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New Integral Transform for Solving Nonlinear Partial Differential Equations of fractional order

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Abstract

In this work, we have applied Elzaki transform and He's homotopy perturbation method to solve partial differential equation (PDEs) with time-fractional derivative. With help He's homotopy perturbation, we can handle the nonlinear terms. Further, we have applied this suggested He's homotopy perturbation method in order to reformulate initial value problem. Some illustrative examples are given in order to show the ability and simplicity of the approach. All numerical calculations in this manuscript were performed on a PC applying some programs written in *Maple*.

Key words: Elzaki transform, homotopy perturbation method, Caputo fractional derivative
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1 Introduction

He's homotopy perturbation method (HPM) was first proposed by He (1998) to deal with linear and nonlinear problems [1]. Various studies have investigated this suggested method since it was proposed. These studies have applied it to solve various linear and nonlinear initial value problems [1,2,3]. A major advantage of HPM is that it addresses a problem directly without the need for any form of transformation, linearization discrimination or any other unrealistic assumption.

This is in contrast with the classical HPM and other series solution methods which form a recurrent scheme of the solution through using only one type of the problem conditions: either the initial conditions or the boundary conditions.

Since fractional differential equations can be extensively used in fluid mechanics, mathematical biology, electro-chemistry, physics, and in other similar fields and because of its various applications, it has been given a lot of attention by researchers in recent years. For instance, fractional derivatives can be used to model the nonlinear oscillation of earthquake. Also, through applying fractional derivatives in fluid-dynamic traffic model, can eliminate the problems and deficiency resulting from the assumption of continuum traffic flow [4]. As a result of the works done by researchers in the related area in recent years, a number of fractional differential equations have been investigated and consequently solutions have been proposed for these equations such as the impulsive fractional differential equations [5], the fractional advection-dispersion equation [6], certain types of time-fractional diffusion equation [7], fractional generalized Burgers' fluid [8], fractional KdV-type equations [9], space-time fractional Whitham-Broer-Kaupand equations [10] and space fractional backward Kolmogorov equation [11].

In a study, Tarig M. Elzaki and Sailh M. Elzaki [12], indicated that the modified Sumudu transform [13,14] or Elzaki transform can be successfully applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations. Elzaki transform is a very efficient tool that can be applied to solve some differential equations that we can not solve through using Sumudu transform [15].

The present paper is organized around the following sections. In section two, we have presented some fundamental definitions of fractional calculus, Elzaki transform of fractional derivative and the classical HPM. Then, in section three, we have introduced and elaborated on homotopy perturbation Elzaki transform method(HPETM). And, finally in section four, we have offered some examples to solve in order to show the validity and efficiency of this approach.

2 Preliminaries

In this part of the paper, we present and define fractional equation, Elzaki transform and we obtain Elzaki transform of Caputo fractional derivative. Then, fractional homotopye perturbation method is introduced and explained in detail.

2.1 Fractional calculus

In this subsection of the paper, we present and define Riemann-Liouville fractional integral and Caputo's fractional derivative that are presented [16].

Definition 1. A real function $f(x)$, $x > 0$, is considered to be in the space C_ν, $(\nu \in R)$, if there exists a real number $n(> \nu)$, so that $f(x) = x^n f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_ν^k if and only if $f^{(k)} \in C_\nu$, $k \in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order of $\alpha > 0$, of a function $f \in C_\nu$, $\nu \geq -1$, is given by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-r)^{\alpha-1} f(r) dr.$$

$$I^\alpha f(x) = I_0^\alpha f(x), \qquad I^0 f(x) = f(x).$$

Definition 3. The Caputo's fractional derivative of f is defined as

$$D^{\alpha}f(x) = I^{k-\alpha}D^{k}f(x) = \frac{1}{\Gamma(k-\alpha)}\int_0^x (x-r)^{k-\alpha-1}f^{(k)}(r)dr, \quad x > 0.$$

where, $f \in C_{-1}^{k}$, $k-1 < \alpha \leq k$ and $k \in \mathbb{N}$.
Property 1. For $k-1 < \alpha \leq k$, $k \in \mathbb{N}$, $f \in C_{\nu}^{k}$, $\nu \geq -1$ and $x > 0$, the following properties satisfy

i) $D_a^{\alpha} I_a^{\alpha} f(x) = f(x)$.
ii) $I_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \sum\limits_{j=0}^{k-1} f^{(j)}(a^{+})\frac{(x-a)^j}{j!}$.

2.2 Elzaki Transform

The basic definitions of modified Sumudu transform or Elzaki transform is given as follows.
Elzaki transform of the function $f(t)$ is:

$$T(v) = E\{f(t), v\} = v\int_0^{\infty} f(t)e^{-\frac{t}{v}}dt, \quad t > 0. \tag{2.1}$$

Tarig M. Elzaki and Sailh M. Elzaki in [12], presented a modified version of Sumudu transform or Elzaki transform. They used this transform to solve partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations. Elzaki transform is a very efficient and powerful tool that can be used to solve some differential equations which can not be solved by Sumudu transform in [17].

In order to obtain Elzaki transform of partial derivative, we have used the integration of parts, and then the result is [17]:

1) $E\{\frac{\partial f(x,t)}{\partial t}\} = \frac{1}{v}T(x,v) - vf(x,0)$,
2) $E\{\frac{\partial^2 f(x,t)}{\partial t^2}\} = \frac{1}{v^2}T(x,v) - f(x,0) - v\frac{\partial f(x,0)}{\partial t}$,
3) $E\{\frac{\partial f(x,t)}{\partial x}\} = \frac{d}{dx}T(x,v)$,

4) $E\{\frac{\partial^2 f(x,t)}{\partial x^2}\} = \frac{d^2}{dx^2}T(x,v)$.

2.3 Elzaki Transform of Caputo fractional derivative

In order to obtain Elzaki transform of Caputo fractional derivative, we have applied Laplace transform formula for the Caputo fractional derivative [16]:

$$L\{D^\alpha f(x), s\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad n-1 < \alpha \leq n \tag{2.2}$$

Theorem 1. Let $T(v)$ is Elzaki transform of $f(t)$:

$$T(v) = E\{f(t), v\} \qquad and \qquad g(t) = \begin{cases} f(t-\tau) \ , t \geq \tau \\ \quad 0 \qquad , t < \tau \end{cases}$$

then

$$E\{g(t)\} = e^{\frac{-\tau}{v}} T(v).$$

proof. See in [12].

The ELzaki transform can certainly be used to deal with all problems that are usually treated by the well- known and widely applied Laplace transform.
In fact, as we can see in the next theorem, the ELzaki transform is closely related to the Laplace transform $F(s)$.
Theorem 2. With Laplace transform $F(s)$, then the Elzaki transform $T(v)$ of $f(t)$ is given by [12]:

$$T(v) = vF(\frac{1}{v}). \tag{2.3}$$

Theorem 3. Suppose $T(v)$ is the Elzaki transform of the function $f(t)$ then:

$$E\{D^{\alpha} f(t), v\} = \frac{T(v)}{v^{\alpha}} - \sum_{k=0}^{n-1} v^{k-\alpha+2} f^{(k)}(0).$$

proof. From the theorem2, we have:

$$E\{D^{\alpha} f(t), v\} = vL\{D^{\alpha} f(t), \frac{1}{v}\}.$$

Now we use of Eq.(2.2), then we have:

$$\begin{aligned} E\{D^{\alpha} f(t), v\} &= v\Big((\frac{1}{v})^{\alpha} F(\frac{1}{v}) - \sum_{k=0}^{n-1} (\frac{1}{v})^{\alpha-k-1} f^{(k)}(0) \Big), \\ &= \frac{vF(\frac{1}{v})}{v^{\alpha}} - \sum_{k=0}^{n-1} v^{k-\alpha+2} f^{(k)}(0). \end{aligned}$$

2.4 Homotopy Perturbation Method

In order to understand the basic idea of He's homotopy perturbation method, lets consider the following general nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.4}$$

with boundary conditions:

$$B\left(u, \partial u/\partial n\right), \quad r \in \Gamma, \tag{2.5}$$

in which A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω.

Now, we can divide the operator A into two parts L and N, where L is linear, and N is nonlinear. Thus, Eq.(2.4) can be written as:

$$L(u) + N(u) - f(r) = 0. \tag{2.6}$$

Then, by using homotopy technique, we can form a homotopy $v(r, p) : \Omega \times [0, 1] \to R$ which satisfies:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0, 1]. \tag{2.7}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{2.8}$$

in which $p \in [0, 1]$ is considered as an embedding parameter, and u_0 is the initial approximation of Eq.(2.4). This will meet the boundary conditions. Then, the result will be:

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{2.9}$$
$$H(v, 1) = A(v) - f(r) = 0. \tag{2.10}$$

The process of deformation involves the changing process of p from zero to unity, which includes the process of $v(r, p)$ changing from $u_0(r)$ to $u(r)$. Also, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If we suppose the embedding parameter p $(0 \leq p \leq 1)$ as a small parameter, using the classical perturbation technique, we can thus suppose that the solution of Eqs. (2.9) and (2.10) can be given as a power series in p, i.e.,

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots, \tag{2.11}$$

and assuming $p = 1$ leads to the approximate solution of Eq.(2.7) as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \dots \ . \tag{2.12}$$

It is interesting to mention that the most important advantage of He's homotopy perturbation method is that one can construct the perturbation equation in various ways, through homotopy in topology, that is this method is problem dependent. Moreover, the initial approximation can be freely selected.

3 Homotopy Perturbation Elzaki Transform Method (HPETM)

In this section, we introduce fractional homotopy perturbation Elzaki transform method.
We consider the following general nonlinear problem, say in two independent variables x and t:

$$D_t^{\alpha} u(x,t) = \mathfrak{R}\ u(x,t) + f(x,t) \tag{3.1}$$

where D_t^{α} is the fractional Capato derivative with respect to t, $\alpha > 0$, $\mathfrak{R}$ is an operator in x, and t which might include derivatives with respect to "x", $u(x,t)$ is an unknown function, and $f(x,t)$ is the source in homogeneous term.
Taking Elzaki transform on both sides of equation (3.1), we get:

$$E\{D_t^{\alpha} u(x,t)\} = E\{\mathfrak{R}\ u(x,t)\} + E\{f(x,t)\} \tag{3.2}$$

Using the differentiation property of Elzaki transforms, we have:

$$E\{u(x,t)\} = \sum_{k=0}^{n-1} v^{k+2} u^{(k)}(x,0) + v^{\alpha}(E\{\mathfrak{R}\ u(x,t)\} + E\{f(x,t)\}) \tag{3.3}$$

Applying the inverse Elzaki transform on both sides of equation (3.3), we find:

$$u(x,t) = G(x,t) - E^{-1}\Big(v^{\alpha}(E\{\mathfrak{R}\ u(x,t)\} + E\{f(x,t)\})\Big), \qquad (3.4)$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions.
Now, we apply He's homotopy perturbation method:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t). \qquad (3.5)$$

And the nonlinear term can be decomposed as:

$$\mathfrak{R}\ u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \qquad (3.6)$$

where $H_n(u)$ are given by:

$$H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{\partial}{\partial p^n}\left[\mathfrak{R}\Big(\sum_{i=0}^{\infty} p^i u_i\Big)\right]. \qquad (3.7)$$

Substituting equations (3.5) and (3.6) in equation (3.4), we get:

$$\begin{aligned}\textstyle\sum_{n=0}^{\infty} p^n u_n(x,t) &= G(x,t) \\ &+p\Big\{E^{-1}\Big(v^{\alpha}(E\{\mathfrak{R}\textstyle\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\})\Big)\Big\}.\end{aligned} \qquad (3.8)$$

This is a Mixture of a Elzaki Transform and Homotopy Perturbation Method for Solving Nonlinear Partial Differential Equations of fractional order.
Comparing the coefficient of like powers of p, the following approxima-

tions are obtained:

$$p^0: \quad u_0(x,t) = G(x,t),$$
$$p^1: \quad u_1(x,t) = E^{-1}\Bigg(v^\alpha(E\{\Re u_0(x,t) + H_0(u)\})\Bigg),$$
$$p^2: \quad u_2(x,t) = E^{-1}\Bigg(v^\alpha(E\{\Re u_1(x,t) + H_1(u)\})\Bigg),$$
$$\cdots.$$

Then the solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots.$$

4 Applications

In this section, we apply the Elzaki transform and Homotopy Perturbation method to solve nonlinear time-fractional advection partial differential equation, time-fractional hyperbolic equation,time-fractional Fisher's equation . All of the plots and computing for this equations have been done on a PC applying some programs written in *Maple*.

Example 1. Consider the nonlinear time-fractional advection partial differential equation:

$$\frac{d^\alpha}{dt^\alpha}u(x,t) + u(x,t)\ u_x(x,t) = x(1+t^2), \tag{4.1}$$

where $t > 0$, $x \in \mathbb{R}$ and $0 < \alpha \leq 1$, with the initial condition:

$$u(x,0) = 0. \tag{4.2}$$

Taking Elzaki transform of equation (3.1) subjected to the initial condition, we have:

$$E\{u(x,t)\} = xv^{\alpha+2} + 2xv^{\alpha+4} - v^\alpha E\{u(x,t)u_x(x,t)\}. \tag{4.3}$$

The inverse Elzaki transform implies that:

$$u(x,t)=\frac{xt^{\alpha}}{\Gamma(\alpha+1)}+\frac{2xt^{\alpha+2}}{\Gamma(\alpha+3)}-E^{-1}\left\{v^{\alpha}E\{u(x,t)u_x(x,t)\right\}. \quad (4.4)$$

Now applying He's homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=\frac{xt^{\alpha}}{\Gamma(\alpha+1)}+\frac{2xt^{\alpha+2}}{\Gamma(\alpha+3)}+p\left\{E^{-1}\left(v^{\alpha}(\sum_{n=0}^{\infty}p^n H_n(u)\})\right)\right\} \quad (4.5)$$

where $H_n(u)$ are He's polynomials that represents the nonlinear terms. The first few components of He's polynomials are given by:

$$H_0(u)=-u_0u_{0x},$$
$$H_1(u)=-(u_0u_{1x}+u_1u_{0x}),$$
$$H_2(u)=-(u_2u_{0x}+u_{2x}u_0+u_1u_{1x})$$
$$\vdots$$

Comparing the coefficients of the same powers of p , we get:

$$p^0: \quad u_0(x,t)=x\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\right),$$
$$p^1: \quad u_1(x,t)=-x\left(\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)}+\frac{4\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)}+\frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)}\right),$$
$$p^2: \quad u_2(x,t)=2x\left(\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)}+\frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)}\right)$$
$$\vdots$$

Then, the third order term approximate solution for Eq.(3.1) is given by:

$$
\begin{aligned}
u(x,t) &= x\Big(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{2t^{\alpha+2}}{\Gamma(\alpha+3)}\Big) - x\Big(\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} \\
&+ \frac{4\Gamma(2\alpha+3)t^{3\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+3)\Gamma(3\alpha+3)} + \frac{4\Gamma(2\alpha+5)t^{3\alpha+4}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)}\Big) \\
&+ 2x\Big(\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(\alpha+1)^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} \frac{8\Gamma(2\alpha+5)\Gamma(4\alpha+7)t^{5\alpha+6}}{\Gamma(\alpha+3)^3\Gamma(3\alpha+5)\Gamma(5\alpha+7)}\Big).
\end{aligned}
$$

The solution that we have found is equivalent to the exact solution in a

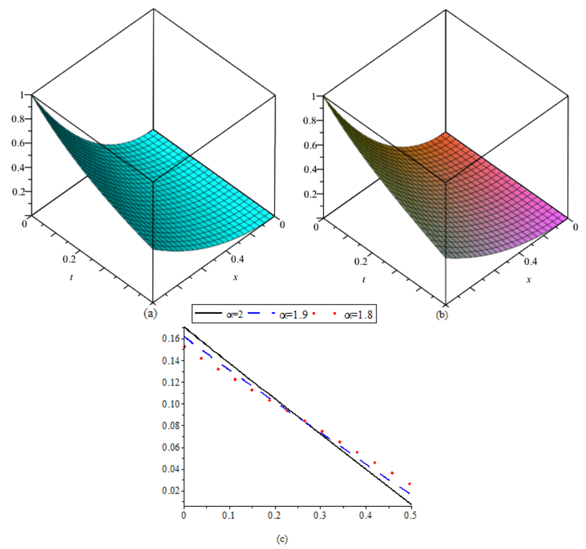

Fig. 1. (a) Exact solution (b) The approximate solution in the case $\alpha = 1.0$ (c)The third-order Eq.(4.3) for different value of α when $x = 0.3$

closed form $u(x,t) = xt$, which is the same third order term approximate solution for Eq.(4.1)-(4.2) obtained from [18] using VIM. We can also solve the nonlinear time-fractional advection partial differential equation (4.1) in [18] through applying ADM.
In table 1, we can see the approximate solutions for $\alpha = 1$, which is derived for different values of x and t using HPETM, HPM and VIM.

Table 1: Numerical values when $\alpha = 1$ for Eq.(3.1)

t	x	u_{VIM}	u_{HPM}	u_{HPETM}	u_{Exact}
0.2	0.25	0.050309	0.0499876	0.050000	0.050000
	0.50	0.100619	0.099978	0.100000	0.100000
	0.75	0.150928	0.149968	0.150001	0.150000
	1.0	0.201237	0.199957	0.200001	0.200000
0.4	0.25	0.101894	0.099645	0.100023	0.100000
	0.50	0.203787	0.199290	0.200046	0.200000
	0.75	0.305681	0.298935	0.300069	0.300000
	1.0	0.407575	0.398580	0.400092	0.400000
0.6	0.25	0.153094	0.147158	0.150411	0.150000
	0.50	0.306188	0.294317	0.300823	0.300000
	0.75	0.459282	0.441475	0.451234	0.450000
	1.0	0.612376	0.588634	0.601646	0.600000

Example 2. Consider the nonlinear time-fractional hyperbolic equation:

$$\frac{d^\alpha}{dt^\alpha}u(x,t) = \frac{\partial}{\partial x}(u(x,t)u_x(x,t)), \quad t > 0, \ x \in \mathbb{R}, \ 1 < \alpha \leq 2, \tag{4.6}$$

with the initial condition:

$$u(x,0) = x^2, \quad u_t(x,0) = -2x^2. \tag{4.7}$$

Taking Elzaki transform of equation (4.6) subjected to the initial condition, we have:

$$E\{u(x,t)\} = x^2v^2 - 2x^2v^3 + v^\alpha E\{u_x^2(x,t)u(x,t)u_{xx}(x,t)\}. \tag{4.8}$$

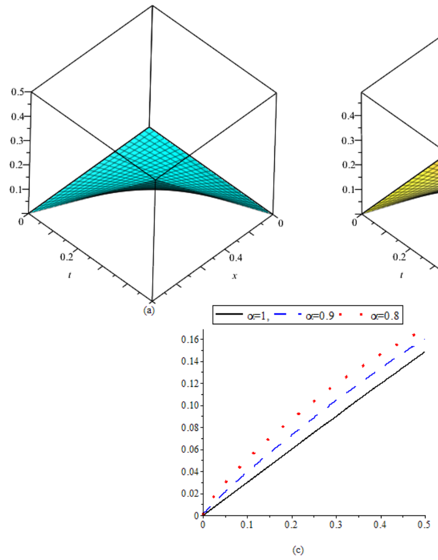

Fig. 2. (a) Exact solution (b) The approximate solution in the case $\alpha = 1.0$ (c)The third-order Eq.(3.1) for different value of α when $x = 0.3$

The inverse Elzaki transform implies that:

$$u(x,t) = x^2 - 2x^2t + E^{-1}\left\{v^\alpha E\{u_x^2(x,t)u(x,t)u_{xx}(x,t)\right\}. \tag{4.9}$$

Now applying He's homotopy perturbation method, we get:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = x^2 - 2x^2t + p\left\{E^{-1}\left(v^\alpha(\sum_{n=0}^{\infty} p^n H_n(u)\})\right)\right\}, \tag{4.10}$$

where

$$H_0(u) = u_{0x}^2 + u_0 u_{0xx},$$
$$H_1(u) = 2u_{0x}u_{1x} + u_0 u_{1xx} + u_1 u_{0xx},$$
$$H_2(u) = 2u_{0x}u_{2x} + u_0 u_{2xx} + u_2 u_{0xx} + u_1 u_{1xx},$$
$$\ldots .$$

Comparing the coefficients of the same powers of p , we get:

$$
\begin{aligned}
p^0: &\quad u_0(x,t) = x^2 - 2tx^2,\\
p^1: &\quad u_1(x,t) = \frac{6x^2t^\alpha}{\Gamma(\alpha+1)} - \frac{24x^2t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{48x^2t^{\alpha+2}}{\Gamma(\alpha+3)},\\
p^2: &\quad u_2(x,t) = 72x^2\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)}\right)\\
&\qquad - 144x^2\left(\frac{\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(2\alpha+2)} - \frac{4\Gamma(\alpha+3)t^{2\alpha+2}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)}\times\right.\\
&\qquad\qquad \left.\frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{82\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}\right),\\
&\ldots .
\end{aligned}
$$

The third order term approximate solution for (4.6) is presented by:

$$
\begin{aligned}
u(x,t) = x^2\Bigg(&1 - 2t + 6\Big(\frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{4t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{8t^{\alpha+2}}{\Gamma(\alpha+3)}\Big)\\
&+ 72\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{4t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)}\right)\\
&- 144\left(\frac{\Gamma(\alpha+2)t^{2\alpha+1}}{\Gamma(\alpha+1)\Gamma(2\alpha+2)} - \frac{4\Gamma(\alpha+3)t^{2\alpha+2}}{\Gamma(\alpha+2)\Gamma(2\alpha+3)}\right.\\
&\left.+ \frac{8t^{2\alpha+2}}{\Gamma(2\alpha+3)} + \frac{82\Gamma(\alpha+4)t^{2\alpha+3}}{\Gamma(\alpha+3)\Gamma(2\alpha+4)}\right)\Bigg).
\end{aligned}
$$

The resulting solution is equivalent to the exact solution in a closed form:
$u(x,t) = (x/t+1)^2$, which is the same third order term approximate solution for (4.3)-(4.4) derived from [18] using VIM. Through applying ADM, time-fractional hyperbolic differential equation (4.3) can also be solved in [18].

Table 2: Numerical values when $\alpha = 2$ for Eq.(4.3)

t	x	u_{VIM}	u_{HPM}	u_{HPETM}	u_{Exact}
0.2	0.25	0.043400	0.043400	0.0433951	0.043403
	0.50	0.173600	0.173600	0.173580	0.173611
	0.75	0.390600	0.390600	0.390556	0.390625
	1.0	0.694400	0.694400	0.694321	0.694444
0.4	0.25	0.031779	0.031779	0.031567	0.031888
	0.50	0.127118	0.127118	0.126268	0.127551
	0.75	0.286015	0.286015	0.284103	0.286990
	1.0	0.508471	0.508471	0.505072	0.508471
0.6	0.25	0.023665	0.023665	0.022005	0.024414
	0.50	0.094660	0.094660	0.088018	0.097656
	0.75	0.212984	0.212984	0.198040	0.219727
	1.0	0.378638	0.378638	0.352071	0.390625

The approximate solutions for $\alpha = 2.0$ obtained for different values of x and t using HPETM, HPM and VIM, is shown in table 2.

5 Conclusion

In this paper, we have proposed a mixed version of Elzaki transform and He's homotopy perturbation method in order to solve nonlinear partial differential equations of fractional order. We have shown that the solution of such equations is simple when we use Adomian decomposition method, but the calculation of Adomian polynomials is complex in this method. The major advantage of this technique in comparison with the decomposition method is that the developed algorithm can solve nonlinear partial differential equations without the need for Adomian's polynomials.

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