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On the strong convergence theorems by the hybrid method for a family of mappings in uniformly convex Banach spaces

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Abstract

Some algorithms for finding common fixed point of a family of mappings is constructed. Indeed, let C be a nonempty closed convex subset of a uniformly convex Banach space X whose norm is Gateaux differentiable and let $\{T_n\}$ be a family of self-mappings on C such that the set of all common fixed points of $\{T_n\}$ is nonempty. We construct a sequence $\{x_n\}$ generated by the hybrid method and also we give the conditions of $\{T_n\}$ under which $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}$.

Keywords: Hybrid method, Common fixed point, Iterative algorithm, Uniformly convex Banach space.

1 Introduction

Let $\{T_n\}_{n=0}^{+\infty}$ be a family of mappings of a real Hilbert space $\mathcal{H}$ into itself and let $F(T_n)$ be the set of all fixed points of T_n. By the assumption that $\bigcap\limits_{n=0}^{+\infty} F(T_n) \neq \emptyset$, Haugazeau

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[4] introduced a sequence $\{x_n\}$ generated by the hybrid method, as following

$$\begin{cases} x_0 \in \mathcal{H} \\ y_n = T_n(x_n) \\ C_n = \{z \in \mathcal{H} : \langle x_n - y_n, y_n - z\rangle \geq 0\} \\ Q_n = \{z \in \mathcal{H} : \langle x_n - z, x_0 - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

In case that C_i is a closed convex subset of $\mathcal{H}$ for $i = 1, \ldots, m$, $\bigcap_{i=1}^{m} C_i \neq \emptyset$ and $T_n = P_{C_{n(mod\ m+1)}}$, he proved a strong convergence theorem. Recently, Solodov and Svaiter [9], Bauschke and Combettes [2], Atsushiba and Takahashi [1], Nakajo and Takahashi [8], Iiduka, Takahashi and Toyoda [5], Nakajo, Shimoji and Takahashi [7], studied the hybrid method in a Hilbert spaces and also Nakajo, Shimoji and Takahashi [6] considered this method for families of mappings in Banach spaces.

Throughout this paper, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let X be a real Banach space with dual space X^*. The *line segment between x and y* is denoted and defined by $[x, y] := \{tx + (1-t)y : t \in [0, 1]\}$. For a set-valued mapping $T : X \multimap Y$, the *domain* of T is $Dom(T) = \{x \in X : T(x) \neq \emptyset\}$, *range* of T is $R(T) = \{y \in Y : \exists x \in X, (x, y) \in T\}$ and the *inverse* T^{-1} of T is $\{(y, x) : (x, y) \in T\}$. For a real number c, let $cT = \{(x, cy) : (x, y) \in T\}$. If S and T are any set-valued mappings, we define $S + T = \{(x, y + z) : (x, y) \in S, (x, z) \in T\}$. Set $R_0^+ = [0, +\infty)$ and

$$\mathcal{G} = \{g : R_0^+ \to R_0^+ : g(0) = 0,\ g \text{ is continuous, strictly increasing and convex}\}. \tag{1.1}$$

Lemma 1.1. *[3] Let C be a nonempty closed convex subset of a uniformly convex Banach space X and let $x \in X$. Then, there exists a unique element $x_0 \in C$ such that $\|x_0 - x\| = \inf_{y \in C} \|y - x\|$. Putting $x_0 = P_C(x)$, we call P_C the metric projection onto C.*

Lemma 1.2. *[10] Let C be a nonempty closed convex subset of a uniformly convex Banach space X whose norm is Gateaux differentiable and let $x \in X$. Then $y = P_C(x)$ if and only if $\langle y - z, J(x - y)\rangle \geq 0$ for all $z \in C$.*

Lemma 1.3. *[10] Suppose X has a Gateaux differentiable norm. Then the duality mapping J is single-valued and $\|x\|^2 - \|y\|^2 \geq 2\langle x - y, Jy\rangle$ for all $x, y \in X$.*

Lemma 1.4. *[11] The Banach space X is uniformly convex if and only if for every bounded subset B of X, there exists $g_B \in \mathcal{G}$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g_B(\|x - y\|) \tag{1.2}$$

for all $x, y \in B$ and all $\lambda \in [0, 1]$.

2 Main results

Let $\{T_n\}_{n=0}^{+\infty}$ be a family of self-mappings of C and $F(T_n)$ be the set of all fixed points of T_n. Assume that $F := \bigcap\limits_{n=0}^{+\infty} F(T_n)$ is a nonempty closed convex subset of C satisfies the following condition,
$\exists x_0 \in C \ \ \exists\{a_n\} \subseteq (0, +\infty)$ *with* $\liminf\limits_n a_n > 0 \ \ \exists\{\alpha_n\} \subseteq [0, 1], \ \exists\{\beta_n\} \subseteq [0, 1]$ such that

$$\langle x - z, J(x - w_n)\rangle \geq a_n\|x - w_n\|^2 \tag{2.1}$$

for all $x \in C$, $z \in F(T_n)$, where, $w_n = \beta_n T_0(x_0) + (1 - \beta_n)T_n(\alpha_n x_0 + (1 - \alpha_n)x)$.

Algorithm 2.1. *Let $\{T_n\}$ be a family of self-mappings of C with $F \neq \emptyset$ which satisfies condition (2.1). Let $\{x_n\}_{n=1}^{+\infty}$ be a sequence generated by the following algorithm.*

$$\begin{cases} x_0 \in C, n \in \mathbb{N}_0 \\ y_n = \alpha_n x_0 + (1 - \alpha_n)x_n \\ z_n = \beta_n T_0(x_0) + (1 - \beta_n)T_n(y_n) \\ C_n = \{z \in C : \langle x_n - z, J(x_n - z_n)\rangle \geq a_n\|x_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, J(x_0 - x_n)\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{2.2}$$

Theorem 2.2. *Suppose C is a nonempty closed convex subset of a uniformly convex Banach space X whose norm is Gateaux differentiable and $\{T_n\}$ is a family of self-mappings of C with $F \neq \emptyset$ which satisfies the condition (2.1). Assume that*

(∗) for every bounded sequence $\{u_n\}$ in C, $\sum\limits_{n=0}^{+\infty} g(\|u_{n+1} - u_n\|) < +\infty$ and $\sum\limits_{n=0}^{+\infty} g(a\|u_n - u\|) < +\infty$ for some $g \in \mathcal{G}$ and some $u \in [T_0(x_0), T_n(w)]$, where $w \in [x_0, u_n]$ and $a > 0$ imply that $w_w(u_n) \subseteq F$. Then the sequence $\{x_n\}$ generated by Algorithm 2.1 converges strongly to $P_F(x_0)$.

Proof. We split the proof into six steps.
Step 1. *$\{x_n\}$ is well defined.*

Notice that C_n and Q_n are closed and convex sets for all $n \in \mathbb{N}_0$. On the other hand, condition (2.1) and the definition of C_n in Algorithm 2.1 imply that $F(T_n) \subseteq C_n$ for all $n \in \mathbb{N}_0$. Hence $F \subseteq C_n$ for all $n \in \mathbb{N}_0$. Since $J(0) = 0$, it follows from the definition of Q_n in Algorithm 2.1 that $Q_0 = C$ which implies that $F \subseteq C_0 \cap Q_0$. Lemma 1.1 guarantees that there exists a unique element $x_1 = P_{C_0 \cap Q_0}(x_0)$. By Lemma 1.2,

$$\langle x_1 - z, J(x_0 - x_1)\rangle \geq 0$$

for all $z \in C_0 \cap Q_0$ and hence by $F \subseteq C_0 \cap Q_0$ we get

$$\langle x_1 - z, J(x_0 - x_1)\rangle \geq 0$$

for all $z \in F$. Therefore, $F \subseteq Q_1$ and so apply the fact that $F \subseteq C_n$ for all $n \in \mathbb{N}_0$ we have $F \subseteq C_1 \cap Q_1$. Again, Lemma 1.1 guarantees that there exists a unique element $x_2 = P_{C_1 \cap Q_1}(x_0)$. Inductively, we find that $\{x_n\}$ is well defined.

Step 2. $\{x_n\}$ *is a bounded sequence.*

From $x_{n+1} = P_{C_n \cap Q_n}(x_0)$ and $F \subseteq C_n \cap Q_n$ for all $n \in \mathbb{N}_0$ we have

$$\|x_{n+1} - x_0\| \leq \|x_0 - P_F(x_0)\| \tag{2.3}$$

for all $n \in \mathbb{N}_0$, which implies that $\{x_n\}$ is a bounded sequence.

Step 3. $\lim_n \|x_n - x_0\|$ *exists.*

Replace terms $x_{n+1} - x_0$ and $x_n - x_o$ respectively with x and y in Lemma 1.3,

$$\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2 - 2\langle x_{n+1} - x_n, J(x_n - x_0)\rangle$$

and hence $x_{n+1} \in Q_n$ implies that $\|x_n - x_0\|^2 \leq \|x_{n+1} - x_0\|^2$ for all $n \in \mathbb{N}_0$; i.e., $\|x_n - x_0\|$ is an increasing sequence and so by Step 2 we find that $\lim_n \|x_n - x_o\|$ exists.

Step 4. $\sum_{n=0}^{+\infty} g(\|x_{n+1} - x_n\|) < +\infty$ *for some* $g \in \mathcal{G}$.

It follows from Lemma 1.4 that there exists $g \in \mathcal{G}$ such that

$$\|\frac{x_n + x_{n+1}}{2} - x_0\|^2 \leq \frac{1}{2}\|x_n - x_0\|^2 + \frac{1}{2}\|x_{n+1} - x_0\|^2 - \frac{1}{4}g(\|x_{n+1} - x_n\|)$$

and hence

$$g(\|x_{n+1} - x_n\|) \leq 2\|x_n - x_0\|^2 + 2\|x_{n+1} - x_0\|^2 - 4\|\frac{x_n + x_{n+1}}{2} - x_0\|^2 \tag{2.4}$$

for all $n \in \mathbb{N}_0$. From Lemma 1.2 and the definition of Q_n we get $x_n = P_{Q_n}(x_0)$ and so by $x_{n+1} \in Q_n$ and convexity of Q_n we get $\frac{x_n + x_{n+1}}{2} \in Q_n$. Again, by $x_n = P_{Q_n}(x_0)$,

$$\|\frac{x_n + x_{n+1}}{2} - x_0\|^2 \geq \|x_n - x_0\|^2. \tag{2.5}$$

It follows from inequalities (2.4) and (2.5) that

$$g(\|x_{n+1} - x_n\|) \leq 2\|x_{n+1} - x_0\|^2 - 2\|x_n - x_0\|^2 \text{ for all } n \in \mathbb{N}_0. \tag{2.6}$$

That $\sum_{n=0}^{+\infty} g(\|x_{n+1} - x_n\|) < +\infty$ follows from (2.6) and Step 3.

Step 5. $\sum_{n=0}^{+\infty} g(a\|x_n - z_n\|) < +\infty$ *for some* $g \in \mathcal{G}$ *and* $a > 0$.

Since $a_n > 0$ for all $n \in \mathbb{N}_0$ and $\liminf_n a_n > 0$, there exists $a > 0$ for which $a_n \geq a$ for all $n \in \mathbb{N}_0$. Now, $x_{n+1} \in C_n$ guarantees that

$$\|x_n - x_{n+1}\|\|x_n - z_n\| \geq \langle x_n - x_{n+1}, J(x_n - z_n)\rangle \geq a_n\|x_n - z_n\|^2$$

and thus

$$a\|x_n - z_n\| \le \|x_{n+1} - x_n\| \tag{2.7}$$

for all $n \in \mathbb{N}_0$. That $\sum_{n=0}^{+\infty} g(a\|x_n - z_n\|) < +\infty$ follows from (2.7), (1.1) and Step 4.

Step 6. $\{x_n\} \to P_F(x_0)$.

It follows from our assumption, Step 4 and Step 5 that $w_w(x_n) \subseteq F$. Let the subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to $w \in F$. Therefore, weakly lower semicontinuity of the norm and (2.3) imply that

$$\|P_F(x_0) - x_0\| \le \|w - x_0\| \le \lim_{i\to+\infty} \|x_{n_i} - x_0\| \le \|P_F(x_0) - x_0\|$$

and hence $x_{n_i} \to w = P_F(x_0)$.

Corollary 2.3. *Suppose C is a nonempty closed convex subset of a uniformly convex Banach space X whose norm is Gateaux differentiable and $\{T_n\}$ is a family of self-mappings of C with $F \neq \emptyset$ which satisfies the following condition.*

(a) $\exists x_0 \in C \ \ \exists\{a_n\} \subseteq (0, +\infty)$ *with* $\liminf_n a_n > 0 \ \ \exists\{\alpha_n\} \subseteq [0, 1]$ *such that*

$$\langle x - z, J(x - T_n(v_n))\rangle \ge a_n\|x - T_n(v_n))\|^2$$

for all $x \in C$, $z \in F(T_n)$, where, $v_n = \alpha_n x_0 + (1 - \alpha_n)x$;

(b) for every bounded sequence $\{u_n\}$ in C, $\sum_{n=0}^{+\infty} g(\|u_{n+1}-u_n\|) < +\infty$ and $\sum_{n=0}^{+\infty} g(a\|u_n-u\|) < +\infty$ for some $g \in \mathcal{G}$ and some $u \in [T_0(x_0), T_n(w)]$, where $w \in [x_0, u_n]$ and $a > 0$ imply that $w_w(u_n) \subseteq F$.

Then the sequence $\{x_n\}$ generated by the following algorithm converges strongly to $P_F(x_0)$.

$$\begin{cases} n \in \mathbb{N}_0 \\ y_n = \alpha_n x_0 + (1 - \alpha_n)x_n \\ z_n = T_n(y_n) \\ C_n = \{z \in C : \langle x_n - z, J(x_n - z_n)\rangle \ge a_n\|x_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, J(x_0 - x_n)\rangle \ge 0\} \\ x_{n+1} = P_{C_n\cap Q_n}(x_0). \end{cases} \tag{2.8}$$

Proof. All conditions of Theorem 2.2 hold for $\beta_n = 0$ and also in this case (2.2) reduces to (2.8). So Theorem 2.2 implies the result.

Corollary 2.4. *Suppose C is a nonempty closed convex subset of a uniformly convex Banach space X whose norm is Gateaux differentiable and $\{T_n\}$ is a family of self-mappings of C with $F \neq \emptyset$ which satisfies the following condition.*

(a) $\exists x_0 \in C \ \ \exists\{a_n\} \subseteq (0, +\infty)$ *with* $\liminf_n a_n > 0 \ \ \exists\{\beta_n\} \subseteq [0, 1]$

$$\langle x - z, J(x - w_n)\rangle \ge a_n\|x - w_n\|^2$$

for all $x \in C$, $z \in F(T_n)$, *where,* $w_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(x)$*;*

(b) for every bounded sequence $\{u_n\}$ *in* C, $\sum_{n=0}^{+\infty} g(\|u_{n+1}-u_n\|) < +\infty$ *and* $\sum_{n=0}^{+\infty} g(a\|u_n - w_n\|) < +\infty$ *for some* $g \in \mathcal{G}$, $w_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(u_n)$, *and* $a > 0$ *imply that* $w_w(u_n) \subseteq F$*.*

Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$*.*

$$\begin{cases} x_0 \in C, n \in \mathbb{N}_0 \\ z_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(x_n) \\ C_n = \{z \in C : \langle x_n - z, J(x_n - z_n)\rangle \geq a_n\|x_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, J(x_0 - x_n)\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases} \tag{2.9}$$

Proof. Similar to Corollary 2.3, all conditions of Theorem 2.2 hold for $\alpha_n = 0$ and so with this assumption, (2.2) collapses to (2.9) which it completes the proof.

Corollary 2.5. *Suppose* C *is a nonempty closed convex subset of a uniformly convex Banach space* X *whose norm is Gateaux differentiable and* $\{T_n\}$ *is a family of self-mappings of* C *with* $F \neq \emptyset$ *which satisfies the following condition.*

(a) $\exists\{a_n\} \subseteq (0,+\infty)$ *with* $\liminf_n a_n > 0$

$$\langle x - z, J(x - T_n(x))\rangle \geq a_n\|x - T_n(x)\|^2$$

for all $x \in C$, $z \in F(T_n)$*;*

(b) for every bounded sequence $\{u_n\}$ *in* C, $\sum_{n=0}^{+\infty} g(\|u_{n+1}-u_n\|) < +\infty$ *and* $\sum_{n=0}^{+\infty} g(a\|u_n - T_n(u_n)\|) < +\infty$ *for some* $g \in \mathcal{G}$ *and* $a > 0$ *imply that* $w_w(u_n) \subseteq F$*.*

Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$*.*

$$\begin{cases} x_0 \in C, n \in \mathbb{N}_0 \\ C_n = \{z \in C : \langle x_n - z, J(x_n - T_n(x_n))\rangle \geq a_n\|x_n - T_n(x_n)\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, J(x_0 - x_n)\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0) \end{cases}$$

Proof. Put $\alpha_n = \beta_n = 0$ in Theorem 2.2.

Corollary 2.6. *Suppose* C *is a nonempty closed convex subset of a real Hilbert space* H *and* $\{T_n\}$ *is a family of self-mappings of* C *with* $F \neq \emptyset$ *which satisfies the following conditions.*

(a) $\exists x_0 \in C$ $\exists\{b_n\} \subseteq (-1,+\infty)$ *with* $\liminf_n b_n > -1$ *and* $\exists\{\alpha_n\} \subseteq [0,1]$, $\exists\{\beta_n\} \subseteq [0,1]$ *such that*

$$\|w_n - z\|^2 \leq \|x - z\|^2 - b_n\|x - w_n\|^2$$

for all $x \in C$, $z \in F(T_n)$, *where,* $v_n = \alpha_n x_0 + (1-\alpha_n)x$ *and* $w_n = \beta_n T_0(x_0) + (1-\beta_n)T_n(v_n)$*;*

(b) for every bounded sequence $\{u_n\}$ *in* C, $\sum\limits_{n=0}^{+\infty}\|u_{n+1}-u_n\|^2<+\infty$ *and* $\sum\limits_{n=0}^{+\infty}(a\|u_n-q_n\|)^2<+\infty$, *where* $q_n=\beta_nT_0(x_0)+(1-\beta_n)T_n(p_n)$, $p_n=\alpha_nx_0+(1-\alpha_n)u_n$ *and* $a>0$ *imply that* $w_w(u_n)\subseteq F$.
Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} y_n=\alpha_nx_0+(1-\alpha_n)x_n\\ z_0=T_0(x_0)\\ z_n=\beta_nz_0+(1-\beta_n)T_n(y_n)\ (n\geq 1)\\ C_n=\{z\in C:\|z_n-z\|^2\leq\|x_n-z\|^2-b_n\|x_n-z_n\|^2\}\\ Q_n=\{z\in C:\langle x_n-z,x_0-x_n\rangle\geq 0\}\\ x_{n+1}=P_{C_n\cap Q_n}(x_0).\end{cases}\tag{2.10}$$

Proof. First we note that, for $x\in C$, $z\in F(T_n)$, $v_n=\alpha_nx_0+(1-\alpha_n)x$ and $w_n=\beta_nT_0(x_0)+(1-\beta_n)T_n(v_n)$, by our assumption we have $\|w_n-z\|^2\leq\|x-z\|^2-b_n\|x-w_n\|^2$ for all $z\in F(T_n)$, if and only if

$$\|w_n-x\|^2+2\langle w_n-x,x-z\rangle+\|x-z\|^2\leq\|x-z\|^2-b_n\|x-w_n\|^2$$

if and only if $\langle x-z,x-w_n\rangle\geq\frac{1+b_n}{2}\|x-w_n\|^2$. Then condition (2.1) satisfies for $a_n=\frac{1+b_n}{2}$. In a real Hilbert space H, we have

$$\|\lambda x+(1-\lambda)y\|^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2$$

for all $x,y\in H$ and $\lambda\in[0,1]$, so, we can consider $g_B(t)=t^2$ for each bounded subset B of H in Lemma 1.4 and hence $(*)$ holds. Then all assumptions of Theorem 2.2 hold which it implies that $\{x_n\}$ converges strongly to $P_F(x_0)$.

By putting $\beta_n=0$, $\alpha_n=0$ and $\alpha_n=\beta_n=0$ in (2.10) we get the following results respectively.

Corollary 2.7. *Suppose* C *is a nonempty closed convex subset of a real Hilbert space* H *and* $\{T_n\}$ *is a family of self-mappings of* C *with* $F\neq\emptyset$ *which satisfies the following conditions.*

(a) $\exists x_0\in C\ \ \exists\{b_n\}\subseteq(-1,+\infty)$ *with* $\liminf\limits_n b_n>-1$ *and* $\exists\{\alpha_n\}\subseteq[0,1]$ *such that*

$$\|T_n(v_n)-z\|^2\leq\|x-z\|^2-b_n\|x-T_n(v_n)\|^2$$

for all $x\in C$, $z\in F(T_n)$, *where,* $v_n=\alpha_nx_0+(1-\alpha_n)x$;

(b) for every bounded sequence $\{u_n\}$ *in* C, $\sum\limits_{n=0}^{+\infty}\|u_{n+1}-u_n\|^2<+\infty$ *and* $\sum\limits_{n=0}^{+\infty}(a\|u_n-T_n(v_n)\|)^2<+\infty$, *where* $v_n=\alpha_nx_0+(1-\alpha_n)u_n$ *and* $a>0$ *imply that* $w_w(u_n)\subseteq F$.
Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} y_n=\alpha_nx_0+(1-\alpha_n)x_n\\ z_n=T_n(y_n)\\ C_n=\{z\in C:\|z_n-z\|^2\leq\|x_n-z\|^2-b_n\|x_n-z_n\|^2\}\\ Q_n=\{z\in C:\langle x_n-z,x_0-x_n\rangle\geq 0\}\\ x_{n+1}=P_{C_n\cap Q_n}(x_0).\end{cases}$$

Corollary 2.8. *Suppose C is a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}$ is a family of self-mappings of C with $F \neq \emptyset$ which satisfies the following conditions.*

(a) $\exists x_0 \in C \;\; \exists \{b_n\} \subseteq (-1, +\infty)$ *with* $\liminf_n b_n > -1$ *and* $\exists \{\beta_n\} \subseteq [0,1]$ *such that*

$$\|w_n - z\|^2 \leq \|x - z\|^2 - b_n \|x - w_n\|^2$$

for all $x \in C$, $z \in F(T_n)$, *where* $w_n = \beta_n T_0(x_0) + (1-\beta_n) T_n(x)$;

(b) for every bounded sequence $\{u_n\}$ *in* C, $\sum_{n=0}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty$ *and* $\sum_{n=0}^{+\infty} (a\|u_n - w_n\|)^2 < +\infty$, *where* $w_n = \beta_n T_0(x_0) + (1-\beta_n) T_n(u_n)$ *and* $a > 0$ *imply that* $w_w(u_n) \subseteq F$.

Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} z_0 = T_0(x_0) \\ z_n = \beta_n z_0 + (1-\beta_n) T_n(x_n) \;\; (n \geq 1) \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - b_n \|x_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

Corollary 2.9. *[6] Suppose C is a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}$ is a family of self-mappings of C with $F \neq \emptyset$ which satisfies the following conditions.*

(a) $\exists \{b_n\} \subseteq (-1, +\infty)$ *with* $\liminf_n b_n > -1$ *such that*

$$\|T_n(x) - z\|^2 \leq \|x - z\|^2 - b_n \|x - T_n(x)\|^2$$

for all $x \in C$, $z \in F(T_n)$;

(b) for every bounded sequence $\{u_n\}$ *in* C, $\sum_{n=0}^{+\infty} \|u_{n+1} - u_n\|^2 < +\infty$ *and* $\sum_{n=0}^{+\infty} \|u_n - T_n u_n\|^2 < +\infty$ *imply that* $w_w(u_n) \subseteq F$.

Then $\{x_n\}$ *generated by the following algorithm converges strongly to* $P_F(x_0)$.

$$\begin{cases} x_0 \in C \\ z_n = T_n(x_n) \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - b_n \|x_n - z_n\|^2\} \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

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