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The new implicit finite difference method for the solution of time fractional advection-dispersion equation

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Abstract

In this paper, a numerical solution of time fractional advection-dispersion equations are presented. The new implicit finite difference methods for solving these equations are studied. We examine practical numerical methods to solve a class of initial-boundary value fractional partial differential equations with variable coefficients on a finite domain. Stability, consistency, and (therefore) convergence of the method are examined and the local truncation error is $O(\Delta t + h)$. This study concerns both theoretical and numerical aspects, where we deal with the construction and convergence analysis of the discretization schemes. The results are justified by some numerical implementations. A numerical example with known exact solution is also presented, and the behavior of the error is examined to verify the order of convergence.

Key words: Implicit finite difference approximation; Stability analysis; Convergence; Fractional derivative; Time Fractional Advection-Dispersion Equation.

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1 Introduction

Fractional order partial differential equations are generalizations of classical partial differential equations. We consider the initial-boundary value problem to a time-dependent time fractional advection-dispersion equation:

$$\begin{aligned} &\tfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \tfrac{\partial u(x,t)}{\partial x} - \tfrac{\partial^2 u(x,t)}{\partial x^2} = f(x,t)\,, \\ &(x,\ t) \in [x_L,\ x_R\,] \times [0,\ T\,],\ \ 0 < \alpha < 1\,. \end{aligned} \tag{1.1}$$

We also assume an initial condition $u(x,0) = u_0(x)$ for $x_L < x < x_R$ and a natural set of boundary conditions for this problem: $u(x_L,t) = 0$ for all $t \geq 0$ and $u(x_R,t) = 0$ for all $t \geq 0$.
Where $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is Caputo fractional order derivative [1,4,5,8]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^T \frac{\partial u(x,\ s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \tag{1.2}$$

where $\Gamma(.)$ is the Gamma function.

Assume that this time fractional advection-dispersion equation has a unique and sufficiently smooth solution.

2 Structure of the new Scheme

To establish the numerical approximation scheme, let $t_k = k\,\Delta t$, $(k = 0, 1, ..., M)$ to be the integration time $0 \leq t_k \leq T$, and $\Delta x = h > 0$ to be the grid size in x-direction, $\Delta x = \dfrac{(x_R - x_L)}{N}$, with $x_i = x_L + i\,h$ for $i = 0, ..., N$. Define $u_i^m = u(x_i, t^m)$, and $f_i^m = f(x_i, t^m)$. Let U_i^m denote the numerical approximation to the exact solution u_i^m. a usual, we take the following finite difference approximation for time fractional derivative appeared in problem 1.1:

$$\begin{aligned}\frac{\partial^\alpha u(x_i,t^{k+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x_i,t^{j+1})-u(x_i,t^j)}{\Delta t}\\ &\times\int_{j\Delta t}^{(j+1)\Delta t}\frac{ds}{(t^{k+1}-s)^\alpha}+O(\Delta t),\end{aligned}\tag{2.1}$$

Therefore, we write

$$\begin{aligned}\frac{\partial^\alpha u(x_i,t^{k+1})}{\partial t^\alpha} &= \frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{u(x_i,t^{k-j+1})-u(x_i,t^{k-j})}{\Delta t}\\ &\times[(j+1)^{1-\alpha}-j^{1-\alpha}]+O(\Delta t),\end{aligned}\tag{2.2}$$

On the other hand, we have

$$\frac{\partial u(x_i,\,t^{k+1})}{\partial x} = \frac{u(x_i,\,t^{k+1})-u(x_{i-1},\,t^{k+1})}{h}+O(h),\tag{2.3}$$

$$\frac{\partial^2 u(x_i,\,t^{k+1})}{\partial x^2} = \frac{u(x_{i+1},\,t^{k+1})-2u(x_i,\,t^{k+1})+u(x_{i-1},\,t^{k+1})}{h^2}+O(h^2),\tag{2.4}$$

Apply (2.2)- (2.4) to (1.1) We have

$$\begin{aligned}&\frac{\Delta t^{1-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{u(x_i,t^{k-j+1})-u(x_i,t^{k-j})}{\Delta t}[(j+1)^{1-\alpha}-j^{1-\alpha}] =\\ &\frac{u(x_{i+1},t^{k+1})-2u(x_i,t^{k+1})+u(x_{i-1},t^{k+1})}{h^2}-\frac{u(x_i,t^{k+1})-u(x_{i-1},t^{k+1})}{h}\\ &+f_i^{k+1}+O(\Delta t+h),\end{aligned}\tag{2.5}$$

Also, we define $\theta_j = (j+1)^{1-\alpha} - j^{1-\alpha}$ for $j = 0,1,2,...,M$ and $p_i = \frac{\Delta t^\alpha\,\Gamma(2-\alpha)}{h^2}$, $r_i = \frac{\Delta t^\alpha\,\Gamma(2-\alpha)}{h}$ Then we have

$$\begin{aligned}\sum_{j=0}^{k}\theta_j(u_i^{k-j+1}-u_i^{k-j}) = p_i\,(u_{i+1}^{k+1}-2u_i^{k+1}+u_{i-1}^{k+1})-r_i\,(u_i^{k+1}+u_{i-1}^{k+1})\\ +\Delta t^\alpha\,\Gamma(2-\alpha)\,f_i^{k+1},\end{aligned}\tag{2.6}$$

Since the local truncation error is $O(\Delta t\ +\ h)$, therefore this method is consistent [9]. We can also written as matrix form:
If $k = 0$,

$$\theta_0(u_i^1 - u_i^0) = p_i(u_{i+1}^1 - 2u_i^1 + u_{i-1}^1) - r_i(u_i^1 + u_{i-1}^1) + \Delta t^\alpha \Gamma(2-\alpha) f_i^1,$$

Where

$$-p_i u_{i+1}^1 + (1+2p_i+r_i)u_i^1 + (-r_i - p_i)u_{i-1}^1 = u_i^0 + \Delta t^\alpha \Gamma(2-\alpha) f_i^1, \quad (2.7)$$

If $k\ > 0$,

$$\begin{aligned}
&- p_i\, u_{i+1}^{k+1}\ + (1 + 2p_i + r_i)\, u_i^{k+1} + (-r_i - p_i)u_{i-1}^{k+1} = \\
&u_i^k + \Delta t^\alpha\, \Gamma(2 -\ \alpha)\, f_i^{k+1}\ + \textstyle\sum_{j=1}^{k}\ \theta_j(u_i^{k-j+1} - u_i^{k-j}) \\
&= (2 - 2^{1-\alpha})u_i^k + \textstyle\sum_{j=1}^{k-1}\ u_i^{k-j}[2(j+1)^{1-\alpha} \qquad (2.8) \\
&-(j+2)^{1-\alpha} - j^{1-\alpha}] + \theta_k u_i^0 + \Delta t^\alpha\, \Gamma(2 -\ \alpha)\, f_i^{k+1} \\
&(i = 1, 2, ..., N - 1\, , k = 1, 2, ..., M - 1).
\end{aligned}$$

Eqs. (2.7) and (2.8) can also written as matrix form:

$$\begin{cases}
\mathbf{A}\mathbf{U}^1 = \mathbf{U}^0 + \Delta t^\alpha\, \Gamma(2 -\ \alpha)\, \mathbf{F}^1 \qquad , \\
\mathbf{A}\mathbf{U}^{k+1} = d_1\mathbf{U}^k + ... + d_k\mathbf{U}^1 + \theta_k\mathbf{U}^0 + \Delta t^\alpha\, \Gamma(2 -\ \alpha)\, \mathbf{F}^{k+1}, k > 0, \\
\mathbf{U}^0 = u_0
\end{cases} \qquad (2.9)$$

And matrix $\mathbf{A} = [\mathbf{A}_{i,j}]_{(N-1)\times(N-1)}$is defined as follows:

$$\mathbf{A}_{i,j}\ = \begin{cases}
-p_i & for\ j = i + 1 \\
1 + 2p_i + r_i & for\ j = i \\
-p_i - r_i & for\ j = i - 1 \\
0 & for\ j < i - 1 \\
0 & for\ j > i + 1
\end{cases} \qquad (2.10)$$

3 Stability and convergence

Theorem 3.1 *The implicit system defined by the linear difference equations (2.7) and (2.8) has unique solution and unconditionally stable for all* $0 < \alpha < 1$ *.*

Proof. we will apply Gerschgorin theorem to conclude that every eigenvalue of the matrix has a magnitude strictly large than 1.
According to the Gerschgorin theorem, the eigenvalue of the matrix $\mathbf{A}$ are in the disks centered at $\mathbf{A}_{i,j} = 1 + 2p_i + r_i$, with radius

$$\mathbf{R}_i = \sum_{j=1\,,\,j\neq i}^{N-1} \mid \mathbf{A}_{i,j} \mid = \mid -p_i \mid + \mid -p_i - r_i \mid = +2p_i + r_i \ ,$$

Hence every eigenvalue λ of the matrix $\mathbf{A}$ has a real part larg than 1 , and therefore a magnitude larger than 1. So the spectral radius of $\mathbf{A}^{-1}$ is less than one. This proves that the scheme has unique solution.To prove the unconditional stability of (2.7) and (2.8), let $u_i^k, \tilde{u}_i^k, (i = 1, 2, ..., N-1, k = 1, 2, ..., M-1)$ be the solution of (2.7) and (2.8) with initial value and respectively, the computation of $f_i^k, (i = 1, 2, ..., N-1, k = 1, 2, ..., M-1)$ is exact. Then error $\varepsilon_i^k = \tilde{u}_i^k - u_i^k$ satisfies:
If $k = 0$,

$$-\,p_i\,\varepsilon_{i+1}^1 + (1 + 2p_i + r_i)\,\varepsilon_i^1 + (-r_i - p_i)\varepsilon_{i-1}^1 = \varepsilon_i^0 \ , \tag{3.1}$$

If $k > 0$,

$$\begin{aligned} &-\,p_i\,\varepsilon_{i+1}^{k+1} + (1 + 2p_i + r_i)\,\varepsilon_i^{k+1} + (-r_i - p_i)\varepsilon_{i-1}^{k+1} = \\ &d_1\varepsilon_i^k + \textstyle\sum_{j=1}^{k-1} d_{j+1}\varepsilon_i^{k-j} + \theta_k\varepsilon_0 \ , \end{aligned} \tag{3.2}$$

It is equivalent to the following matrix form:

$$\begin{cases} \mathbf{AE}^1 = \mathbf{E}^0 \quad , \\ \\ \mathbf{AE}^{k+1} = d_1\mathbf{E}^k + ... + d_k\mathbf{E}^1 + \theta_k\mathbf{E}^0, \ k > 0, \\ \end{cases}$$

Where $\mathbf{E}^k = [\varepsilon_1^k, \varepsilon_2^k, ..., \varepsilon_{N-1}^k]^T$. Let us use mathematical induction method to prove:

$$\|\mathbf{E}^k\|_\infty \ \leq \ \|\mathbf{E}^0\|_\infty \ , \ k = 1, 2, ...$$

In fact, if $k \ = 1$, suppose $|\varepsilon_l^1| = \max_{1\leq i\leq N-1} |\varepsilon_i^1|$, note that p_i , $r_i > 0$ we have:

$$\|\mathbf{E}^1\|_\infty \ = |\varepsilon_l^1| \leq |\varepsilon_l^1| + r_l(|\varepsilon_l^1| - |\varepsilon_{l-1}^1|) + p_i(|\varepsilon_l^1| - |\varepsilon_{l-1}^1|) \leq$$

$$- p_l \,|\varepsilon_{l+1}^1| \ + (1 + 2p_l + r_l)\, |\varepsilon_l^1| + (-r_l - p_l)|\varepsilon_{l-1}^1| \leq$$
$$| - \ p_l \, \varepsilon_{l+1}^1 \ + (1 + 2p_l + r_l) \, \varepsilon_l^1 + (-r_l - p_l)\varepsilon_{l-1}^1| \ = |\varepsilon_l^0| \leq \|\mathbf{E}^0\|_\infty \, .$$

Therefore $\|\mathbf{E}^1\|_\infty \ \leq \|\mathbf{E}^0\|_\infty$.
Suppose if $k \ \leq \ s$, $\|\mathbf{E}^s\|_\infty \ \leq \ \|\mathbf{E}^0\|_\infty$ hold, then when $k \ = \ s+1$, let $|\varepsilon_l^{s+1}| \ = \max_{1\leq i\leq N-1} |\varepsilon_i^{s+1}|$,

similar to former estimate, we have:

$$
\begin{aligned}
\|\mathbf{E}^{s+1}\|_\infty &= |\varepsilon_l^{s+1}| \\
&\le |\varepsilon_l^{s+1}| + r_l(|\varepsilon_l^{s+1}| - |\varepsilon_{l-1}^{s+1}|) \\
&\quad + p_l(|\varepsilon_l^{s+1}| - |\varepsilon_{l-1}^{s+1}|) \\
&\le -\, p_l\,|\varepsilon_{l+1}^{s+1}| \ + (1+2p_l+r_l)\,|\varepsilon_l^{s+1}| \\
&\quad + (-r_l-p_l)|\varepsilon_{l-1}^{s+1}| \\
&\le |-\ p_l\,\varepsilon_{l+1}^{s+1} \ + (1+2p_l+r_l)\,\varepsilon_l^{s+1} \\
&\quad + (-r_l-p_l)\varepsilon_{l-1}^{s+1}| \\
&\le \|\mathbf{A}\mathbf{E}^{s+1}\|_\infty \\
&\le d_1|\varepsilon_l^s| + \textstyle\sum_{j=1}^{s-1} d_{j+1}|\varepsilon^{s-j}| \\
&\quad + \theta_s|\varepsilon_l^0| \\
&\le d_1\|\mathbf{E}^s\|_\infty + \textstyle\sum_{j=1}^{s-1} d_{j+1}\|\mathbf{E}^{s-j}\|_\infty \\
&\quad + \theta_s\|\mathbf{E}^0\|_\infty \\
&\le (d_1 + \textstyle\sum_{j=1}^{s-1} d_{j+1} + \theta_s\,)\|\mathbf{E}^0\|_\infty \\
&= \ \|\mathbf{E}^0\|_\infty.
\end{aligned}
$$

Therefore $\|\mathbf{E}^{s+1}\|_\infty \ \le \ \|\mathbf{E}^0\|_\infty$. So the implicit scheme defined by the linear difference equations (2.7) and (2.8) is unconditionally stable and finished the proof of Theorem 3.1.

Theorem 3.2 *Suppose that $u(x_i,t_k)$ is the exact solution of (1.1) at grid point (x_i,t_k), u_i^k is the difference solution of (2.7) and (2.8), then there exists positive $\boldsymbol{M}$, such that*

$$\|e^k\|_\infty \ \le \theta_{k-1}^{-1}\boldsymbol{M}(\Delta t^{1+\alpha} \ + \ \Delta t^\alpha h)\,, (k=1,2,...,M) \tag{3.3}$$

where $\|e^k\|_\infty = \max_{1\le i\le N-1}|e_i^k|$ and $\boldsymbol{M}$ is a constant independent of $h, \Delta t$.

Proof. Since $u_i^k = u(x_i,t_k) - e_i^k$, notice that $e^0=0$, We have from (2.7) and (2.8):

If $k = 0$,

$$- p_i\, e^1_{i+1} \ + (1 + 2p_i + r_i)\, e^1_i + (-r_i - p_i) e^1_{i-1} = \ R^1_i \ ,$$

If $k \ > 0$,

$$- p_i\, e^{k+1}_{i+1} \ + (1 + 2p_i + r_i)\, e^{k+1}_i + (-r_i - p_i) e^{k+1}_{i-1} =$$

$$d_1 e^k_i + \sum_{j=1}^{k-1} d_{j+1} e^{k-j}_i + R^{k+1}_i \ ,$$

Where $|R^{k+1}_i| \leq \mathbf{M}(\Delta t^{1+\alpha} + \Delta t^{\alpha} h), (k = 1, 2, ..., M-1, i = 1, 2, ..., N-1)$ and $\mathbf{M}$ is a constant independent of $h, \Delta t$.
Lets use mathematical induction method to prove the theorem.
if $k \ = 1$, suppose $\|e^1\|_\infty \ = |e^1_l| = \max_{1\leq i \leq N-1} |e^1_i|$, we have:

$$\|e^1\|_\infty \ = |e^1_l| \leq$$

$$- p_l\, |e^1_{l+1}| \ + (1 + 2p_l + r_l)\, |e^1_l| + (-r_l - p_l)|e^1_{l-1}| \leq$$
$$|- \ p_l\, e^1_{l+1} \ + (1 + 2p_l + r_l)\, e^1_l + (-r_l - p_l) e^1_{l-1}| \ = |R^1_l| \leq$$
$$\mathbf{M}(\Delta t^{1+\alpha} \ + \ \Delta t^{\alpha} h) = \theta_0^{-1} \mathbf{M}(\Delta t^{1+\alpha} \ + \ \Delta t^{\alpha} h).$$

Suppose that $k \leq s$, $\|e^s\|_\infty \ \leq \theta^{-1}_{s-1} \mathbf{M}(\Delta t^{1+\alpha} \ + \ \Delta t^{\alpha} h)$ hold,
then when $k \ = \ s + 1$, let $|e^{s+1}_l| \ = \ \max_{1\leq i \leq N-1} |e^{s+1}_i|$, notice that $\theta_j^{-1} \leq \theta_k^{-1}$, $j = 0, 1, 2, ..., k$.

therefore

$$\|e^{s+1}\|_\infty \ = |e_l^{s+1}| \le$$

$$d_1\|e^s\|_\infty + \sum_{j=1}^{s-1} d_{j+1}\|e^{s-j}\|_\infty + \mathbf{M}(\Delta t^{1+\alpha} \ + \ \Delta t^\alpha h) =$$

$$\sum_{j=0}^{s-1} d_{j+1}\|e^{s-j}\|_\infty + \mathbf{M}(\Delta t^{1+\alpha} \ + \ \Delta t^\alpha h) \le$$

$$(d_1\theta_{s-1}^{-1} + d_2\theta_{s-2}^{-1} + d_3\theta_{s-3}^{-1} + ... + d_s\theta_0^{-1} + 1)\mathbf{M}(\Delta t^{1+\alpha} \ + \ \Delta t^\alpha h) \le$$

$$\theta_s^{-1}(\sum_{j=0}^{s-1} d_j + \theta_s)\mathbf{M}(\Delta t^{1+\alpha} \ + \ \Delta t^\alpha h) = \theta_s^{-1}\mathbf{M}(\Delta t^{1+\alpha} \ + \ \Delta t^\alpha h)$$

Therefore Theorem 3.2 is proved. □
Since

$$\lim_{k\to\infty} \frac{\theta_k^{-1}}{k^\alpha} = lim_{k\to\infty}\frac{k^\alpha}{(k+1)^{1-\alpha} - (k)^{1-\alpha}} = lim_{k\to\infty}\frac{k^{-1}}{(1+\frac{1}{k})^{1-\alpha} - 1} = \frac{1}{1-\alpha},$$

Hence there exists constant $\mathbf{C} > 0$, such that

$$\|e^k\|_\infty \le k^\alpha\mathbf{C}(\Delta t^{1+\alpha} + \Delta t^\alpha h) = (k\Delta t)^\alpha.\mathbf{C}(\Delta t + h), k = 1, 2, ..., M.$$

When $k\Delta t \le T$, We get the following theorem:

Theorem 3.3 *Suppose that $u(x_i, t_k)$ is the exact solution of (1.1) at grid point (x_i, t_k) is implicity difference solution of (2.7) and (2.8), then there exists positive constant $\boldsymbol{C}$, such that*

$$|\ u(x_i, t_k) - u_i^k\ | \le \boldsymbol{C}(\Delta t \ + \ h)\,, (i = 1, 2, ..., N\ , k = 1, 2, ..., M)\,.$$

4 Numerical experiments

In this section, we carry out numerical experiments to investigate the performance and convergence behavior of The implicit finite difference method for time fractional advection-dispersion equation.

Example. We start with the following fractional $\alpha = 0.2$ advection-dispersion equation with
$\alpha = 0.2\,,\ 0 < x < 2\,,\ 0 \le\ t \le 1.$
forcing function

$$f(x,t) \;=\; 2(\frac{t^{2-\alpha}}{\Gamma(3-\alpha)})\,x^2\,(2-x)^2 + (1+t^2)(4x^3 - 24x^2 + 32x - 8)\,,$$

the initial condition

$$u(x,0) \;=\; x^2\;(2-x)^2\;,$$

and the boundary condition

$$u(0,t) \;=\; u(2,t) \;=\; 0\;,$$

The exact solution of this fractional advection-dispersion flow equation is given by

$$u(x,t) \;=\; x^2\;(2-x)^2(1+t^2)\;,$$

We have shown the exact and numerical solutions with $\alpha = 0.2$ in figure 1.

5 Conclusion and Suggestions

We have developed the implicit finite difference method, for solving the fractional partial differential equation. For this method, we drive convergence rates. However, focusing on theoretical aspects, we do not deal with couple equations in this paper. We plan to address this issue and some other approaches in a forthcoming paper.

Table 1
Maximum error behavior for versus grid size reduction for the Example problem with $\alpha = 0.2$ at time $t = 1.0$.

M	N	$Maximum Error$	$Error\ rate$
10	10	0.248776	–
20	20	0.143465	1.73
40	40	0.077287	1.86
80	80	0.0399791	1.93
160	160	0.020316	1.97
320	320	0.010232	1.98

Fig. 1. Numerical solutions and exact solution at time $t = 1.0$. The solid line corresponds to the exact solution, the stared line corresponds to numerical solution of The implicit finite difference method with $\alpha = 0.2$, $\Delta t = \frac{1}{80}$ and $h = \frac{1}{40}$.

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