**Mathematics Scientific Journal**
Vol. 7, No. 1, (2011), 89-95

# Approximating fixed points of generalized non-expansive non-self mappings in $CAT(0)$ spaces

**Saeed Shabani**[a,1], **S. J. Hoseini Ghoncheh** [b,2]

[a] *Department of Mathematics, Izeh Branch, Islamic Azad University, Izeh, Iran.*
[b] *Department of Mathematics, Takestan Branch, Islamic Azad University, Takestan, Iran.*
*Received 11 February 2011; Accepted 23 June 2011*

---

**Abstract**

Suppose $K$ is a nonempty closed convex subset of a complete $CAT(0)$ space $X$ with the nearest point projection $P$ from $X$ onto $K$. Let $T : K \to X$ be a nonself mapping, satisfying condition $(C)$ with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Suppose $\{x_n\}$ is generated iteratively by $x_1 \in K$, $x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n])$,$n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Then $\{x_n\}$ is $\Delta-$convergence to some point $x^*$ in $F(T)$. This work extends a result of Laowang and Panyanak [5] to the case of generalized nonexpansive nonself mappings.

***Keywords***: $CAT(0)$ spaces, fixed point, generalized nonexpansive nonself mappings.

---

## 1 Introduction

In 2010 Laowang and Panyanak [5] studied the iterative scheme define as follows: Let $K$ is a nonempty closed convex subset of a complete $CAT(0)$ space $X$ with the nearest point projection $P$ from $X$ onto $K$ and $T : K \to X$ be a nonexpansive nonself mapping with nonempty fixed point set,and if $\{x_n\}$ is generated iteratively by

$$x_1 \in K, \quad x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n]), \tag{1.1}$$

[1]Shabani60@gmail.com
[2]Sjhghoncheh@gmail.com

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Then $\{x_n\}$ is $\Delta-$convergence to a fixed point of $T$. In this paper we extend the result of Laowang and Panyanak to the case of generalized nonexpansive nonself mappings.

Let $K$ be a nonempty subset of a $CAT(0)$ space $X$ and let $T : K \to X$ be a mapping. A point $x \in K$ is called a fixed point of $T$ if $x = Tx$. We shall denote by $F(T)$ the set of fixed points of $T$. $T$ is called nonexpansive if for each $x, y \in K$, $d(Tx, Ty) \leq d(x, y)$.

In 2008, suzuki [6] introduced condition $(C)$: $T$ is said to satisfy condition $(C)$ if $\frac{1}{2}d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$.

**Proposition 1.1.** *[7, Proposition 1.1] Every nonexpansive mapping satisfies condition $(C)$, but the inverse is not true.*

Let $(X, d)$ be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from $x$ to $y$) is a map $c$ from a closed interval $[0, l] \subset R$ to $X$ such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, $c$ is an isometry and $d(x, y) = l$. The image $\alpha$ of $c$ is called a geodesic ( or metric) segment joining $x$ and $y$. When it is unique, this geodesic is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ to $y$, for each $x, y \in X$. A subset $Y \subset X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space $(X, d)$ consists of three points in $X$ (the vertices of $\Delta$) and a geodesic segment between each pair of vertices (the edges of $\Delta$). A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in the Euclidean plane $E^2$ such that $d_{E^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a $CAT(0)$ space (see [1]), if all geodesic triangles of appropriate size satisfy the following comparison axiom. Let $\Delta$ be a geodesic triangle in $X$ and let $\bar{\Delta}$ be a comparison triangle for $\Delta$. Then $\Delta$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{E^2}(\bar{x}, \bar{y}). \tag{1.2}$$

If $x, y_1, y_2$ are points in a $CAT(0)$ space and if $y_0$ is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2 \tag{CN}$$

In fact [1, page 163], a geodesic space is a $CAT(0)$ space if and only if it satisfies the $(CN)$ inequality.

**Proposition 1.2.** *[6, Proposition 2.2] Let $K$ be a bounded closed convex subset of a complete $CAT(0)$ space $X$ and $T : K \to X$ satisfies condition $(C)$. then*

$$d(x, Ty) \leq 3d(Tx, x) + d(x, y)$$

*holds, for all $x, y \in K$.*

**Lemma 1.3.** *Let $(X, d)$ be a $CAT(0)$ space.*

1. *[1, Proposition 2.4] Let $K$ be a convex subset of $X$ which is complete in the induced metric. Then for every $x \in X$, there exists a unique point $P(x) \in K$ such that $d(x, P(x)) = \inf\{d(x, y) : y \in K\}$. Moreover, the map $x \to P(x)$ is a nonexpansive retract from $X$ onto $K$.*

2. *[3, Lemma 2.1(iv)] For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*
$$d(x, z) = td(x, y), \qquad d(y, z) = (1 - t)d(x, y)$$
*one uses the notation $(1 - t)x \oplus ty$ for the unique point $z$.*

3. *[3, Lemma 2.4] For $x, y, z \in X$ and $t \in [0, 1]$, one has*
$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z).$$

4. *[3, Lemma 2.5] For $x, y, z \in X$ and $t \in [0, 1]$, one has*
$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2.$$

Let $\{x_n\}$ be a bounded sequence in a $CAT(0)$ space $X$. For $x \in X$, we set
$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$
The asymptotic radius
$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$
and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set
$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$
It is known [2, Proposition 7] that in a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point.

**Definition 1.4.** [4, Definition 3.1] A sequence $\{x_n\}$ in a $CAT(0)$ space $X$ is said to $\Delta-$convergence to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case one writes $\Delta - \lim_n x_n = x$ and call $x$ the $\Delta - \lim$ of $\{x_n\}$.

**Lemma 1.5.** *Let $(X, d)$ be a $CAT(0)$ space.*

1. *[4, p. 3690] Every bounded sequence in $X$ has a $\Delta-$convergent subsequence.*

2. *[8, Proposition 2.1] If $K$ is a closed convex subset of $X$ and if $\{x_n\}$ is a bounded sequence in $K$, then the asymptotic center of $\{x_n\}$ is in $K$.*

3. *[3, Lemma 2.8] If $\{x_n\}$ is a bounded sequence in $X$ with $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

## 2 Main results

The following lemma was proved by Dhompongsa and Panyanak in the case of non-expansive mappings(see [3, Lemma 2.10]).

**Lemma 2.1.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$, and let $T : K \to X$ be a nonself mapping, satisfying condition $(C)$. Suppose $\{x_n\}$ is a bounded below sequence in $K$ such that $\lim_n d(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subset F(T)$. Here $\omega_w(x_n) := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

*Proof.* Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By part (1) and (2) Lemma 1.5, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - lim_n v_n = v \in K$. We show $v \in F(T)$. In order to prove this, by the condition $(C)$,

$$d(x_n, Tv) \leq 3d(Tx_n, x_n) + d(x_n, v).$$

Therefore

$$\begin{aligned}\limsup_n d(x_n, Tv) \leq & \limsup_n (3d(Tx_n, x_n) + d(x_n, v)) \\ = & \limsup_n d(x_n, v).\end{aligned}$$

The uniqueness of asymptotic center, imply $v \in K$ and $T(v) = v$. By part (3) Lemma 1.5, $u = v$. Therefore $\omega_w(x_n) \subset F(T)$. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subset F(T)$, $\{d(x_n, v)\}$ converges. By part (3) Lemma 1.5, $x = u$. This shows that $\omega_w(x_n)$ consists of exactly one point. □

**Theorem 2.2.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$, and $T : K \to X$ be a nonself mapping, satisfying condition $(C)$ with $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by $x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n])$,$n \geq 1$. then $\lim_{n\to\infty} d(x_n, x^*)$ exists.*

*Proof.* By part (1) of Lemma 1.3, the nearest point projection $P$ from $X$ onto $K$ is nonexpansive. Then

$$\begin{aligned}
d(x_{n+1}, x^*) = & d(P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n]), Px^*) \\
\leq & d((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n], x^*) \\
\leq & (1-\alpha_n)d(x_n, x^*) + \alpha_n d(TP[(1-\beta_n)x_n \oplus \beta_n Tx_n], Tx^*) \\
= & (1-\alpha_n)d(x_n, x^*) + \alpha_n d(TP[(1-\beta_n)x_n \oplus \beta_n Tx_n], x^*) \\
\leq & (1-\alpha_n)d(x_n, x^*) + \alpha_n(3d(Tx^*, x^*) + d(P[(1-\beta_n)x_n \oplus \beta_n Tx_n], x^*)) \\
\leq & (1-\alpha_n)d(x_n, x^*) + \alpha_n[(1-\beta_n)d(x_n, x^*) + \beta_n d(Tx_n, Tx^*)] \\
\leq & (1-\alpha_n)d(x_n, x^*) + \alpha_n[(1-\beta_n)d(x_n, x^*) + \beta_n d(x_n, x^*)] \\
= & d(x_n, x^*).
\end{aligned}$$

Consequently we have for all $n \in N$,

$$d(x_n, x^*) \leq d(x_1, x^*).$$

This implies that $\{d(x_n, x^*)\}_{n=1}^{\infty}$ is bounded and decreasing. Hence $\lim_{n\to\infty} d(x_n, x^*)$ exists. □

**Theorem 2.3.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$, and $T : K \to X$ be a nonself mapping, satisfying condition $(C)$ with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0,1)$. Starting from arbitrary $x_1 \in K$, define the sequence $\{x_n\}$ by $x_{n+1} = P((1-\alpha_n)x_n \oplus \alpha_n TP[(1-\beta_n)x_n \oplus \beta_n Tx_n])$,$n \geq 1$. then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

*Proof.* Let $x^* \in F(T)$. Then by Theorem 2.2, $\lim_{n\to\infty} d(x_n, x^*)$ exists. Let

$$\lim_{n\to\infty} d(x_n, x^\star) = r.$$

If $r = 0$ then by the condition $(C)$, we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x^*, x_n) + d(x^*, Tx_n) \\ &\leq d(x^*, x_n) + 3d(x^*, Tx^*) + d(x^*, x_n) \end{aligned}$$

Therefore $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.
If $r > 0$, we let $y_n = P[(1-\beta_n)x_n \oplus \beta_n Tx_n]$. By part (4) of Lemma 1.3 and condition $(C)$, we have

$$\begin{array}{rcl} d(y_n, x^*)^2 &=& d(P[(1-\beta_n)x_n \oplus \beta_n Tx_n], Px^*)^2 \\ &\leq& d((1-\beta_n)x_n \oplus \beta_n Tx_n], x^*)^2 \\ &\leq& (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2 - \beta_n(1-\beta_n)d(x_n, Tx_n)^2 \\ &\leq& (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, x^*)^2 \\ &=& (1-\beta_n)d(x_n, x^*)^2 + \beta_n d(Tx_n, Tx^*)^2 \\ &\leq& (1-\beta_n)d(x_n, x^*)^2 + \beta_n(3d(Tx^*, x^*) + d(x_n, x^*))^2 \\ &=& d(x_n, x^*)^2 \end{array}$$

Therefore

$$d(y_n, x^*) \leq d((1-\beta_n)x_n \oplus \beta_n Tx_n], x^*) \leq d(x_n, x^*) \tag{2.1}$$

By part (4) of Lemma 1.3, we have

$$\begin{array}{rcl} d(x_{n+1}, x^*)^2 &=& d(P[(1-\alpha_n)x_n \oplus \alpha_n Ty_n], Px^*)^2 \\ &\leq& d((1-\alpha_n)x_n \oplus \alpha_n Ty_n, x^*)^2 \\ &\leq& (1-\alpha_n)d(x_n, x^*)^2 + \alpha_n d(Ty_n, x^*)^2 - \alpha_n(1-\alpha_n)d(x_n, Ty_n)^2 \\ &\leq& (1-\alpha_n)d(x_n, x^*)^2 + \alpha_n(3d(Tx^*, x^*) + d(y_n, x^*))^2 \\ && \qquad -\alpha_n(1-\alpha_n)d(x_n, Ty_n)^2 \\ &=& (1-\alpha_n)d(x_n, x^*)^2 + \alpha_n d(y_n, x^*)^2 - \alpha_n(1-\alpha_n)d(x_n, Ty_n)^2 \\ &\leq& (1-\alpha_n)d(x_n, x^*)^2 + \alpha_n d(x_n, x^*)^2 - \alpha_n(1-\alpha_n)d(x_n, Ty_n)^2 \\ &=& d(x_n, x^*)^2 - \alpha_n(1-\alpha_n)d(x_n, Ty_n)^2. \end{array}$$

Therefore

$$d(x_{n+1},x^*)^2 \leq d(x_n,x^*)^2 - W(\alpha_n)d(x_n,Ty_n)^2,$$

where $W(\alpha)=\alpha(1-\alpha)$. Since $\alpha\in[\epsilon,1-\epsilon]$, $W(\alpha_n)\geq\epsilon^2$. Therefore

$$\epsilon^2\sum_{n=1}^{\infty} d(x_n,Ty_n)^2 \leq d(x_1,x^*)^2 < \infty.$$

This implies $\lim_{n\to\infty} d(x_n,Ty_n)=0$. By the condition $(C)$, we have

$$\begin{aligned} d(x_n,x^*) \leq&\quad d(x_n,Ty_n)+d(Ty_n,x^*) \\ \leq&\quad d(x_n,Ty_n)+3d(Tx^*,x^*)+d(y_n,x^*) \\ =&\quad d(x_n,Ty_n)+d(y_n,x^*). \end{aligned}$$

Hence

$$r \leq \liminf_{n\to\infty} d(y_n,x^*).$$

On the other hand, from (2.1), we have

$$\limsup_{n\to\infty} d(y_n,x^*) \leq r.$$

This implies $\lim_{n\to\infty} d(y_n,x^*)=r$. Thus from (2.1) we have

$$\lim_{n\to\infty} d((1-\beta_n)x_n\oplus\beta_nTx_n],x^*)=r.$$

Since $T$ satisfies in condition $(C)$ we have

$$\begin{aligned} d(Tx_n,x^*) \leq&\quad 3d(Tx^*,x^*)+d(x_n,x^*) \\ =&\quad d(x_n,x^*) \end{aligned}$$

Thus

$$\limsup_{n\to\infty} d(Tx_n,x^*) \leq r.$$

Now, by [5, Lemma 2.9], $\lim_{n\to\infty} d(x_n,Tx_n)=0$. □

**Theorem 2.4.** *Let $K$ be a nonempty closed convex subset of a complete $CAT(0)$ space $X$, and $T:K\to X$ be a nonself mapping, satisfying condition $(C)$ with $F(T)\neq\emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[\epsilon,1-\epsilon]$ for some $\epsilon\in(0,1)$. Starting from arbitrary $x_1\in K$, define the sequence $\{x_n\}$ by $x_{n+1}=P((1-\alpha_n)x_n\oplus\alpha_nTP[(1-\beta_n)x_n\oplus\beta_nTx_n])$,$n\geq 1$. then $\{x_n\}$ $\Delta-$ converges to some point $x^*$ in $F(T)$.*

*Proof.* By Theorem 2.3, $\lim_{n\to\infty} d(x_n,Tx_n)=0$. It follows from the proof of the Theorem 2.2 that $\{d(x_n,v)\}$ is bounded and decreasing for each $v\in F(T)$, and so it is convergent. By Lemma 2.1, $\omega_w(x_n)$ consists exactly one point and is a fixed point of $T$. Consequently the sequence $\{x_n\}$ $\Delta-$ converges to some point $x^*$ in $F(T)$. □

### Acknowledgment

This article is taken from the research project number 4779 date 2010/06/02, with the title of "existence of fixed points for some mappings in CAT(0) spaces" with approach of "Innovation and Science Production", which has been supported by the Izeh Branch, Islamic Azad University, Izeh, Iran. The first author would like to thank this support.

## References

[1] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature,* vol. 319 of Fundamental Principles of Mathematical Sciences, Berlin, Germany, 1999.

[2] S. Dhompongsa and W. Kirk and B. Sims, *fixed point of uniformly lipschitzian mappings,* Nonlinear Anal. **65** (2006), 762–772.

[3] S. Dhompongsa and B. Panyanak, *On $\Delta$-convergence theorems in $CAT(0)$ spaces.* Computers and Mathematics with Applications. **56** (2008), 2572–2579.

[4] W. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces.* Nonlinear Anal. **68** (2008), 3689–3696.

[5] W. Laowang and B. Panyanak, *Approximating fixed points of nonexpansive nonself mappings in $CAT(0)$ spaces.* Fixed Point Theory Appl. **Article ID 367274** (2010), 11 pages.

[6] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mapping,* Math. Anal. Appl. **340** (2008), 1088–1095.

[7] A. Razani and H. Salahifard, *Invariant approximation for $CAT(0)$ spaces,* Nonlinear Anal. **72** (2010), 2421–2425.

[8] S. Dhompongsa, W. A. Kirk and B. Panyanak, *Nonexpansive set-valued mappings in metric and Banach spaces,* J. Nonlinear and Convex Anal. **8** (2007), 35-45.