Theory of Approximation and Applications

Vol. 10, No. 2, (2016), 61-73

Analytical solution of the Hunter-Saxton equation using the reduced differential transform method

H. Rouhparvar *

Department of Mathematics, College of Technical and Engineering, Saveh Branch, Islamic Azad University, Saveh, Iran.

Received 19 July 2015; accepted 27 October 2015

Abstract

In this paper, the reduced differential transform method is investigated for a nonlinear partial differential equation modeling nematic liquid crystals, it is called the Hunter-Saxton equation. The main advantage of this method is that it can be applied directly to nonlinear differential equations without requiring linearization, discretization, or perturbation. It is a semi analytical-numerical method that formulizes Taylor series in a very different manner. The numerical results denote that reduced differential transform method is efficient and accurate for Hunter-Saxton equation.

Key words: Reduced differential transform method, Hunter-Saxton equation, Taylor series.
2010 AMS Mathematics Subject Classification : 35G25; 65M99; 65D99.

* Corresponding's E-mail:rouhparvar59@gmail.com.

1 Introduction

Nonlinear wave phenomena appears in various scientific and engineering fields, such as fluid mechanics, plasma physics, biology, chemical physics. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations.

In this paper we study the Hunter-Saxton equation

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0, \qquad t \geq 0, x \in \mathbb{R}, \tag{1.1}$$

where describes the propagation of waves in a massive director field of a nematic liquid crystal [10] with the orientation of the molecules, x being the space variable in a reference frame moving with the linearized wave velocity, and t being a slow time variable. The liquid crystal state is a distinct phase of matter observed between the solid and liquid states. More specifically, liquids are isotropic (that is, with no directional order) and without a positional order of their molecules, whereas the molecules in solids are constrained to point only in certain directions and to be only in certain positions with respect to each other. The liquid crystal phase exists between the solid and the liquid phase - the molecules in a liquid crystal do not exhibit any positional order, but they do possess a certain degree of orientational order. Not all substances can have a liquid crystal phase e.g. water molecules melt directly from solid crystalline ice to liquid water. The equation (1.1) is also relevant in other physical situations, e.g. it is a high-frequency limit of the Camassa-Holm equation [7], a nonlinear shallow water equation [7,13] modeling solitons [7] as well as breaking waves [9].

The present paper investigates for the first time the applicability and effectiveness of the reduced differential transform method (RDTM) on the Hunter-Saxton equation.

The remaining structure of this article is organized as follows: Section 2 is a brief introduction to the RDTM for finding approximate solution. In Section 3, we illustrate this method in detail with the Hunter-Saxton equation. In Section 4, several examples are exhibited for illustration and in Section 5, some conclusions are given.

2 Differential transform method (DTM)

The DTM was first proposed by Zhou [20], who solved linear and nonlinear initial value problems in electric circuit analysis, and was used heavily in the literature successfully applied to one-dimensional planar Bratu problem [1], integro-differential equations [4], higher-order initial value problems [2], systems of differential equations [3,5,11], eigenvalue problems [8], partial differential equation [12], high index differential-algebraic equations [19].

2.1 One-dimensional differential transform method

The differential transform of the function $g(t)$ is defined as the following

$$G(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}g(t)\right]_{t=0}, \tag{2.1}$$

where $g(t)$ is the original function and $G(k)$ is the transformed function. Here $\frac{d^k}{dt^k}$ means the k derivative as

$$g(t) = \sum_{k=0}^{\infty} G(k)t^k. \tag{2.2}$$

Substituting (2.1) to (2.2) we have

$$g(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{d^k}{dt^k}g(t)\right]_{t=0} t^k.$$

By attention to (2.1) and (2.2), the basic mathematical operations are readily be obtained and given in Table 2.1.

Table 2.1. One-dimensional differential transformation

Original Function	**Transformed Function**
$g(t) \pm z(t)$	$G(k) \pm Z(k)$
$cg(t)$	$cG(k)$
$\frac{d^m g(t)}{dt^m}$	$\frac{(k+m)!}{k!}U(k+m)$
$g(t)z(t)$	$\sum_{r=0}^{k} G(r)Z(k-r)$

2.2 Two-dimensional differential transform method

The basic definitions and fundamental operations of the two- dimensional differential transform are introduced in [14] as the following

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}u(x,t)\right]_{(0,0)}, \tag{2.3}$$

where $u(x,t)$ is the original function and $U(k,h)$ is the transformed function. The differential inverse transform of $U(k,h)$ is

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h, \tag{2.4}$$

and from Eqs. (2.3) and (2.4) can be concluded

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial t^h}u(x,t)\right]_{(0,0)} x^k t^h. \tag{2.5}$$

In Table 2.2 has listed the fundamental mathematical operations of two-dimensional differential transform. The proofs of Table 2.2 are available in [6].

2.3 The reduced differential transform method

The basic definitions and operations of the RDTM [15–17] are defined as follows:

Table 2.2. Two-dimensional differential transformation

Original Function	**Transformed Function**
$u(x,t) \pm v(x,t)$	$U(k,h) \pm V(k,h)$
$cu(x,t)$	$cU(k,h)$
$\frac{\partial u(x,t)}{\partial x}$	$(k+1)U(k+1,h)$
$\frac{\partial u(x,t)}{\partial t}$	$(h+1)U(k,h+1)$
$\frac{\partial^{r+s} u(x,t)}{\partial x^r \partial t^s}$	$\frac{(k+r)!}{k!}\frac{(h+s)!}{h!}U(k+r,h+s)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s)$

Definition 2.1 *If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let*

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{2.6}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function.

Definition 2.2 *The reduced differential transform of a sequence $\{U_k(x)\}_{k=0}^{\infty}$ is introduced as follows:*

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{2.7}$$

To combining equation (2.6) and (2.7), we have

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k. \tag{2.8}$$

Some basic properties of the reduced differential transformation obtained from definitions (2.6) and (2.8) are summarized in Table 2.3. The proofs of Table 2.3 and the basic definitions of the RDTM are available in [18].

Table 2.3. Basic operations of RDTM

Original Function	**Transformed Function**
$u(x,t)$	$U_k(x)$
$u(x,t) \pm v(x,t)$	$U_k(x) \pm V_k(x)$
$cu(x,t)$	$cU_k(x)$ c is a cons.
$x^m t^n$	$x^m \delta(k-n)$
$x^m t^n u(x,t)$	$x^m U_{k-n}(x)$
$\frac{\partial}{\partial x} u(x,t)$	$\frac{\partial}{\partial x} U_k(x)$
$\frac{\partial^r}{\partial t^r} u(x,t)$	$\frac{(k+r)!}{k!} U_{k+r}(x)$
$u(x,t)v(x,t)$	$\sum_{r=0}^{k} U_r(x) V_{k-r}(x)$

3 The RDTM for the Hunter-Saxton equation

In this section, we will illustrate as an application the proposed method for the Hunter-Saxton equation. Consider the following equation

$$u_{txx} + 2u_x u_{xx} + u u_{xxx} = 0, \qquad t \geq 0, x \in \mathbb{R}, \tag{3.1}$$

with the initial conditions

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(a,t) &= g(t), \\ u_{tx}(a,t) &= z(t), \end{aligned} \tag{3.2}$$

where a is constant. The approximate solution using the conditions (3.2) is given by

$$\begin{aligned} u(x,t) &= f(x) + (x-a) \int_0^t \left(g(\tau) + z(\tau)\right) d\tau \\ &- \int_0^t \int_a^x \int_a^\lambda \left(2u_s(s,\tau) u_{ss}(s,\tau) + u(s,\tau) u_{sss}(s,\tau)\right) ds d\lambda d\tau. \end{aligned} \tag{3.3}$$

According to the RDTM, we consider the transformations of the functions $u(x,t)$, $g(t)$ and $z(t)$ as the following

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k, \qquad g(t) = \sum_{k=0}^{\infty} G(k)t^k,$$
$$z(t) = \sum_{k=0}^{\infty} Z(k)t^k,$$

where

$$\begin{aligned} U_k(x) &= \tfrac{1}{k!}\left[\tfrac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \qquad G(k) = \tfrac{1}{k!}\left[\tfrac{d^k}{dt^k}g(t)\right]_{t=0}, \\ Z(k) &= \tfrac{1}{k!}\left[\tfrac{d^k}{dt^k}z(t)\right]_{t=0}, \quad k = 0,1,2,\dots . \end{aligned} \tag{3.4}$$

To substitute the relations (3.4) into Eq. (3.3), we have

$$\begin{aligned} \sum_{k=0}^{\infty} U_k(x)t^k &= f(x) + (x-a)\int_0^t \sum_{k=0}^{\infty} \left(G(k)+Z(k)\right)\tau^k d\tau \\ &\quad - \int_0^t \int_a^x \int_a^\lambda \sum_{k=0}^{\infty} \left(2N_k(s)+M_k(s)\right)\tau^k ds d\lambda d\tau, \end{aligned} \tag{3.5}$$

where by Table 2.3, $N_k(s)$ and $M_k(s)$ are as follows

$$N_k(s) = \sum_{r=0}^{k} \frac{\partial}{\partial s}U_r(s)\frac{\partial^2}{\partial s^2}U_{k-r}(s), \qquad M_k(s) = \sum_{r=0}^{k} U_r(s)\frac{\partial^3}{\partial s^3}U_{k-r}(s).$$

We now carry out the integration with respect to t on the Eq. (3.5) to write

$$\begin{aligned} \sum_{k=0}^{\infty} U_k(x)t^k &= f(x) + (x-a)\sum_{k=0}^{\infty} \left(G(k)+Z(k)\right)\tfrac{t^{k+1}}{k+1} \\ &\quad - \int_a^x \int_a^\lambda \sum_{k=0}^{\infty} \left(2N_k(s)+M_k(s)\right)\tfrac{t^{k+1}}{k+1} ds d\lambda. \end{aligned} \tag{3.6}$$

At last, equation coefficients of the same powers of t, we obtain the recursive formula for coefficients as the following

$$U_0(x) = f(x), \tag{3.7}$$

and

$$U_{k+1}(x) = \tfrac{1}{k+1}(x-a)\left(G(k)+Z(k)\right)$$
$$-\tfrac{1}{k+1}\int_a^x\int_a^\lambda \left(2N_k(s)+M_k(s)\right)dsd\lambda, \quad k=0,1,2,\ldots. \tag{3.8}$$

Substituting (3.7) into (3.8) and by a straight forward iterative calculations, we obtain the following $U_k(x)$ values. So, the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{p}$ give approximate solution as

$$u_p(x,t) \approx \sum_{k=0}^{p} U_k(x)t^k,$$

where p is order of approximation solution. In result,the exact solution of problem is given by

$$u(x,t) = \lim_{p\to\infty} u_p(x,t).$$

Let us consider the error functional for p-order approximate solution as the following

$$Error(x,t) = |u_{txx(p)} + 2u_{x(p)}u_{xx(p)} + u_{(p)}u_{xxx(p)}|. \tag{3.9}$$

4 Applications

In this section, we apply the recursive formula (3.8) with (3.7) for the three cases of the conditions (3.2).

Case 1:
Consider the Hunter-Saxton equation (3.1) with the conditions

$$\begin{aligned} u(x,0) &= 0.3x^2, \\ u_t(a,t) &= -te^{t^2}, \\ u_{tx}(a,t) &= te^{t^2}. \end{aligned} \tag{4.1}$$

We assume that $a = -1$ and by (3.4) have

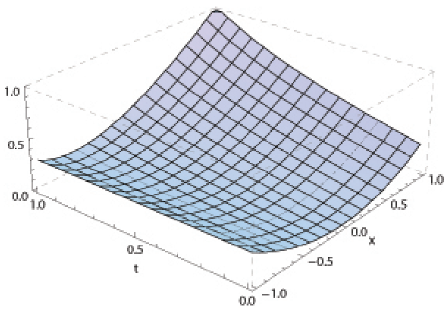
$$G(0) = 0,\ G(1) = -1,\ G(2) = 0,\ G(3) = -1,\ G(4) = 0,$$

$$G(5) = -\tfrac{1}{2},\ \ldots,$$

$$Z(0) = 0,\ Z(1) = 1,\ Z(2) = 0,\ Z(3) = 1,\ Z(4) = 0,$$

$$Z(5) = \tfrac{1}{2},\ Z(6) = 0, \ldots.$$

Therefore, by attention to the recursive formula (3.8) for $p = 24$, the approximate solution and error functional have been shown in figures (1) and (2), respectively.



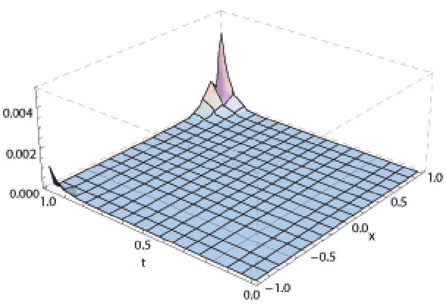


Fig. 1. Plot of $u(x,t)$ in the case 1. Fig. 2. Plot of $Error(x,t)$ in the case 1.

Case 2:
Consider the Hunter-Saxton equation (3.1) with the conditions

$$\begin{aligned} &u(x,0) = 0.08 \sin x, \\ &u_t(-\pi, t) = 0, \\ &u_{tx}(-\pi, t) = \sin t \cos t. \end{aligned} \tag{4.2}$$

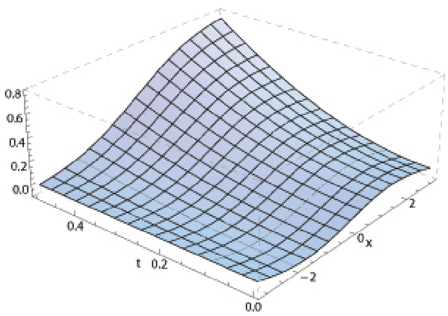
By attention to (3.4), we have

$$G(0) = 0,\ G(1) = 0,\ G(2) = 0, \ldots,$$

$$Z(0) = 0,\ Z(1) = 1,\ Z(2) = 0,\ Z(3) = -\tfrac{2}{3},\ Z(4) = 0,$$

$$Z(5) = \tfrac{2}{15},\ Z(6) = 0, \ldots.$$

In result, by the recursive formula (3.8) for $p = 7$, the approximate solution and error functional have been shown in figures (3) and (4), respectively.



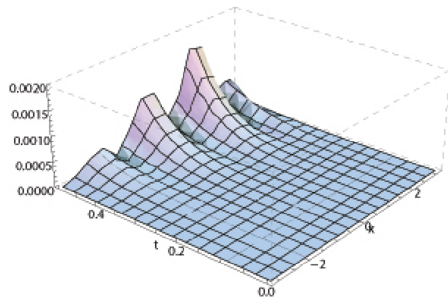


Fig. 3. Plot of $u(x,t)$ in the case 2. Fig. 4. Plot of $Error(x,t)$ in the case 2.

Case 3:
Consider the Hunter-Saxton equation (3.1) with the conditions

$$\begin{aligned} &u(x,0) = 0.1e^{-x}, \\ &u_t(,t) = 0.1e^{-t}, \\ &u_{tx}(,t) = 0.1e^{-t^2}. \end{aligned} \tag{4.3}$$

By attention to (3.4), we have

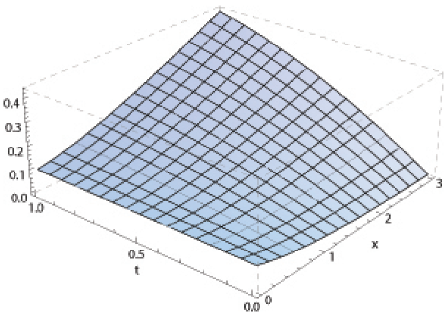
$$G(0) = 0.1, G(1) = -0.1, G(2) = 0.05, G(3) = -0.0166667,$$

$$G(4) = 0.00416667, \ldots,$$

$$Z(0) = 0.1,\ Z(1) = 0,\ Z(2) = -0.1,\ Z(3) = 0,\ Z(4) = 0.05,$$

$$Z(5) = 0, \ldots.$$

So, by the recursive formula (3.8) for $p = 10$, the approximate solution and error functional have been shown in figures (5) and (6), respectively.



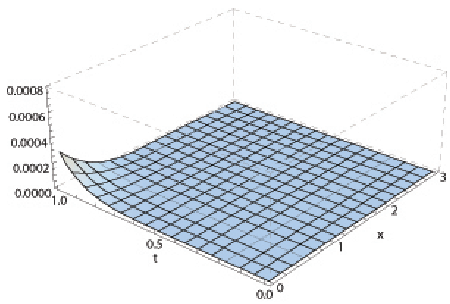


Fig. 5. Plot of $u(x,t)$ in the case 3. Fig. 6. Plot of $Error(x,t)$ in the case 3.

5 Conclusion

In this article, we applied the RDTM for solving the Hunter-Saxton equation. The obtained results denoted that the proposed method is applied for solving equation very easy and efficient.

Acknowledge

The author would like to appreciate the anonymous reviewers for their helpful comments.

References

[1] I. H. Abdel-Halim Hassan, Vedat Suat Ertrk, Applying differential transformation method to the one-dimensional planar Bratu problem, Int. J. Contemp. Math. Sciences, 2(30) (2007), 1493-1504.

[2] I. H. Abdel-Halim Hassan, Differential transformation technique for solving higher-order initial value problems, Appl. Math. Comput., 154 (2004), 299-311.

[3] I. H. Abdel-Halim Hassan, Application to differential transformation method for solving systems of differential equations, Appl. Math. Modell., 32(12) (2008), 2552-2559.

[4] A. Arikoglu and I. Ozkol, Solution of boundary value problems for integro-differential equations by using differential transform method, Appl. Math. Comput., 168 (2005), 1145-1158.

[5] F. Ayaz, Solutions of the systems of differential equations by differential transform method, Appl. Math. Comput., 147 (2004), 547-567.

[6] F. Ayaz, On the two-dimensional differential transform method, Appl. Math. Comput., 143 (2003) 361-374.

[7] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett., 71 (1993), 1661-1664.

[8] C. K. Chen and S. H. Ho, Application of differential transformation to eigenvalue problems, Appl. Math. Comput., 79 (1996), 173-188.

[9] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Mathematics, 181 (1998), 229-243.

[10] J. K. Hunter and R. Saxton, Dynamics of director fields, SIAM J. Appl. Math, 51 (1991), 1498-1521.

[11] M. J. Jang and C. L. Chen, Analysis of the response of a strongly nonlinear damped system using a differential transformation technique, Appl. Math. Comput., 88 (1997), 137-151.

[12] M. J. Jang and C. K. Chen, Two-dimensional differential transformation method for partial differantial equations, Appl. Math. Copmut., 121 (2001), 261-270.

[13] R. S. Johnson and Camassa-Holm, Korteweg-de Vries and related models for water waves, J. Fluid Mech., 455 (2002), 63-82.

[14] F. Kangalgil and F. Ayaz, Solitary wave solutions for the KdV and mKdV equations by differential transform method, Chaos Solitons & Fractals, 41 (2009), 464-472.

[15] Y. Keskin and G. Oturanc, Reduced Differential Transform Method for Partial Differential Equations, International Journal of Nonlinear Sciences and Numerical Simulation, 10 (6) (2009), 741-749.

[16] Y. Keskin and G. Oturanc, Reduced differential transform method for solving linear and nonlinear wave equations, Iranian Journal of Science & Technology, Transaction A, 34(A2) 2010, 113-122.

[17] Y. Keskin and G. Oturanc, Reduced Differential Transform Method for fractional partial differential equations, Nonlinear Science Letters A, 1(2) (2010), 61-72.

[18] Y. Keskin, Ph.D. Thesis, Selcuk University (to appear).

[19] H. Liu and Yongzhong Song, Differential transform method applied to high index differential algebraic equations, Appl. Math. Comput.,184 (2007), 748-753.

[20] J. K. Zhou, Differential transformation and its application for electrical circuits, Huarjung University PressWuuhahn, China (1986), (in Chinese).