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Positive solution for boundary value problem of fractional differential equation

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Abstract

In this paper, we prove the existence of the solution for boundary value problem(BVP) of fractional differential equations of order $q \in (2, 3]$. The Krasnoselskii's fixed point theorem is applied to establish the results. In addition, we give an detailed example to demonstrate the main result.

Key words: Fractional differential equation; Krasnoselskii's fixed point theorem; Boundary value problem.

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1 Introduction

Fractional differential equations are the generalization of ordinary differential equation to arbitrary non-integer order, and have received more and more interest due to their wide applications in various sciences, such as physics, chemistry, biophysics, capacitor theory, blood flow phenomena, electrical circuits, control theory, etc, also recent investigations have demonstrated that the dynamics of many systems are described more accurately by using fractional differential equations. So fractional differential equations have attracted many authors.
In [1], Nickolai was concerned with the nonlinear differential equation of fractional order

$$D_{0+}^{q}u(t) = f(t, u(t), u'(t)) \quad a.e.\ t \in (0,1),$$

where D_{0+}^{q} is Riemann-Liouville(R-L) fractional order derivative, subject to the boundary conditions $u(0) = u(1) = 0$. The author obtained the existence of at least one solution by using the Leray-Schauder Continuation Principle.
In [2], Zhang has given the existence of positive solution to the equation

$$\begin{cases} {}^{c}D^{q}u(t) + f(t, u(t)) = 0,\ 0 < t < 1, \\ u(0) + u'(0) = u(1) + u'(1) = 0, \end{cases}$$

by the use of classical fixed point theorems, where${}^{c}D^{q}$ denotes Caputo fractional derivative with $1 < q \leq 2$. Very recently, Chen (see[3]) considered the existence of three positive solutions to three-point boundary value problem of the following fractional differential equation

$$\begin{cases} D_{0+}^{q}u(t) + f(t, u(t)) = 0,\ 0 < t < 1, \\ u(0) = 0,\ D_{0+}^{p}u(t)\mid_{t=1} = \alpha D_{0+}^{p}u(t)\mid_{t=\xi}, \end{cases}$$

where $1 < q \leq 2$, $0 < p < 1$, $1 + p \leq q$, and D_{0+}^{q} is the R-L fractional order derivative. The multiplicity results of positive solutions to the equations are obtained by using the well-known Leggett-Williams fixed-point theorem on a convex cone. The other excellent studies of fractional dif-

ferential equations can be founded in [4,5,6,7,8].
Motivated by the paper mentioned above, we study the existence of positive solution to two-point BVP of nonlinear fractional equation

$$\begin{cases} D_{0+}^{q}u(t)+\lambda f(t,u(t))=0,\ 0<t<1, \\ u(0)=\ D_{0+}^{p}u(t)\mid_{t=0}=D_{0+}^{p}u(t)\mid_{t=1}=0, \end{cases} \tag{1.1}$$

where $q,p\in R$, $2<q\leq 3$, $1<p\leq 2$, $1+p\leq q$, D_{0+}^{q} is the R-L fractional order derivative, and $f\in C([0,1]\times[0,\infty),[0,\infty))$, $\lambda>0$. By using Krasnoselskii's fixed point theorem, the positive solution to the equations (1) is obtained.

2 Preliminaries

In this section, we present some definitions and preliminary results.

Definition 2.1 *(see equation (2.1.1) in [4]) The R-L fractional integrals $I_{0+}^{p}f$ of order $p\in R$ $(p>0)$ is defined by*

$$I_{0+}^{p}f(x):=\frac{1}{\Gamma(p)}\int_0^x\frac{f(t)dt}{(x-t)^{1-p}},\quad (x>0).$$

Here $\Gamma(p)$ is the Gamma function.

Definition 2.2 *(see equation (2.1.5) in [4]) The R-L fractional derivative $D_{0+}^{p}f$ of order $p\in R$ $(p>0)$ is defined by*

$$\begin{aligned} D_{0+}^{p}f(x)=&(\frac{d}{dx})^nI_{0+}^{n-p}f(x)\\ =&\frac{1}{\Gamma(n-p)}(\frac{d}{dx})^n\int_0^x\frac{f(t)dt}{(x-t)^{p-n+1}},\quad (n=[p]+1,\ x>0), \end{aligned}$$

where $[p]$ means the integral part of p.

Lemma 2.3 *(see Lemma 2.4 and property 2.2 in [4]) If* $q_1 > q_2 > 0$*, then, for* $f(x) \in L_p(0,1), (1 \le p \le \infty)$*, the relations*

$$D^{q_2}_{0+} I^{q_1}_{0+} f(x) = I^{q_1 - q_2}_{0+} f(x),$$
$$I^{q_1}_{0+} I^{q_2}_{0+} f(x) = I^{q_1+q_2}_{0+} f(x) \text{ and } D^{q_1}_{0+} I^{q_1}_{0+} f(x) = f(x)$$

hold a.e. on [0,1].

Lemma 2.4 *(see Lemma 2.5 in [4]) Let* $q > 0$*,* $n = [q] + 1$*,* $f(x) \in L_1(0,1)$*, then the equality*

$$I^{q}_{0+} D^{q}_{0+} f(x) = f(x) + \sum_{i=1}^{n} C_i t^{q-n}.$$

Lemma 2.5 *Let* $y \in C[0,1]$*,* $2 < q \le 3$*,* $1 < p \le 2$*,* $1 + p \le q$*, then the problem*

$$D^{q}_{0+} u(t) + y(t) = 0,\ 0 < t < 1, \tag{2.1}$$

subject to the boundary conditions

$$u(0) = \ D^{p}_{0+} u(t) \mid_{t=0} = D^{p}_{0+} u(t) \mid_{t=1} = 0, \tag{2.2}$$

has the unique solution $u(t) = \int_0^1 G(t,s)ds$*, where*

$$G(t,s) = \frac{1}{\Gamma(q)} \begin{cases} t^{q-1}(1-s)^{q-p-1} - (t-s)^{q-1}, & 0 \le s \le t \le 1, \\ t^{q-1}(1-s)^{q-p-1}, & 0 \le t \le s \le 1. \end{cases}$$

And that $G(t,s)$ *has the following properties*
I) $G(t,s) \in C([0,1] \times [0,1])$*, and* $G(t,s) > 0$ *for* $t, s \in (0,1)$ *and*
$\max\limits_{0 \le t \le 1} G(t,s) = G(s,s)$ *where* $s \in (0,1)$*.*
II)There exists a positive function $\varphi \in C((0,1) \times (\tau, +\infty))$ *such that*

$$\min_{\frac{1}{4} \le t \le \frac{3}{4}} G(t,s) = \varphi(s) \widetilde{G}(s,s) \ge \inf_{0<s<1} \varphi(s) \max_{0 \le t \le 1} G(t,s) = \tau G(s,s),$$

where

$$\widetilde{G}(s,s) = \frac{s^{q-p}(1-s)^{q-p-1}}{\Gamma(q)}, \quad s, \tau \in (0,1), \tau = \inf_{0<s<1} \varphi(s).$$

Proof.. Applying the operator I_{0+}^{q} to both sides of the equation (2), and using Lemma 2, we have

$$u(t) = -I_{0+}^{q} y(t) + C_1 t^{q-1} + C_2 t^{q-2} + C_3 t^{q-3}. \tag{2.3}$$

In view of the boundary condition $u(0) = 0$, we find that $C_3 = 0$, hence

$$u(t) = -I_{0+}^{q} y(t) + C_1 t^{q-1} + C_2 t^{q-2},$$

then, noting the relation $D_{0+}^{q_2} I_{0+}^{q_1} f(x) = I_{0+}^{q_1 - q_2} f(x)$ in Lemma 1, we obtain

$$D_{0+}^{p} u(t) = -I_{0+}^{q-p} y(t) + C_1 \frac{\Gamma(q)}{\Gamma(q-p)} t^{q-p-1} + C_2 \frac{\Gamma(q-1)}{\Gamma(q-p-1)} t^{q-p-2},$$

in accordance with the equations (3) ,we can calculate out that

$$C_1 = \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-p-1} y(s) ds, \ C_2 = 0.$$

Substituting the values of C_1,C_2 and C_3 in (4), we have

$$\begin{aligned}
u(t) =& -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds + \frac{t^{q-1}}{\Gamma(q)} \int_0^1 (1-s)^{q-p-1} y(s) ds \\
=& \frac{1}{\Gamma(q)} \{ \int_0^t [t^{q-1}(1-s)^{q-p-1} - (t-s)^{q-1}] y(s) ds \\
&+ \int_t^1 [t^{q-1}(1-s)^{q-p-1}] y(s) ds \} \\
=& \int_0^1 G(t,s) y(s) ds.
\end{aligned}$$

Next we prove the properties of $G(t,s)$.
For a given $s \in (0,1)$, $G(t,s)$ is decreasing with respect to t for $s \leq t$ while increasing for $t \leq s$. Thus, we have

$$\max_{0 \leq t \leq 1} G(t,s) = G(s,s) = \frac{s^{q-1}(1-s)^{q-p-1}}{\Gamma(q)} \leq \frac{s^{q-p}(1-s)^{q-p-1}}{\Gamma(q)} = \widetilde{G}(s,s),$$

for $s \in (0,1)$. Then we set

$$g_1(t,s) = \frac{t^{q-1}(1-s)^{q-p-1} - (t-s)^{q-1}}{\Gamma(q)}, \quad g_2(t,s) = \frac{t^{q-1}(1-s)^{q-p-1}}{\Gamma(q)},$$

from the two equation above we have

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} G\left(t,s\right)=\frac{1}{\Gamma(q)}\begin{cases}0.75^{q-1}(1-s)^{q-p-1}-(0.75-s)^{q-1}, & 0<s\leq r,\\ 0.25^{q-1}(1-s)^{q-p-1}, & r\leq s<1,\end{cases}$$

where $\frac{1}{4}<r<\frac{3}{4}$ is the unique solution of the equation

$$0.75^{q-1}(1-s)^{q-p-1}-(0.75-s)^{q-1}=0.25^{q-1}(1-s)^{q-p-1}.$$

Finally, we consider a function $\varphi(s)$ defined by

$$\varphi(s)=\frac{\min\limits_{\frac{1}{4}\leq t\leq\frac{3}{4}} G\left(t,s\right)}{\widetilde{G}(s,s)}=\begin{cases}\frac{0.75^{q-1}(1-s)^{q-p-1}-(0.75-s)^{q-1}}{s^{q-p}(1-s)^{q-p-1}}, & 0<s\leq r,\\ \frac{0.25^{q-1}}{s^{q-p}}, & r\leq s<1.\end{cases}$$

When $q>p-1$ we find from the continuity of $\varphi(s)$ and $\lim\limits_{s\to 0^+}\varphi\left(s\right)=+\infty$ that there exists $\tilde{r}$ small enough such that $\varphi'(s)<0$ for $s\in(0,\tilde{r}]$, hence, we set

$$0<\tau=\inf_{0<s<1}\varphi\left(s\right)=min\{\varphi(\tilde{r}),m,\frac{1}{4^{q-1}}\}<1,$$

here, $m=\min\limits_{\tilde{r}\leq s\leq r}\varphi\left(s\right)$.
When $q=p-1$, we have $\lim\limits_{s\to 0^+}\varphi\left(s\right)=\frac{4}{3}(q-1)$, then we set

$$0<\tau=\inf_{0<s<1}\varphi\left(s\right)=min\{\inf_{0<s\leq r}\varphi\left(s\right),\frac{4}{3}(q-1),\frac{1}{4^{q-1}}\}<1.$$

Thus,

$$\min_{\frac{1}{4}\leq t\leq\frac{3}{4}} G\left(t,s\right)\geq\varphi(s)\widetilde{G}(s,s)\geq\inf_{0<s<1}\varphi\left(s\right)\max_{0\leq t\leq 1} G\left(t,s\right)=\tau G(s,s).$$

This completes the proof. Therefore, the solution $u\in C_{[0,1]}$ of the problem (1) can be written by

$$u(t)=\lambda\int_0^1 G(t,s)f(s,u(s))ds.$$

Lemma 2.6 *(see[9]) Let E be a Banach space and $P \subset E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $A : P \cap (\overline{\Omega_2} \backslash \Omega_1) \to P$ be a completely continuous operator. In addition suppose either*
(1) $\|Au\| \leq \|u\|$, $\forall u \in P \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|$, $\forall u \in P \cap \partial\Omega_2$ or
(2) $\|Au\| \leq \|u\|$, $\forall u \in P \cap \partial\Omega_2$ and $\|Au\| \geq \|u\|$, $\forall u \in P \cap \partial\Omega_1$
holds. Then A has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.
Define P to be a cone in $C_{[0,1]}$(with norm $\|u\| = \max\limits_{0 \leq t \leq 1} |u\,(t)\,|$) by

$$P = \{u \in C_{[0,1]} \mid u(t) \geq 0, t \in [0,1] \textit{ and } \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} u\,(t) \geq \tau \|u\|\},$$

and the operator $A : P \to C_{[0,1]}$ by

$$Au(t) = \lambda \int_0^1 G(t,s) f(s, u(s)) ds. \tag{2.4}$$

Lemma 2.7 *If A is defined by (5), then $A : P \to P$ is completely continuous.*
***Proof.**. First, assume that $f \in C([0,1] \times [0,\infty), [0,\infty))$, $u \in P$, and from Lemma 3, we have*

$$\begin{aligned}
\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} Au\,(t) &= \min_{\frac{1}{4} \leq t \leq \frac{3}{4}} \lambda \int_0^1 G(t,s) f(s,u(s)) ds \\
&\geq \max_{0 \leq t \leq 1} \lambda \int_0^1 \inf_{0<s<1} \varphi\,(s)\, G(t,s) f(s,u(s)) ds \\
&= \tau \max_{0 \leq t \leq 1} \lambda \int_0^1 G(t,s) f(s,u(s)) ds \\
&= \tau \|Au(t)\|,
\end{aligned}$$

thus $A : P \to P$.
Second, $\forall N > 0$, Let $\Omega = \{\Omega \subset P : \|u\| \leq N, u \in \Omega\}$ and $M = \max\limits_{(t,u) \in [0,1] \times [0,N]} f(t, u(t))$, and noting the property (II) of $G(t,s)$, we can easily obtain $A(\Omega)$ is bounded.

Third, for each $u \in \Omega$, let $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$, then we have

$$\begin{aligned}
|Au(t_2) - Au(t_1)| =& \lambda \Big| \int_0^1 G(t_2, s) f(s, u(s)) ds - \int_0^1 G(t_1, s) f(s, u(s)) ds \Big| \\
=& \frac{\lambda}{\Gamma(q)} \Big| \int_0^{t_1} \{[t_2^{q-1}(1-s)^{q-p-1} - (t_2 - s)^{q-1}] \\
& - [t_1^{q-1}(1-s)^{q-p-1} - (t_1 - s)^{q-1}]\} f(s, u(s)) ds \\
& + \int_{t_1}^{t_2} \{[t_2^{q-1}(1-s)^{q-p-1} - (t_2 - s)^{q-1}] \\
& - t_1^{q-1}(1-s)^{q-p-1}\} f(s, u(s)) ds \\
& + \int_{t_2}^{1} [t_2^{q-1}(1-s)^{q-p-1} - t_1^{q-1}(1-s)^{q-p-1}] f(s, u(s)) ds \Big| \\
<& \frac{\lambda}{\Gamma(q)} \int_0^1 (t_2^{q-1} - t_1^{q-1})(1-s)^{q-p-1} f(s, u(s)) ds \\
<& \frac{\lambda M}{\Gamma(q)} (t_2^{q-1} - t_1^{q-1}) \int_0^1 (1-s)^{q-p-1} ds \\
=& \frac{\lambda M}{\Gamma(q)(q-p)} (t_2^{q-1} - t_1^{q-1}) \\
=& \frac{\lambda M (q-1)}{\Gamma(q)(q-p)} [t_1 + \theta(t_2 - t_1)]^{q-2} (t_2 - t_1), \quad (0 < \theta < 1) \\
<& \frac{\lambda M (q-1)}{\Gamma(q)(q-p)} 2^{q-2} (t_2 - t_1).
\end{aligned}$$

Thus, $\forall \varepsilon > 0$, $\exists \delta = \varepsilon \frac{\Gamma(q)(q-p)}{2^{q-2}\lambda M(q-1)}$, we have $|Au(t_2) - Au(t_1)| < \varepsilon$ for $t_2 - t_1 < \delta$. Therefore, $A(\Omega)$ is equivalent-continuous, so the Arzela-Ascoli theorem implies that the operator $A : P \to P$ is completely continuous. This completes the proof.

3 Main Results

In this section, we study the existence of the positive solution to BVP of equations (1). Suppose (H_1) $\lim\limits_{u \to 0+} \sup\limits_{0 \le t \le 1} \dfrac{f(t,u)}{u} = 0$, (H_2) $\lim\limits_{u \to +\infty} \inf\limits_{0 \le t \le 1} \dfrac{f(t,u)}{u} = +\infty$,

(H_3) $\lim_{u\to 0+}\inf_{0\le t\le 1}\dfrac{f(t,u)}{u}=+\infty$, (H_4) $\lim_{u\to +\infty}\sup_{0\le t\le 1}\dfrac{f(t,u)}{u}=0$.

Theorem 3.1 *If (H_1) and (H_2) hold, then for all $\lambda>0$, the equations (1) have a positive solution.*

Theorem 3.2 *If (H_3) and (H_4) hold, then for all $\lambda>0$, the equations (1) have a positive solution.*

The Proof of Theorem 1. From (H_1), there exists $L_1\in(0,1)$ such that $f(t,u)\le\eta_1 u$ for $(t,u)\in[0,1]\times(0,L_1]$, where $\eta_1>0$ satisfying $\lambda\eta_1\int_0^1 G(s,s)ds\le 1$. Then let $\Omega_1=\{u\in P:\|u\|<L_1\}$, $\partial\Omega_1=\{u\in P:\|u\|=L_1\}$, for $u\in\partial\Omega_1$, we have

$$\begin{aligned}Au(t)&=\lambda\int_0^1 G(t,s)f(s,u(s))ds\\&\le\lambda\max_{0\le t\le 1}\int_0^1 G(t,s)f(s,u(s))ds\\&\le\lambda\eta_1\int_0^1 G(s,s)u(s)ds\\&\le\lambda\eta_1\int_0^1 G(s,s)ds\|u\|\le\|u\|,\end{aligned}$$

which implies that

$$\|Au\|\le\|u\|,\ \ for\ u\in\partial\Omega_1. \tag{3.1}$$

On the other hand, from (H_2), there exists $L_2>L_1$ such that $f(t,u)\ge\eta_2 u$ for $(t,u)\in[0,1]\times[L_2,\infty)$, where $\eta_2>0$ satisfying $\lambda\eta_2\tau^2\int_{\frac14}^{\frac34}G(s,s)ds\ge 1$. Then let $\Omega_2=\{u\in P:\|u\|<L_2\}$, $\partial\Omega_2=\{u\in P:\|u\|=L_2\}$, for

$u \in \partial\Omega_2$, we have

$$\begin{aligned}\min_{\frac{1}{4}\le t\le\frac{3}{4}} Au\,(t) =& \min_{\frac{1}{4}\le t\le\frac{3}{4}} \lambda \int_0^1 G(t,s)f(s,u(s))ds \\ \geq& \lambda\tau \int_0^1 G(s,s)f(s,u(s))ds \\ \geq& \lambda\tau\eta_2 \int_0^1 G(s,s)u(s)ds \\ \geq& \lambda\tau\eta_2 \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)u(s)ds \\ \geq& \lambda\tau^2\eta_2 \int_{\frac{1}{4}}^{\frac{3}{4}} G(s,s)ds\|u\| \geq \|u\|,\end{aligned}$$

which implies that

$$\|Au\| \geq \|u\|, \ for\ u \in \partial\Omega_2. \tag{3.2}$$

Then from (6), (7) and Lemma 4, the operator A has a fixed point in $P \cap (\overline{\Omega_2}\backslash\Omega_1)$.

The Proof of Theorem 2. By the similar method of the proof of Theorem 1, we can easily obtain $\Omega_3 = \{u \in P : \|u\| < L_3\}$, $\partial\Omega_3 = \{u \in P : \|u\| = L_3\}$, and $\Omega_4 = \{u \in P : \|u\| < L_4\}$, $\partial\Omega_4 = \{u \in P : \|u\| = L_4\}$, and satisfying

$$\|Au\| \geq \|u\|, \ for\ u \in \partial\Omega_3, \tag{3.3}$$

and

$$\|Au\| \leq \|u\|, \ for\ u \in \partial\Omega_4, \tag{3.4}$$

respectively. Then from (8), (9) and Lemma 4, we obtain a fixed point of operator A in $P \cap (\overline{\Omega_4}\backslash\Omega_3)$.

4 Example

We consider the following problem

$$\begin{cases} D_{0+}^{\frac{5}{2}}u(t) + (t+1)u^2 = 0, \ 0 < t < 1, \\ u(0) = \ D_{0+}^{\frac{3}{2}}u(t) \mid_{t=0} = D_{0+}^{\frac{3}{2}}u(t) \mid_{t=1} = 0. \end{cases} \tag{4.1}$$

Then $f(t,u)=(t+1)u^2$, $\lambda=1$, and $\lim\limits_{u\to 0+}\sup\limits_{0\le t\le 1}\dfrac{(t+1)u^2}{u}=0$, $\lim\limits_{u\to+\infty}\inf\limits_{0\le t\le 1}\dfrac{(t+1)u^2}{u}=\infty$, so the condition (H_1) and (H_2) hold. On the other hand, substituting the equations $q=\frac{5}{2}$ and $p=\frac{3}{2}$ in $G(t,s)$ and $\varphi(s)$, we have

$$G(t,s)=\frac{1}{\Gamma(\frac{5}{2})}\begin{cases} t^{\frac{3}{2}}-(t-s)^{\frac{3}{2}}, & 0\le s\le t\le 1,\\ t^{\frac{3}{2}}, & 0\le t\le s\le 1,\end{cases}$$

and

$$\varphi(s)=\begin{cases} \frac{(0.75)^{\frac{3}{2}}-(0.75-s)^{\frac{3}{2}}}{s}, & 0<s\le r,\\ \frac{(0.25)^{\frac{3}{2}}}{s}, & r\le s<1,\end{cases}$$

where r is the unique solution of the equation

$$(0.75)^{\frac{3}{2}}-(0.75-s)^{\frac{3}{2}}=(0.25)^{\frac{3}{2}}.$$

By calculating the minimum of $\varphi(s)$, we obtain $\tau=\frac{1}{8}$. Thus, we set $L_1=\frac{1}{2}$, $\eta_1=2\le\frac{1}{\int_0^1 G(s,s)ds}=\frac{5}{2}\Gamma(\frac{5}{2})$, then $f(t,u)=(t+1)u^2\le 2u^2\le\eta_1 u$, for $(t,u)\in[0,1]\times[0,L_1]$. Therefore, we derive

$$\Omega_1=\{u\in P:\|u\|<\frac{1}{2}\}. \tag{4.2}$$

Next we set $L_2=\frac{5120\Gamma(\frac{5}{2})}{3^{\frac{5}{2}}-1}$, $\eta_2\ge\frac{1}{\tau^2\int_{0.25}^{0.75}G(s,s)ds}=\frac{5120\Gamma(\frac{5}{2})}{3^{\frac{5}{2}}-1}$, then $f(t,u)=(t+1)u^2\ge u^2\ge\eta_2 u$, for $(t,u)\in[0,1]\times[L_2,+\infty)$. Therefore, we derive

$$\Omega_2=\{u\in P:\|u\|<\frac{5120\Gamma(\frac{5}{2})}{3^{\frac{5}{2}}-1}\}. \tag{4.3}$$

According to (11) and (12), from Theorem 1, we obtain a positive solution u of (10) such that $\frac{1}{2}\le\|u\|\le\frac{5120\Gamma(\frac{5}{2})}{3^{\frac{5}{2}}-1}$.

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