x_n \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences of positive numbers in* $[0,1]$*. This iteration is Thianwan's new two step iteration. And if the iterates* $\{x_n\}$ *is defined as*

$$\begin{cases} x_{n+1} = \alpha_n T y_n \oplus (1-\alpha_n) y_n, \\ y_n \quad\;\; = \beta_n T z_n \oplus (1-\beta_n) z_n, \\ z_n \quad\;\; = \gamma_n T x_n \oplus (1-\gamma_n) x_n, \end{cases}$$

for all $n \geq 1$*, where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences of positive numbers in* $[0,1]$*. Then it is known as SP-iteration.*

Remark 1.12 *If* $\gamma_n = 0$ *then (1.5) reduces to the iterative process (1.4).*

2 Main Results

Now we are all set to prove our main results. We start with proving key lemmas for later use.

Lemma 2.1 ([9]) *Let* C *be a nonempty subset of a* $CAT(0)$ *space* X*. Let* $T : C \to X$ *be an* α*-nonexpansive mapping for some real number* $\alpha < 1$ *such that* $F(T) \neq \phi$*. Then* T *is quasi-nonexpansive.*

Proof. Let $x \in C$ and $z \in F(T)$. Then we have

$$\begin{aligned} d(Tx,z)^2 &= d(Tx,Tz)^2 \\ &\leq \alpha d(Tx,z)^2 + \alpha d(x,Tz)^2 + (1-2\alpha)d(x,z)^2 \\ &= \alpha d(Tx,z)^2 + \alpha d(x,z)^2 + (1-2\alpha)d(x,z)^2 \\ &= \alpha d(Tx,z)^2 + (1-\alpha)d(x,z)^2. \end{aligned}$$

Therefore, $d(Tx,z) \leq d(x,z)$. This inequality ensures the closedness of $F(T)$.

Lemma 2.2 ([9]) *Let C be a nonempty closed and convex subset of a $CAT(0)$ space X. Let $T : C \to X$ be an α-nonexpansive mapping for some $\alpha < 1$. Then the following conditions hold:*

(i) If $0 \leq \alpha < 1$, then

$$\begin{aligned} d(x,Ty)^2 \leq\ & \frac{1+\alpha}{1-\alpha} d(x,Tx)^2 \\ &+ \frac{2}{1-\alpha}\{\alpha d(x,y) + d(Tx,Ty)\} d(x,Tx) + d(x,y)^2, \end{aligned}$$

for all $x, y \in C$.

(ii) If $\alpha < 0$, then

$$\begin{aligned} d(x,Ty)^2 \leq\ & d(x,Tx)^2 \\ &+ \frac{2}{1-\alpha}\{(-\alpha)d(Tx,y) + d(Tx,Ty)\} d(x,Tx) + d(x,y)^2, \end{aligned}$$

for all $x, y \in C$.

Proof. (i) Observe

$$
\begin{aligned}
d(x,Ty)^2 \leq\ & [d(x,Tx)+d(Tx,Ty)]^2 \\
=\ & d(x,Tx)^2 + d(Tx,Ty)^2 + 2d(x,Tx)d(Tx,Ty) \\
\leq\ & d(x,Tx)^2 + \alpha d(Tx,y)^2 + \alpha d(x,Ty)^2 + (1-2\alpha)d(x,y)^2 \\
& + 2d(x,Tx)d(Tx,Ty) \\
\leq\ & d(x,Tx)^2 + \alpha[d(Tx,x)+d(x,y)]^2 + \alpha d(x,Ty)^2 \\
& + (1-2\alpha)d(x,y)^2 + 2d(x,Tx)d(Tx,Ty)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & d(x,Tx)^2 + \alpha d(Tx,x)^2 + \alpha d(x,y)^2 + 2\alpha d(Tx,x)d(x,y) \\
& + \alpha d(x,Ty)^2 + (1-2\alpha)d(x,y)^2 + 2d(x,Tx)d(Tx,Ty) \\
=\ & (1+\alpha)d(x,Tx)^2 + 2\alpha d(Tx,x)d(x,y) + \alpha d(x,Ty)^2 \\
& + (1-\alpha)d(x,y)^2 + 2d(x,Tx)d(Tx,Ty)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
d(x,Ty)^2 \leq\ & \frac{1+\alpha}{1-\alpha} d(x,Tx)^2 \\
& + \frac{2}{1-\alpha}\{\alpha d(x,y) + d(Tx,Ty)\} d(x,Tx) \\
& + d(x,y)^2
\end{aligned}
$$

(ii) Observe

$$
\begin{aligned}
d(x,Ty)^2 \leq & \quad [d(x,Tx)+d(Tx,Ty)]^2 \\
= & \quad d(x,Tx)^2+d(Tx,Ty)^2+2d(x,Tx)d(Tx,Ty) \\
\leq & \quad d(x,Tx)^2+\alpha d(Tx,y)^2 \\
+ & \quad \alpha d(x,Ty)^2+(1-2\alpha)d(x,y)^2 \\
& + 2d(x,Tx)d(Tx,Ty) \\
= & \quad d(x,Tx)^2+\alpha d(Tx,y)^2+\alpha d(x,Ty)^2 \\
+ & \quad (1-\alpha)d(x,y)^2 \\
& - \alpha d(x,y)^2+2d(x,Tx)d(Tx,Ty) \\
\leq & \quad d(x,Tx)^2+\alpha d(Tx,y)^2+\alpha d(x,Ty)^2 \\
+ & \quad (1-\alpha)d(x,y)^2 \\
& - \alpha[d(x,Tx)^2+d(Tx,y)^2+2d(x,Tx)d(Tx,y)] \\
& + 2d(x,Tx)d(Tx,Ty) \\
= & \quad (1-\alpha)d(x,Tx)^2+\alpha d(x,Ty)^2 \\
& + (1-\alpha)d(x,y)^2-2\alpha d(x,Tx)d(Tx,y) \\
+ & \quad 2d(x,Tx)d(Tx,Ty) \\
= & \quad (1-\alpha)d(x,Tx)^2+\alpha d(x,Ty)^2 \\
& + (1-\alpha)d(x,y)^2 \\
+ & \quad 2[(-\alpha)d(Tx,y)+d(Tx,Ty)]d(x,Tx)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
d(x,Ty)^2 \leq d(x,Tx)^2 + \frac{2}{1-\alpha}[(-\alpha)d(Tx,y) \\
+ d(Tx,Ty)]d(x,Tx)+d(x,y)^2.
\end{aligned}
$$

Lemma 2.3 *Let C be a nonempty closed and convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be an α-nonexpansive mapping for* *some $\alpha < 1$. If $\{x_n\}$ is a sequence in C such that $d(Tx_n, x_n) \to 0$ and $\Delta\text{-}\lim_{n\to\infty} x_n = z$ for some z in X then $z \in C$ and $Tz = z$.*

Proof. It follows from Lemma 1.10 that $z \in C$. Let $0 \leq \alpha < 1$. By Lemma 2.2(i), we deduce that

$$\begin{aligned} d(x_n, Tz)^2 \leq\ & \frac{1+\alpha}{1-\alpha} d(x_n, Tx_n)^2 \\ &+ \frac{2}{1-\alpha}\{\alpha d(x_n, z) + d(Tx_n, Tz)\} d(x_n, Tx_n) + d(x_n, z)^2 \end{aligned}$$

for all n in N. Thus we have $\lim_{n\to\infty} \sup d(x_n, Tz) \leq \lim_{n\to\infty} \sup d(x_n, z)$. Let $\alpha < 0$. Then by Lemma 2.2(ii), we have

$$\begin{aligned} d(x_n, Tz)^2 \leq\ & d(x_n, Tx_n)^2 \\ &+ \frac{2}{1-\alpha}\{(-\alpha) d(Tx_n, z) + d(Tx_n, Tz)\} d(x_n, Tx_n) + d(x_n, z)^2 \end{aligned}$$

for all n in N. This implies again that

$$\lim_{n\to\infty} \sup d(x_n, Tz) \leq \lim_{n\to\infty} \sup d(x_n, z).$$

By the uniqueness of asymptotic centers, $Tz = z$.

Lemma 2.4 *Let C be a nonempty closed and convex subset of a $CAT(0)$ space X and $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let a sequence $\{x_n\}$ with x_1 in C be defined by (1.5) such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are arbitrary sequences of positive numbers in $[0, 1]$. Let $p \in F(T)$. Then the following assertions hold:*

(i) $\max\{d(x_{n+1}, p), d(y_n, p)\} \leq d(x_n, p)$ *for* $n = 1, 2, \ldots$.

(ii) $\lim_{n\to\infty} d(x_n, p)$ *exists.*

(iii) $\lim_{n\to\infty} d(x_n, F(T))$ *exists.*

Proof. Consider

$$\begin{aligned} d(y_n, p) &= d(\beta_n T z_n \oplus (1-\beta_n) z_n, p) \\ &\le \beta_n d(T z_n, p) + (1-\beta_n) d(z_n, p) \\ &\le \beta_n d(z_n, p) + (1-\beta_n) d(z_n, p) \\ &= d(z_n, p) \end{aligned}$$

$$\begin{aligned} &= d(\gamma_n T x_n \oplus (1-\gamma_n) x_n, p) \\ &\le \gamma_n d(T x_n, p) + (1-\gamma_n) d(x_n, p) \\ &\le \gamma_n d(x_n, p) + (1-\gamma_n) d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Consequently,

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n T y_n \oplus (1-\alpha_n) y_n, p) \\ &\le \alpha_n d(T y_n, p) + (1-\alpha_n) d(y_n, p) \\ &\le \alpha_n d(y_n, p) + (1-\alpha_n) d(y_n, p) \\ &= d(y_n, p) \\ &\le d(x_n, p). \end{aligned}$$

This implies that $\{d(x_n, p)\}$ is a bounded and non-increasing sequence. Thus $\lim_{n\to\infty} d(x_n, p)$ exists. In the same manner, we see that $\{d(x_n, F(T))\}$ is also a bounded non-increasing real sequence and thus converges.

Lemma 2.5 *Let C be a nonempty closed and convex subset of a $CAT(0)$ space X and $T : C \to C$ be an α-nonexpansive mapping for some $\alpha < 1$. Let a sequence $\{x_n\}$ with x_1 in C be defined by (1.5) such that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are arbitrary sequences of positive numbers in $[0, 1]$. Let $p \in F(T)$. Then $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.*

Proof. From the above Lemma 2.4(iii), $\lim_{n\to\infty} d(x_n, F(T))$ exists.
Let $p \in F(T)$ and

$$\lim_{n\to\infty} d(x_n, p) = c\,. \tag{2.1}$$

Now we prove that $\lim_{n\to\infty} d(y_n, p) = c$.

Since $d(x_{n+1}, p) \le d(y_n, p)$. This implies

$$\lim_{n\to\infty} d(x_{n+1}, p) \le \lim_{n\to\infty} d(y_n, p) \quad \text{or} \quad c \le \lim_{n\to\infty} d(y_n, p). \tag{2.2}$$

But $d(y_n, p) \le d(x_n, p)$. This implies

$$\lim_{n\to\infty} \sup d(y_n, p) \le c. \tag{2.3}$$

From (2.2) and (2.3), we get

$$\lim_{n\to\infty} d(y_n, p) = c\,. \tag{2.4}$$

Similarly we can get

$$\lim_{n\to\infty} d(z_n, p) = c\,. \tag{2.5}$$

Now

$$\begin{aligned} d(z_n, p)^2 &= d(\gamma_n T x_n \oplus (1-\gamma_n)x_n, p)^2 \\ &\le \gamma_n d(Tx_n, p)^2 + (1-\gamma_n) d(x_n, p)^2 - \gamma_n(1-\gamma_n) d(Tx_n, x_n)^2 \\ &\le d(x_n, p)^2 - \gamma_n(1-\gamma_n) d(Tx_n, x_n)^2. \end{aligned}$$

Thus

$$\gamma_n(1-\gamma_n) d(Tx_n, x_n)^2 \le d(x_n, p)^2 - d(z_n, p)^2,$$

so that

$$d(Tx_n, x_n)^2 \le \frac{1}{\gamma_n(1-\gamma_n)}[d(x_n, p)^2 - d(z_n, p)^2].$$

By (2.1) and (2.5), $\lim\sup d(Tx_n, x_n) \le 0$ and hence $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Theorem 2.6 *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be an α-nonexpansive mapping with*

$F(T) \neq \phi$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$. From arbitrary $x_1 \in C$, define the sequence $\{x_n\}$ by the recursion (1.5). Then $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. It follows from Lemma 2.5 that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. We now let $\omega_w(x_n) = \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subseteq F(T)$. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 1.9, 1.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in C$. Since $\lim_n d(v_n, Tv_n) = 0$, then $v \in F(T)$ by Lemma 2.3. We claim that $u = v$. Suppose not, by the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_n d(v_n, v) &< \limsup_n d(v_n, u) \\
&\leq \limsup_n d(u_n, u) \\
&= \limsup_n d(u_n, v) \\
&= \limsup_n d(x_n, v) \\
&= \limsup_n d(v_n, v),
\end{aligned}$$

which is a contradiction and hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 1.9, 1.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in C$. Let $A(\{u_n\}\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $u = v$ and $v \in F(T)$. We can complete the proof by showing that $x = v$. If not, since $\{d(x_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_n d(v_n, v) &< \limsup_n d(v_n, v) \\
&\leq \limsup_n d(x_n, x) \\
&< \limsup_n d(x_n, v) \\
&\leq \limsup_n d(v_n, x)
\end{aligned}$$

which is a contradiction and hence the conclusion follows.

Remark 2.7 *As we know that Thianwan's iteration is the special case for SP-iteration for $\gamma_n = 0$. So, the above Δ-convergence result is also hold for Thianwan's iterative process.*

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