Theory of Approximation and Applications

Vol.12, No.1, (2018), 43-64

New Generalized Interval Valued Intuitionistic Fuzzy Numbers

E. Baloui Jamkhaneh [a,*] A. saeidifar [b]

[a] *Department of Statistics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran*

[b] *Department of Statistics, Arak Branch, Islamic Azad University, Arak, Iran*

Received 12 March 2017; accepted 09 January 2018

Abstract

The aim of this paper is investigate the notion of a generalized interval valued intuitionistic fuzzy number ($GIVIFN_B$), which extends the interval valued intuitionistic fuzzy number. Firstly, the concept of $GIVIFN_Bs$ is introduced. Arithmetic operations and cut sets over $GIVIFN_Bs$ are investigated. Then, the values and ambiguities of the membership degree and the non-membership degree and the value index and ambiguity index for $GIVIFN_Bs$ are defined. Finally, we develop a value and ambiguity-based ranking method.

Key words: Generalized interval valued intuitionistic fuzzy sets, generalized interval valued intuitionistic fuzzy numbers, cut set, value index, ambiguity index.

* Corresponding author's E-mail: e_baloui2008@yahoo.com

1 Introduction

Later on introduce fuzzy sets theory in Zadeh (1965), the concept of fuzzy numbers and its arithmetic operations were first investigated by Chang and Zadeh (1972) and others. The notion of fuzzy numbers was introduced by Dubois and Prade (1978) as a fuzzy subset of the real line. Subsequently, Atanassov (1986) introduced concept intuitionistic fuzzy sets (IFSs) as a generalization of fuzzy sets. IFS can be used to deal with uncertainty by taking both degree of membership and degree of non-membership. Burillo et al. (1994) proposed the definition of intuitionistic fuzzy number. Mahapatra and Roy (2009) presented triangular intuitionistic fuzzy number and used it for reliability evaluation. Wang and Zhang (2009) defined the trapezoidal intuitionistic fuzzy number (TrIFN) and their operational laws. Also Mahapatra and Mahapatra (2010) defined trapezoidal intuitionistic fuzzy number and arithmetic operations of TrIFN based on (α, β)-cut method. Parvathi (2012) introduced symmetric trapezoidal intuitionistic fuzzy numbers (STrIFNs) and discussed their desirable properties and arithmetic operations based on (α, β)-cut. Atanassov and Gargov (1989) proposed the notion of the interval valued intuitionistic fuzzy sets (IVIFSs) as combined IFS concept with interval valued fuzzy sets concept, which is characterized by membership function and non-membership function whose values are interval rather than exact numbers. They are very useful in the process of decision making since in many real word decision problems the values membership function and non-membership function in an IFS are difficult to be expressed as an exact numbers. Yuan and Li (2009) and Adak and Bhowmik (2011) defined different types of cut sets on IVIFSs.
Based on IVIFS, Xu (2007) defined the notion of interval valued intuitionistic fuzzy number (IVIFN) and introduced some operations on IVIFNs. Xu (2007) proposed score function and accuracy function to rank IVIFNs. Ye (2009) also proposed a novel accuracy function to rank IVIFNs. Garg (2016) considered a new generalized improved score function of IVIFSs and applications in expert systems.
The literature review shows, notions of IVIFNs is very useful in modeling real life problems with imprecision or uncertainty and they have been applied to many different fields. For example, some recent appli-

cations of IVIFNs have been: multiple attribute group decision making (MAGDM) using elimination and choice translation reality (ELECTRE) method (Li et al. (2012); Veeramachaneni and Kandikonda, (2016)); extension of the TOPSIS method to solve multiple attribute group decision making problems (Park et al. (2011), Sudha et al. (2015)); interval valued intuitionistic fuzzy analytic hierarchy process (Abdullah and Najib, (2016)); The selection of technology forecasting method (Intepe et al. (2013)) and etc.
Baloui Jamkhaneh and Nadarajah (2015) extended IFSs with the introduction of the concept of new generalized intuitionistic fuzzy sets ($GIFS_B$) and introduced some operators over $GIFS_B$. Shabani and Baloui Jamkhaneh (2014) introduced generalized intuitionistic fuzzy numbers ($GIFN_B$) base on $GIFS_B$. Baloui Jamkhaneh (2016) study the concepts of values and ambiguities of the degree of membership and the degree of non-membership for $GIFN_B$. Baloui Jamkhaneh (2015) considered new generalized interval value intuitionistic fuzzy sets ($GIVIFS_B$) and introduced some operators over $GIVIFS_B$. The aim of this paper is to introduction the generalized interval valued intuitionistic fuzzy number ($GIVIFN_B$) based on generalization of the IVIFS related to Baloui Jamkhaneh (2015) and to drive their specifications. The derived specifications include: i) (α_1, α_2)-cut, ii) (β_1, β_2)-cut, iii) $(\overrightarrow{\alpha}, \overleftarrow{\beta})$-cut, iv) values of the membership degree and the non-membership degree, v) ambiguities of the membership degree and the non-membership degree, vi) the value index, vii) the ambiguity index, viii) ranking function of $GIVIFN_B$.
In order to achieve this, the remainder of this paper is organized as follows: In Section 2, we briefly introduce IFS and its generalizes. In Section 3 define new generalized interval valued intuitionistic fuzzy sets and arithmetic operations. In Section 4 introduce cut sets on $GIVIFN_B$. In Section 5 define values and ambiguities of the membership degree and the non-membership degree, the value index, ambiguity index and value and ambiguity-based ranking method for $GIVIFN_Bs$ are defined. The paper is concluded in Section 6.

2 Preliminaries

We some basic definitions of IFSs and IVIFSs are introduced to facilitate the discussion.

Definition 2.1 *(Atanassov,1986) An IFS A in X is defined as an object of the form* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *where the functions* $\mu_A : X \to [0,1]$ *and* ν_A*:* $X \to [0,1]$ *denotes the degree of membership and non-membership functions of A, respectively and* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ *for each* $x \in X$*.*

Definition 2.2 *(Atanassov and Gargov, 1989) Let* X *be a non-empty set. Interval valued intuitionistic fuzzy sets* $(IVIFS)$ A *in* X*, is defined as an object of the form* $A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in X\}$ *where the functions* $M_A(x) : X \to [\ I\ ]$ *and* $N_A(x) : X \to [I]$*, denote the degree of membership and degree of non-membership of A respectively, where* $M_A(x) = [M_{AL}(x), M_{AU}(x)]$*,* $N_A(x) = [N_{AL}(x), N_{AU}(x)]$*,* $0 \leq M_{AU}(x) + N_{AU}(x) \leq 1$ *for each* $x \in X$*.*

Definition 2.3 *(Baloui Jamkhaneh and Nadarajah (2015)) Let* X *be a non-empty set. Generalized intuitionistic fuzzy sets* $(GIFS_B)$ A *in* X*, is defined as an object of the form* $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ *where the functions* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$*, denote the degree of membership and degree of non-membership functions of A respectively, and* $0 \leq \mu_A(x)^{\delta} + v_A(x)^{\delta} \leq 1$ *for each* $x \in X$ *and* $\delta = n$ *or* $\frac{1}{n}, n = 1, 2, \ldots, N$*.*

Definition 2.4 *Let* $[I]$ *be the set of all closed subintervals of the interval* $[0,1]$ *and* $M_A(x) = [M_{AL}(x), M_{AU}(x)] \in [I]$ *and* $N_A(x) = [N_{AL}(x), N_{AU}(x)] \in [I]$ *then* $N_A(x) \leq M_A(x)$ *if and only if* $N_{AL}(x) \leq M_{AL}(x)$ *and* $N_{AU}(x) \leq M_{AU}(x)$*.*

Definition 2.5 *(Baloui Jamkhaneh(2015)) Let* X *be a non-empty set. Generalized interval valued intuitionistic fuzzy sets* $(GIVIFS_Bs)$ A *in* X*, is defined as an object of the form* $A = \{\langle x, M_A(x), N_A(x)\rangle \mid x \in X\}$ *where the functions* $M_A(x) : X \to [I]$ *and* $N_A(x) : X \to [I]$*, denote the degree of membership and degree of non-membership of A respectively,* *and* $M_A(x) = [M_{AL}(x), M_{AU}(x)]$, $N_A(x) = [N_{AL}(x), N_{AU}(x)]$, *where* $0 \leq M_{AU}(x)^{\delta} + N_{AU}(x)^{\delta} \leq 1$, *for each* $x \in X$ *and* $\delta = n$ *or* $\frac{1}{n}$, $n = 1, 2, \ldots, N$. *The collection of all* $GIVIFS_B(\delta)$ *is denoted by* $GIVIFS_B(\delta, X)$.

3 New Generalized Interval Valued Intuitionistic Fuzzy Numbers

In this section, $GIVIFN_B$ and their operations are defined as follows.

Definition 3.1 *In general, our generalized interval valued intuitionistic fuzzy number* A *can be described as any* $GIVIFS_B(X)$ *of the real line* $\mathbb{R}$ *whose membership function* $\mu_A(x) = [\underline{\mu}_A(x), \overline{\mu}_A(x)]$ *and non-membership function* $\nu_A(x) = [\underline{\nu}_A(x), \overline{\nu}_A(x)]$ *are defined as follows:*

$$\underline{\mu}_A(x) = \begin{cases} \left(\frac{(x-a)\underline{\mu}}{b-a}\right)^{\frac{1}{\delta}}, & a \leq x \leq b \\ \underline{\mu}^{\frac{1}{\delta}}, & b \leq x \leq c \\ \left(\frac{(d-x)\underline{\mu}}{d-c}\right)^{\frac{1}{\delta}}, & c \leq x \leq d \\ 0, & o.w. \end{cases}, \quad \bar{\mu}_A(x) = \begin{cases} \left(\frac{(x-a)\bar{\mu}}{b-a}\right)^{\frac{1}{\delta}}, & a \leq x \leq b \\ \bar{\mu}^{\frac{1}{\delta}}, & b \leq x \leq c \\ \left(\frac{(d-x)\bar{\mu}}{d-c}\right)^{\frac{1}{\delta}}, & c \leq x \leq d \\ 0, & o.w. \end{cases},$$

$$\underline{\nu}_A(x) = \begin{cases} \left(\frac{(b-x)+\underline{\nu}(x-a_1)}{b-a_1}\right)^{\frac{1}{\delta}}, & a_1 \leq x \leq b \\ \underline{\nu}^{\frac{1}{\delta}}, & b \leq x \leq c \\ \left(\frac{(x-c)+\underline{\nu}(d_1-x)}{d_1-c}\right)^{\frac{1}{\delta}}, & c \leq x \leq d_1 \\ 1, & o.w. \end{cases}, \quad \bar{\nu}_A(x) = \begin{cases} \left(\frac{(b-x)+\bar{\nu}(x-a_1)}{b-a_1}\right)^{\frac{1}{\delta}}, & a_1 \leq x \leq b \\ \bar{\nu}^{\frac{1}{\delta}}, & b \leq x \leq c \\ \left(\frac{(x-c)+\bar{\nu}(d_1-x)}{d_1-c}\right)^{\frac{1}{\delta}}, & c \leq x \leq d_1 \\ 1, & o.w. \end{cases}$$

where $a_1 \leq a \leq b \leq c \leq d \leq d_1$ and $0 \leq \underline{\mu} \leq \bar{\mu} \leq 1$, $0 \leq \underline{\nu} \leq \bar{\nu} \leq 1$, $\bar{\mu} + \bar{\nu} \leq 1$. The $GIVIFN_B A$ is denoted as $A = (a_1, a, b, c, d, d_1, [\underline{\mu}, \bar{\mu}], [\underline{\nu}, \bar{\nu}], \delta)$.

Definition 3.2 *A* $GIVIFN_B$ *is said to be symmetric* $GIVIFN_B$ *if* $b - a = d - c$ *and* $b - a_1 = d_1 - c$.

Figure 1 shows membership and non-membership functions of $A = (0.25, 1, 2, 3, 4, 4.75, [0.54, 0.64], [0.26, 0.36], \delta)$ with $\delta = 2$ respectively. Figure 3 shows membership and non-membership functions of $A = (0.25, 1, 2, 3, 4,$

$4.75, [0.54, 0.64], [0.26, 0.36], \delta)$ with $\delta = 1$.

$$\underline{\mu}_A(x) = \begin{cases} (0.54x - 0.54)^{\frac{1}{\delta}}, & 1 \le x \le 2 \\ 0.54^{\frac{1}{\delta}}, & 2 \le x \le 3 \\ (2.16 - 0.54x)^{\frac{1}{\delta}}, & 3 \le x \le 4 \\ 0, & o.w. \end{cases},$$

$$\bar{\mu}_A(x) = \begin{cases} (0.64x - 0.64)^{\frac{1}{\delta}}, & 1 \le x \le 2 \\ 0.64^{\frac{1}{\delta}}, & 2 \le x \le 3 \\ (2.56 - 0.64x)^{\frac{1}{\delta}}, & 3 \le x \le 4 \\ 0, & o.w. \end{cases},$$

$$\underline{\nu}_A(x) = \begin{cases} (1.1057143 - 0.4228571x)^{\frac{1}{\delta}}, & 0.25 \le x \le 2 \\ 0.26^{\frac{1}{\delta}}, & 2 \le x \le 3 \\ (0.4228571x - 1.0085714)^{\frac{1}{\delta}}, & 3 \le x \le 4.75 \\ 1, & o.w. \end{cases},$$

$$\bar{\nu}_A(x) = \begin{cases} (1.0914286 - 0.3657143x)^{\frac{1}{\delta}}, & 0.25 \le x \le 2 \\ 0.36^{\frac{1}{\delta}}, & 2 \le x \le 3 \\ (0.3657143x - 0.7371429)^{\frac{1}{\delta}}, & 3 \le x \le 4.75 \\ 1, & o.w. \end{cases}$$

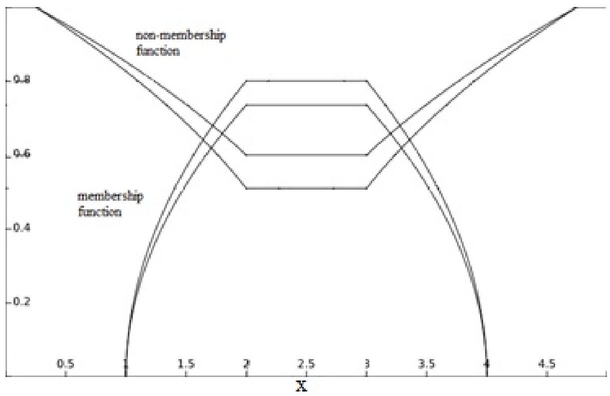

Fig. 1. Membership and non-membership functions of $A = (0.25, 1, 2, 3, 4, 4.75, [0.54, 0.64], [0.26, 0.36])$

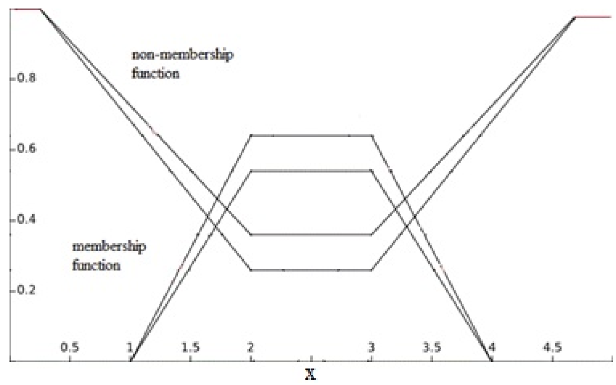

Fig. 2. Membership and non-membership functions of $A = (0.25, 1, 2, 3, 4, 4.75, [0.54, 0.64], [0.26, 0.36])$

Definition 3.3 *Let* $A = (a'_1, a_1, b_1, c_1, d_1, d'_1, [\underline{\mu}_1, \bar{\mu}_1], [\underline{\nu}_1, \bar{\nu}_1], \delta)$ *and* $B = (a'_2, a_2, b_2, c_2, d_2, d'_2, [\underline{\mu}_2, \bar{\mu}_2], [\underline{\nu}_2, \bar{\nu}_2], \delta)$ *be two* $GIVIFN_B s$*, then,*

$$\begin{aligned} A + B = &(a'_1 + a'_2, a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2, d'_1 + d'_2 \\ &, [\underline{\mu}_1 + \underline{\mu}_2 - \underline{\mu}_1\underline{\mu}_2, \bar{\mu}_1 + \bar{\mu}_2 - \bar{\mu}_1\bar{\mu}_2], [\underline{\nu}_1\underline{\nu}_2, \bar{\nu}_1\bar{\nu}_2], \delta), \end{aligned}$$

$$\begin{aligned} kA = &(ka'_1, ka_1, kb_1, kc_1, kd_1, kd'_1, [1 - (1 - \underline{\mu}_1)^k, 1 - (1 - \bar{\mu}_1)^k] \\ &, [\underline{\nu}_1^k, \bar{\nu}_1^k], \delta), k = 2, 3, \cdots \end{aligned}$$

$$\begin{aligned} kA = &(kd'_1, kd_1, kc_1, kb_1, ka_1, ka'_1, [1 - (1 - \underline{\mu}_1)^{|k|}, 1 - (1 - \bar{\mu}_1)^{|k|}] \\ &, [\underline{\nu}_1^{|k|}, \bar{\nu}_1^{|k|}], \delta), k = -2, -3, \cdots \end{aligned}$$

$$-A = (-d'_1, -d_1, -c_1, -b_1, -a_1, -a'_1, [\underline{\mu}_1, \bar{\mu}_1], [\underline{\nu}_1, \bar{\nu}_1], \delta),$$

$$\begin{aligned} A - B = &(a'_1 - d'_2, a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2, d'_1 - a'_2 \\ &, [\underline{\mu}_1 + \underline{\mu}_2 - \underline{\mu}_1\underline{\mu}_2, \bar{\mu}_1 + \bar{\mu}_2 - \bar{\mu}_1\bar{\mu}_2], [\underline{\nu}_1\underline{\nu}_2, \bar{\nu}_1\bar{\nu}_2], \delta), \end{aligned}$$

$$A \cdot B = (a'_1 \cdot a'_2, a_1 \cdot a_2, b_1 \cdot b_2, c_1 \cdot c_2, d_1 \cdot d_2, d'_1 \cdot d'_2$$

$$, [\underline{\mu}_1\underline{\mu}_2, \bar{\mu}_1\bar{\mu}_2], [\underline{\nu}_1 + \underline{\nu}_2 - \underline{\nu}_1\underline{\nu}_2, \bar{\nu}_1 + \bar{\nu}_2 - \bar{\nu}_1\bar{\nu}_2], \delta),$$

$$A^k = ({a'_1}^k, a_1^k, b_1^k, c_1^k, d_1^k, {d'_1}^k, [\underline{\mu}_1^k, \bar{\mu}_1^k], [1-(1-\underline{\nu}_1)^k, 1-(1-\bar{\nu}_1)^k], \delta), k > 0.$$

4 Cut Sets on $GIVIFN_B$

Definition 4.1 *A $\vec{\alpha}$- cut set ($\vec{\alpha} = (\alpha_1, \alpha_2)$) of a $GIVIFN_B A$ is a crisp subset of $\mathbb{R}$, which defined is as*

$$A[\vec{\alpha}, \delta] = \left\{\langle x, \underline{\mu}_A(x) \geq \alpha_1, \bar{\mu}_A(x) \geq \alpha_2\rangle : x \in X\right\},\ 0 \leq \alpha_1 \leq \underline{\mu}^{\frac{1}{\delta}}, 0 \leq \alpha_2 \leq \bar{\mu}^{\frac{1}{\delta}}$$

According to the definition of membership function of $GIVIFN_B$, it can be shown that

$$A[\vec{\alpha}, \delta] = [L_1(\vec{\alpha}), U_1(\vec{\alpha})] = [\max(\underline{L}_1(\alpha_1), \bar{L}_1(\alpha_2)), \min(\underline{U}_1(\alpha_1), \bar{U}_1(\alpha_2))],$$

$$\underline{L}_1(\alpha_1) = a + \frac{(b-a)\alpha_1^\delta}{\underline{\mu}}, \quad \bar{L}_1(\alpha_2) = a + \frac{(b-a)\alpha_2^\delta}{\bar{\mu}},$$
$$\underline{U}_1(\alpha_1) = d - \frac{(d-c)\alpha_1^\delta}{\underline{\mu}}, \quad \bar{U}_1(\alpha_2) = d - \frac{(d-c)\alpha_2^\delta}{\bar{\mu}}.$$

Theorem 4.1 *Let $A[\vec{\alpha}, \delta]$ be $\vec{\alpha}$- cut set of $GIVIFN_B A$, then*

i. If $\alpha_2 \geq \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^{\frac{1}{\delta}} \alpha_1$, then $A[\vec{\alpha}, \delta] = [\bar{L}_1(\alpha_2), \bar{U}_1(\alpha_2)]$,

ii. If $\alpha_2 < \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^{\frac{1}{\delta}} \alpha_1$, then $A[\vec{\alpha}, \delta] = [\underline{L}_1(\alpha_1), \underline{U}_1(\alpha_1)]$.

Proof. (i) If $\alpha_2 \geq \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^{\frac{1}{\delta}} \alpha_1 \Rightarrow \frac{\alpha_2^\delta}{\bar{\mu}} \geq \frac{\alpha_1^\delta}{\underline{\mu}} \Rightarrow a + \frac{(b-a)\alpha_2^\delta}{\bar{\mu}} \geq a + \frac{(b-a)\alpha_1^\delta}{\underline{\mu}}$
$\Rightarrow \bar{L}_1(\alpha_2) \geq \underline{L}_1(\alpha_1)$
If $\alpha_2 \geq \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^{\frac{1}{\delta}} \alpha_1 \Rightarrow \frac{\alpha_2^\delta}{\bar{\mu}} \geq \frac{\alpha_1^\delta}{\underline{\mu}} \Rightarrow d - \frac{(d-c)\alpha_2^\delta}{\bar{\mu}} \leq d - \frac{(d-c)\alpha_1^\delta}{\underline{\mu}} \Rightarrow \bar{U}_1(\alpha_2) \leq \underline{U}_1(\alpha_1)$
Finally, we have

$$A[\vec{\alpha}, \delta] = [\max(\underline{L}_1(\alpha_1), \bar{L}_1(\alpha_2)), \min(\underline{U}_1(\alpha_1), \bar{U}_1(\alpha_2))] = [\bar{L}_1(\alpha_2), \bar{U}_1(\alpha_2)]$$

Proof (ii) is similar to (i).

Corollary 4.1 *Let $A[\vec{\alpha},\delta]$ be $\vec{\alpha}$- cut set of $GIVIFN_BA$, if $\alpha_2=\alpha_1$, using Theorem 4.1-ii result is*

$$A[\vec{\alpha},\delta]=[\underline{L}_1(\alpha_1),\underline{U}_1(\alpha_1)].$$

Corollary 4.2 *Let $A[\vec{\alpha},\delta]$ be $\vec{\alpha}$- cut set of $GIVIFN_BA$, if $\alpha_2=\bar{\mu}^{\frac{1}{\delta}}$, $\alpha_1=\underline{\mu}^{\frac{1}{\delta}}$, using Definition 4.1 result is $A[\vec{\alpha},\delta]=[b,c]$.*

Theorem 4.2 *Let $A[\vec{\alpha},\delta]$ be $\vec{\alpha}$- cut set of $GIVIFN_BA$, then*

i. If $1<\delta_1\leq\delta_2$ then $A[\vec{\alpha},\delta_1]\subset A[\vec{\alpha},\delta_2]$,
ii. If $\delta_1\leq\delta_2<1$ then $A[\vec{\alpha},\delta_2]\subset A[\vec{\alpha},\delta_1]$,

Proof. (i) Since for $1\leq\delta$, $\underline{L}_1(\alpha_1)$ and $\bar{L}_1(\alpha_2)$ are decreasing, also $\underline{U}_1(\alpha_1)$ and $\bar{U}_1(\alpha_2)$ are increasing, proof (i) is clear.
(ii) Since for $\delta\leq 1$, $\underline{L}_1(\alpha_1)$ and $\bar{L}_1(\alpha_2)$ are increasing, also $\underline{U}_1(\alpha_1)$ and $\bar{U}_1(\alpha_2)$ are decreasing, proof (ii) is clear.

Remark 4.1 *Let $\vec{\tau}=(\tau_1,\tau_2)<\vec{\gamma}=(\gamma_1,\gamma_2)$ (This means that $\tau_i<\gamma_i, i=1,2$). Then $A[\vec{\gamma},\delta]\subseteq A[\vec{\tau},\delta]$.*

Since $\underline{L}_1(\alpha_1)$ and $\bar{L}_1(\alpha_2)$ aspect to α_1 and α_2 are increasing, also $\underline{U}_1(\alpha_1)$ and $\bar{U}_1(\alpha_2)$ aspect to α_1 and α_2 are decreasing, proof is clear.

Definition 4.2 *A $\overleftarrow{\beta}$- cut set ($\overleftarrow{\beta}=(\beta_1,\beta_2)$) of a $GIVIFN_BA$ is a crisp of $\mathbb{R}$, which defined is as*

$$A[\overleftarrow{\beta},\delta]=\left\{\langle x,\underline{\nu}_A(x)\leq\beta_1,\bar{\nu}_A(x)\leq\beta_2\rangle : x\in X\right\},\ \underline{\nu}^{\frac{1}{\delta}}\leq\beta_1\leq 1,\bar{\nu}^{\frac{1}{\delta}}\leq\beta_2\leq 1$$

According to the definition of non-membership function of $GIVIFN_B$ it can be shown that

$$A[\overleftarrow{\beta},\delta]=[L_2(\overleftarrow{\beta}),U_2(\overleftarrow{\beta})]=[\max(\underline{L}_2(\beta_1),\bar{L}_2(\beta_2)),\min(\underline{U}_2(\beta_1),\bar{U}_2(\beta_2))],$$

$$\underline{L}_2(\beta_1)=\frac{b(1-{\beta_1}^{\delta})+a_1({\beta_1}^{\delta}-\underline{\nu})}{1-\underline{\nu}},\quad \bar{L}_2(\beta_2)=\frac{b(1-{\beta_2}^{\delta})+a_1({\beta_2}^{\delta}-\bar{\nu})}{1-\bar{\nu}},$$
$$\underline{U}_2(\beta_1)=\frac{c(1-{\beta_1}^{\delta})+d_1({\beta_1}^{\delta}-\underline{\nu})}{1-\underline{\nu}},\quad \bar{U}_2(\beta_2)=\frac{c(1-{\beta_2}^{\delta})+d_1({\beta_2}^{\delta}-\bar{\nu})}{1-\bar{\nu}}.$$

Theorem 4.3 *Let $A[\overleftarrow{\beta},\delta]$ be $\overleftarrow{\beta}$- cut set of $GIVIFN_BA$, then*

i. If $\frac{\beta_1^\delta}{1-\underline{\nu}}-\frac{\beta_2^\delta}{1-\bar{\nu}} \geq \frac{b-a_1\underline{\nu}}{(b-a_1)(1-\underline{\nu})}-\frac{b-a_1\bar{\nu}}{(b-a_1)(1-\bar{\nu})}$, *then* $A[\overleftarrow{\beta},\delta]=[\bar{L}_2(\beta_2),\bar{U}_2(\beta_2)]$,

ii. If $\frac{\beta_1^\delta}{1-\underline{\nu}}-\frac{\beta_2^\delta}{1-\bar{\nu}} < \frac{b-a_1\underline{\nu}}{(b-a_1)(1-\underline{\nu})}-\frac{b-a_1\bar{\nu}}{(b-a_1)(1-\bar{\nu})}$, *then* $A[\overleftarrow{\beta},\delta]=[\underline{L}_2(\beta_1),\underline{U}_2(\beta_1)]$.

Proof. (i) $\frac{\beta_1^\delta}{1-\underline{\nu}}-\frac{\beta_2^\delta}{1-\bar{\nu}} \geq \frac{b-a_1\underline{\nu}}{(b-a_1)(1-\underline{\nu})}-\frac{b-a_1\bar{\nu}}{(b-a_1)(1-\bar{\nu})} \Rightarrow \frac{(a_1-b)\beta_1^\delta+(b-a_1\underline{\nu})}{1-\underline{\nu}}-\frac{(a_1-b)\beta_2^\delta+(b-a_1\bar{\nu})}{1-\bar{\nu}} \leq 0$,
$\Rightarrow \underline{L}_2(\beta_1)-\bar{L}_2(\beta_2)\leq 0 \Rightarrow \max(\underline{L}_2(\beta_1),\bar{L}_2(\beta_2))=\bar{L}_2(\beta_2)$,
Since $\frac{b-a_1\underline{\nu}}{(b-a_1)(1-\underline{\nu})}-\frac{b-a_1\bar{\nu}}{(b-a_1)(1-\bar{\nu})}=\frac{c-d_1\underline{\nu}}{(c-d_1)(1-\underline{\nu})}-\frac{c-d_1\bar{\nu}}{(c-d_1)(1-\bar{\nu})}$ then

$\frac{\beta_1^\delta}{1-\underline{\nu}}-\frac{\beta_2^\delta}{1-\bar{\nu}} \geq \frac{c-d_1\underline{\nu}}{(c-d_1)(1-\underline{\nu})}-\frac{c-d_1\bar{\nu}}{(c-d_1)(1-\bar{\nu})} \Rightarrow \frac{(d_1-c)\beta_1^\delta+(c-d_1\underline{\nu})}{1-\underline{\nu}}-\frac{(d_1-c)\beta_2^\delta+(c-d_1\bar{\nu})}{1-\bar{\nu}} \geq 0$,

$$\underline{U}_2(\beta_1)-\bar{U}_2(\beta_2)\geq 0 \Rightarrow \min(\underline{U}_2(\beta_1),\bar{U}_2(\beta_2))=\bar{U}_2(\beta_2).$$

Proof is complete.

Proof (ii) is similar to (i).

Corollary 4.3 *Let $A[\overleftarrow{\beta},\delta]$ be $\overleftarrow{\beta}$- cut set of $GIVIFN_BA$, if $\beta_2=\beta_1$, using Definition 4.2 result is $A[\overleftarrow{\beta},\delta]=[\bar{L}_2(\beta_2),\bar{U}_2(\beta_2)]$.*

Corollary 4.4 *Let $A[\overleftarrow{\beta},\delta]$ be $\overleftarrow{\beta}$- cut set of $GIVIFN_BA$, if $\beta_2=\bar{\nu}^{\frac{1}{\delta}}$, $\beta_1=\underline{\nu}^{\frac{1}{\delta}}$, using Definition 4.2 result is $A[\overleftarrow{\beta},\delta]=[b,c]$.*

Theorem 4.4 *Let $A[\overleftarrow{\beta},\delta]$ be $\overleftarrow{\beta}$- cut set of $GIVIFN_BA$, then*

i. If $1\leq\delta_1\leq\delta_2$ then $A[\overleftarrow{\beta},\delta_2]\subset A[\overleftarrow{\beta},\delta_1]$,

ii. If $\delta_1\leq\delta_2\leq 1$ then $A[\overleftarrow{\beta},\delta_1]\subset A[\overleftarrow{\beta},\delta_2]$.

Proof. (i) Since for $1\leq\delta$, $\underline{L}_2(\beta_1)$ and $\bar{L}_2(\beta_2)$ are increasing, also $\underline{U}_2(\beta_1)$ and $\bar{U}_2(\beta_2)$ are decreasing, proof (i) is clear.
(ii) Since for $\delta\leq 1$, $\underline{L}_2(\beta_1)$ and $\bar{L}_2(\beta_2)$ are decreasing, also $\underline{U}_2(\beta_1)$ and $\bar{U}_2(\beta_2)$ are increasing, proof (ii) is clear.

Remark 4.2 *Let $\overleftarrow{\xi}=(\xi_1,\xi_2)<\overleftarrow{\eta}=(\eta_1,\eta_2)$ (This means that $\xi_i<\eta_i$, $i=1,2$). Then $A[\overleftarrow{\xi},\delta]\subseteq A[\overleftarrow{\eta},\delta]$.*

Since $\underline{L}_1(\beta_1)$ and $\bar{L}_1(\beta_2)$ aspect to β_1 and β_2 are decreasing, also $\underline{U}_1(\beta_1)$ and $\bar{U}_1(\beta_2)$ aspect to β_1 and β_2 are increasing, proof is clear.

Corollary 4.5 *Let* $A = (a_1, a, b, c, d, d_1, [\underline{\mu}_1, \bar{\mu}_1], [\underline{\nu}_1, \bar{\nu}_1], \delta)$ *and* $B = (a_1, a, b, c, d, d_1, [\underline{\mu}_2, \bar{\mu}_2], [\underline{\nu}_2, \bar{\nu}_2], \delta)$ *be two* $GIVIFN_Bs$*, then*

i. If $[\underline{\mu}_1, \bar{\mu}_1] \leq [\underline{\mu}_2, \bar{\mu}_2]$ *then* $A[\vec{\alpha}, \delta] \subset B[\vec{\alpha}, \delta]$,

ii. If $[\underline{\nu}_2, \bar{\nu}_2] \leq [\underline{\nu}_1, \bar{\nu}_1]$ *then* $A[\overleftarrow{\beta}, \delta] \subset B[\overleftarrow{\beta}, \delta]$.

Definition 4.3 *Let* $\alpha_i, \beta_i \in [0,1]$ *be fixed numbers such that* $0 \leq \alpha_1 \leq \underline{\mu}^{\frac{1}{\delta}}$, $0 \leq \alpha_2 \leq \bar{\mu}^{\frac{1}{\delta}}$, $\underline{\nu}^{\frac{1}{\delta}} \leq \beta_1 \leq 1$, $\bar{\nu}^{\frac{1}{\delta}} \leq \beta_2 \leq 1$, $0 \leq \alpha_2^{\delta} + \beta_2^{\delta} \leq 1$, $0 \leq \alpha_1^{\delta} + \beta_1^{\delta} \leq 1$, $\vec{\alpha} = (\alpha_1, \alpha_2)$, $\overleftarrow{\beta} = (\beta_1, \beta_2)$.
A $(\vec{\alpha}, \overleftarrow{\beta})$*- cut set generated by a* $GIVIFN_B A$ *is defined by:*

$$A[\vec{\alpha}, \overleftarrow{\beta}, \delta] = \left\{ \langle x, \underline{\mu}_A(x) \geq \alpha_1, \bar{\mu}_A(x) \geq \alpha_2, \underline{\nu}_A(x) \leq \beta_1, \bar{\nu}_A(x) \leq \beta_2 \rangle : x \in X \right\},$$

$A[\vec{\alpha}, \overleftarrow{\beta}, \delta]$ *is defined as the crisp set of elements x which belong to A at least to the degree* α_1 *or* α_2 *and which does not belong to A at most to the degree* β_1 *or* β_2*. Therefore the* $(\vec{\alpha}, \overleftarrow{\beta})$ *-cut of a* $GIVIFN_B$ *is given by*

$$A[\vec{\alpha}, \overleftarrow{\beta}, \delta] = \left\{ x, x \in [L_1(\vec{\alpha}), U_1(\vec{\alpha})] \cap [L_2(\overleftarrow{\beta}), U_2(\overleftarrow{\beta})] \right\} = [L(\vec{\alpha}, \overleftarrow{\beta}), U(\vec{\alpha}, \overleftarrow{\beta})].$$

Corollary 4.6 *Let* $A[\vec{\alpha}, \overleftarrow{\beta}, \delta]$ *be a* $(\vec{\alpha}, \overleftarrow{\beta})$*- cut set generated by a* $GIVIFN_B A$*, the using Theorem 4.1 and Theorem 4.3 result is:*

Case I. *If* $\alpha_2 \geq \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^{\frac{1}{\delta}} \alpha_1$ *and* $\frac{\beta_1^{\delta}}{1-\underline{\nu}} - \frac{\beta_2^{\delta}}{1-\bar{\nu}} \geq \frac{b-a_1\underline{\nu}}{(b-a_1)(1-\underline{\nu})} - \frac{b-a_1\bar{\nu}}{(b-a_1)(1-\bar{\nu})}$ *then we have*

$$L(\vec{\alpha}, \overleftarrow{\beta}) = \begin{cases} \bar{L}_1(\alpha_2), & \alpha_2 \geq \left(\frac{\bar{\mu}}{1-\bar{\nu}} \times \frac{b(1-\beta_2^{\delta})+a_1(\beta_2^{\delta}-\bar{\nu})-a(1-\bar{\nu})}{b-a}\right)^{\frac{1}{\delta}} \\ \bar{L}_2(\beta_2), & \alpha_2 < \left(\frac{\bar{\mu}}{1-\bar{\nu}} \times \frac{b(1-\beta_2^{\delta})+a_1(\beta_2^{\delta}-\bar{\nu})-a(1-\bar{\nu})}{b-a}\right)^{\frac{1}{\delta}} \end{cases},$$

and

$$U(\vec{\alpha}, \overleftarrow{\beta}) = \begin{cases} \bar{U}_1(\alpha_2), & \alpha_2 \geq \left(\frac{\bar{\mu}}{1-\bar{\nu}} \times \frac{c(1-\beta_2^{\delta})+d_1(\beta_2^{\delta}-\bar{\nu})-d(1-\bar{\nu})}{c-d}\right)^{\frac{1}{\delta}} \\ \bar{U}_2(\beta_2), & \alpha_2 < \left(\frac{\bar{\mu}}{1-\bar{\nu}} \times \frac{c(1-\beta_2^{\delta})+d_1(\beta_2^{\delta}-\bar{\nu})-d(1-\bar{\nu})}{c-d}\right)^{\frac{1}{\delta}} \end{cases}.$$

In special case $a_1 = a,\ d_1 = d$

$$A[\vec{\alpha}, \overleftarrow{\beta}, \delta] = \begin{cases} [\bar{L}_1(\alpha_2), \bar{U}_1(\alpha_2)], & \alpha_2 \geq \left(\frac{\bar{\mu}}{1-\bar{\nu}} \times (1-\beta_2^\delta)\right)^{\frac{1}{\delta}} \\ [\bar{L}_2(\beta_2), \bar{U}_2(\beta_2)], & \alpha_2 < \left(\frac{\bar{\mu}}{1-\bar{\nu}} \times (1-\beta_2^\delta)\right)^{\frac{1}{\delta}} \end{cases}.$$

Case II. *If* $\alpha_2 < \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^{\frac{1}{\delta}} \alpha_1$ *and* $\frac{\beta_1^\delta}{1-\underline{\nu}} - \frac{\beta_2^\delta}{1-\bar{\nu}} \geq \frac{b-a_1\underline{\nu}}{(b-a_1)(1-\underline{\nu})} - \frac{b-a_1\bar{\nu}}{(b-a_1)(1-\bar{\nu})}$ *then we have*

$$L(\vec{\alpha}, \overleftarrow{\beta}) = \begin{cases} \underline{L}_1(\alpha_1), & \alpha_1 \geq \left(\frac{\underline{\mu}}{1-\bar{\nu}} \times \frac{b(1-\beta_2^\delta)+a_1(\beta_2^\delta-\bar{\nu})-a(1-\bar{\nu})}{b-a}\right)^{\frac{1}{\delta}} \\ \bar{L}_2(\beta_2), & \alpha_1 < \left(\frac{\underline{\mu}}{1-\bar{\nu}} \times \frac{b(1-\beta_2^\delta)+a_1(\beta_2^\delta-\bar{\nu})-a(1-\bar{\nu})}{b-a}\right)^{\frac{1}{\delta}} \end{cases},$$

and

$$U(\vec{\alpha}, \overleftarrow{\beta}) = \begin{cases} \underline{U}_1(\alpha_1), & \alpha_1 \geq \left(\frac{\underline{\mu}}{1-\bar{\nu}} \times \frac{c(1-\beta_2^\delta)+d_1(\beta_2^\delta-\bar{\nu})-d(1-\bar{\nu})}{c-d}\right)^{\frac{1}{\delta}} \\ \bar{U}_2(\beta_2), & \alpha_1 < \left(\frac{\underline{\mu}}{1-\bar{\nu}} \times \frac{c(1-\beta_2^\delta)+d_1(\beta_2^\delta-\bar{\nu})-d(1-\bar{\nu})}{c-d}\right)^{\frac{1}{\delta}} \end{cases}.$$

In special case $a_1 = a,\ d_1 = d$

$$A[\vec{\alpha}, \overleftarrow{\beta}, \delta] = \begin{cases} [\underline{L}_1(\alpha_1), \underline{U}_1(\alpha_1)], & \alpha_1 \geq \left(\frac{\underline{\mu}}{1-\bar{\nu}} \times (1-\beta_2^\delta)\right)^{\frac{1}{\delta}} \\ [\bar{L}_2(\beta_2), \bar{U}_2(\beta_2)], & \alpha_1 < \left(\frac{\underline{\mu}}{1-\bar{\nu}} \times (1-\beta_2^\delta)\right)^{\frac{1}{\delta}} \end{cases}.$$

Case III. *If* $\alpha_2 \geq \left(\frac{\bar{\mu}}{\underline{\mu}}\right)^{\frac{1}{\delta}} \alpha_1$ *and* $\frac{\beta_1^\delta}{1-\underline{\nu}} - \frac{\beta_2^\delta}{1-\bar{\nu}} < \frac{b-a_1\underline{\nu}}{(b-a_1)(1-\underline{\nu})} - \frac{b-a_1\bar{\nu}}{(b-a_1)(1-\bar{\nu})}$ *then we have*

$$L(\vec{\alpha}, \overleftarrow{\beta}) = \begin{cases} \bar{L}_1(\alpha_2), & \alpha_2 \geq \left(\frac{\bar{\mu}}{1-\underline{\nu}} \times \frac{b(1-\beta_1^\delta)+a_1(\beta_1^\delta-\underline{\nu})-a(1-\underline{\nu})}{b-a}\right)^{\frac{1}{\delta}} \\ \underline{L}_2(\beta_1), & \alpha_2 < \left(\frac{\bar{\mu}}{1-\underline{\nu}} \times \frac{b(1-\beta_1^\delta)+a_1(\beta_1^\delta-\underline{\nu})-a(1-\underline{\nu})}{b-a}\right)^{\frac{1}{\delta}} \end{cases},$$

and

$$U(\vec{\alpha},\overleftarrow{\beta})=\begin{cases}\bar{U}_1(\alpha_2), & \alpha_2\geq\left(\frac{\bar{\mu}}{1-\underline{\nu}}\times\frac{c(1-\beta_1^\delta)+d_1(\beta_1^\delta-\underline{\nu})-d(1-\underline{\nu})}{c-d}\right)^{\frac{1}{\delta}}\\ \underline{U}_2(\beta_1), & \alpha_2<\left(\frac{\bar{\mu}}{1-\underline{\nu}}\times\frac{c(1-\beta_1^\delta)+d_1(\beta_1^\delta-\underline{\nu})-d(1-\underline{\nu})}{c-d}\right)^{\frac{1}{\delta}}\end{cases}.$$

In special case $a_1=a,\ d_1=d$

$$A[\vec{\alpha},\overleftarrow{\beta},\delta]=\begin{cases}[\bar{L}_1(\alpha_2),\bar{U}_1(\alpha_2)], & \alpha_2\geq\left(\frac{\bar{\mu}}{1-\underline{\nu}}\times(1-\beta_1^\delta)\right)^{\frac{1}{\delta}}\\ [\underline{L}_2(\beta_1),\underline{U}_2(\beta_1)], & \alpha_2<\left(\frac{\bar{\mu}}{1-\underline{\nu}}\times(1-\beta_1^\delta)\right)^{\frac{1}{\delta}}\end{cases}.$$

Case IV. *If* $\alpha_2<\left(\frac{\bar{\mu}}{\underline{\mu}}\right)^{\frac{1}{\delta}}\alpha_1$ *and* $\frac{\beta_1^\delta}{1-\underline{\nu}}-\frac{\beta_2^\delta}{1-\bar{\nu}}<\frac{b-a_1\underline{\nu}}{(b-a_1)(1-\underline{\nu})}-\frac{b-a_1\bar{\nu}}{(b-a_1)(1-\bar{\nu})}$ *then we have*

$$L(\vec{\alpha},\overleftarrow{\beta})=\begin{cases}\underline{L}_1(\alpha_1), & \alpha_1\geq\left(\frac{\underline{\mu}}{1-\underline{\nu}}\times\frac{b(1-\beta_1^\delta)+a_1(\beta_1^\delta-\underline{\nu})-a(1-\underline{\nu})}{b-a}\right)^{\frac{1}{\delta}}\\ \underline{L}_2(\beta_1), & \alpha_1<\left(\frac{\underline{\mu}}{1-\underline{\nu}}\times\frac{b(1-\beta_1^\delta)+a_1(\beta_1^\delta-\underline{\nu})-a(1-\underline{\nu})}{b-a}\right)^{\frac{1}{\delta}}\end{cases},$$

and

$$U(\vec{\alpha},\overleftarrow{\beta})=\begin{cases}\underline{U}_1(\alpha_1), & \alpha_1\geq\left(\frac{\underline{\mu}}{1-\underline{\nu}}\times\frac{c(1-\beta_1^\delta)+d_1(\beta_1^\delta-\underline{\nu})-d(1-\underline{\nu})}{c-d}\right)^{\frac{1}{\delta}}\\ \underline{U}_2(\beta_1), & \alpha_1<\left(\frac{\underline{\mu}}{1-\underline{\nu}}\times\frac{c(1-\beta_1^\delta)+d_1(\beta_1^\delta-\underline{\nu})-d(1-\underline{\nu})}{c-d}\right)^{\frac{1}{\delta}}\end{cases}.$$

In special case $a_1=a,\ d_1=d$

$$A[\vec{\alpha},\overleftarrow{\beta},\delta]=\begin{cases}[\underline{L}_1(\alpha_1),\underline{U}_1(\alpha_1)], & \alpha_1\geq\left(\frac{\underline{\mu}}{1-\underline{\nu}}\times(1-\beta_1^\delta)\right)^{\frac{1}{\delta}}\\ [\underline{L}_2(\beta_1),\underline{U}_2(\beta_1)], & \alpha_1<\left(\frac{\underline{\mu}}{1-\underline{\nu}}\times(1-\beta_1^\delta)\right)^{\frac{1}{\delta}}\end{cases}.$$

Example 4.1 *Let* $A=(0.5,1,2,3,4,4.5,[0.49,0.64],[0.25,0.36],2)$ *be a* $GIVIFN_B$. *Then for* $0\leq\alpha_1\leq0.7$, $0\leq\alpha_2\leq0.8$, $0.5\leq\beta_1\leq1$, $0.6\leq\beta_2\leq1$, $0\leq\alpha_i^2+\beta_i^2\leq1$, $i=1,2$, *we get following* $\vec{\alpha}$- *cut and* $\overleftarrow{\beta}$-*cut set as*

$$\underline{L}_1(\alpha_1)=1+\frac{\alpha_1^2}{0.49},\bar{L}_1(\alpha_2)=1+\frac{\alpha_2^2}{0.64},\underline{U}_1(\alpha_1)=4-\frac{\alpha_1^2}{0.49},\bar{U}_1(\alpha_2)=4-\frac{\alpha_2^2}{0.64}.$$

If $\alpha_2 \geq 1.1429\alpha_1$ *then* $A[\vec{\alpha},\delta] = \left[1+\frac{\alpha_2^2}{0.64}, 4-\frac{\alpha_2^2}{0.64}\right]$, *otherwise* $A[\vec{\alpha},\delta] = \left[1+\frac{\alpha_1^2}{0.49}, 4-\frac{\alpha_1^2}{0.49}\right]$, *for example* $\alpha_2 = 0.5$ *and* $\alpha_1 = 0.3$ *then* $A[\alpha,\delta] = [1.3906, 3.6094]$.
Also

$$\underline{L}_2(\beta_1) = 2.5 - 2\beta_1^2, \quad \bar{L}_2(\beta_2) = 2.8437 - 2.3437\beta_2^2,$$

$$\underline{U}_2(\beta_1) = 2.5 + 2\beta_1^2, \quad \bar{U}_2(\beta_2) = 2.1562 + 2.3437\beta_2^2.$$

If $0.75\beta_2^2 - 0.64\beta_1^2 \leq 0.11$ *then* $A[\overleftarrow{\beta},\delta] = [2.8437 - 2.3437\beta_2^2, 2.1562 + 2.3437\beta_2^2]$ *otherwise* $A[\overleftarrow{\beta},\delta] = [2.5-2\beta_1^2, 2.5+2\beta_1^2]$, *for example* $\beta_1 = 0.6$, $\beta_2 = 0.7$ *then* $A[\overleftarrow{\beta},\delta] = [1.78, 3.22]$, *therefore for* $\alpha_1 = 0.3$, $\alpha_2 = 0.5$, $\beta_1 = 0.6$, $\beta_2 = 0.7$, *we have* $A[\vec{\alpha},\overleftarrow{\beta},\delta] = [1.78, 3.22]$.

5 Indices of a $GIVIFN_B$

Definition 5.1 *Let A be a $GIVIFN_B$. Then the values of the membership function A and the non membership function A are defined as follows:*

$$V_\mu(A,\delta) = \frac{1}{2}\int_0^{\mu^{\frac{1}{\delta}}} (L(\alpha)+U(\alpha))f(\alpha)d\alpha, \quad f(\alpha) = \frac{2\alpha}{\mu^{\frac{1}{\delta}}},$$

where $L(\alpha) = a + \frac{(b-a)\alpha^\delta}{\mu}$, $U(\alpha) = d - \frac{(d-c)\alpha^\delta}{\mu}$, $\mu = \frac{\bar{\mu}+\underline{\mu}}{2}$,

$$V_\nu(A,\delta) = \frac{1}{2}\int_{\nu^{\frac{1}{\delta}}}^1 (L(\beta)+U(\beta))f(\beta)d\beta, \quad f(\beta) = \frac{2(1-\beta)}{1-\nu^{\frac{1}{\delta}}},$$

where $L(\beta) = \frac{b(1-\beta^\delta)+a_1(\beta^\delta-\nu)}{1-\nu}$, $U(\beta) = \frac{c(1-\beta^\delta)+d_1(\beta^\delta-\nu)}{1-\nu}$, $\nu = \frac{\bar{\nu}+\underline{\nu}}{2}$.

Corollary 5.1 *Let A be a $GIVIFN_B$, then using Definition 5.1*

$$V_\mu(A,\delta) = \mu^{\frac{1}{\delta}}(\frac{a}{2}+\frac{d}{2}+\frac{(b-a)-(d-c)}{\delta+2}),$$

$$V_\nu(A,\delta)=\frac{b+c}{(1-\nu)(1-\nu^{\frac{1}{\delta}})}\left[\frac{1}{2}-\frac{1}{(\delta+1)(\delta+2)}-\nu^{\frac{1}{\delta}}+\frac{\nu^{\frac{\delta+1}{\delta}}}{\delta+1}+\frac{\nu^{\frac{2}{\delta}}}{2}-\frac{\nu^{\frac{\delta+2}{\delta}}}{\delta+2}\right],$$
$$+\frac{a_1+d_1}{(1-\nu)(1-\nu^{\frac{1}{\delta}})}\left[\frac{1}{(\delta+1)(\delta+2)}-\frac{\nu}{2}+\frac{\delta\nu^{\frac{\delta+1}{\delta}}}{\delta+1}-\frac{\delta\nu^{\frac{\delta+2}{\delta}}}{2(\delta+1)}\right].$$

In this case $V_\mu(-A,\delta)=-V_\mu(A,\delta)$ and $V_\nu(-A,\delta)=-V_\nu(A,\delta)$.

Theorem 5.1 *Let $A=(a_1,a,b,c,d,d_1,[\underline{\mu},\bar{\mu}],[\underline{\nu},\bar{\nu}],\delta)$, then*

i. $a\mu^{\frac{1}{\delta}}\le V_\mu(A,\delta)\le d\mu^{\frac{1}{\delta}}$,
ii. $(1-\nu^{\frac{1}{\delta}})a_1\le V_\nu(A,\delta)\le(1-\nu^{\frac{1}{\delta}})d_1$.

Proof. (i)

$$\begin{aligned}V_\mu(A,\delta)&=\frac{1}{2}\int_0^{\mu^{\frac{1}{\delta}}}(L(\alpha)+U(\alpha))f(\alpha)d\alpha,\\&=\frac{1}{\mu^{\frac{1}{\delta}}}\int_0^{\mu^{\frac{1}{\delta}}}\left(a+\frac{(b-a)\alpha^\delta}{\mu}+d-\frac{(d-c)\alpha^\delta}{\mu}\right)\alpha d\alpha,\\&\ge\frac{1}{\mu^{\frac{1}{\delta}}}\int_0^{\mu^{\frac{1}{\delta}}}2a\alpha d\alpha=a\mu^{\frac{1}{\delta}}.\end{aligned}$$

$$\begin{aligned}V_\mu(A,\delta)&=\frac{1}{2}\int_0^{\mu^{\frac{1}{\delta}}}(L(\alpha)+U(\alpha))f(\alpha)d\alpha,\\&=\frac{1}{\mu^{\frac{1}{\delta}}}\int_0^{\mu^{\frac{1}{\delta}}}\left(a+\frac{(b-a)\alpha^\delta}{\mu}+d-\frac{(d-c)\alpha^\delta}{\mu}\right)\alpha d\alpha,\\&\le\frac{1}{\mu^{\frac{1}{\delta}}}\int_0^{\mu^{\frac{1}{\delta}}}2d\alpha d\alpha=d\mu^{\frac{1}{\delta}}.\end{aligned}$$

Proof is complete. (ii)

$$\begin{aligned}
V_\nu(A,\delta) &= \tfrac{1}{1-\nu^{\frac{1}{\delta}}}\int_{\nu^{\frac{1}{\delta}}}^{1}\left(\tfrac{b(1-\beta^\delta)+a_1(\beta^\delta-\nu)}{1-\nu}+\tfrac{c(1-\beta^\delta)+d_1(\beta^\delta-\nu)}{1-\nu}\right)\\
&(1-\beta)d\beta\\
&= \tfrac{1}{1-\nu^{\frac{1}{\delta}}}\int_{\nu^{\frac{1}{\delta}}}^{1}\left(\tfrac{(b-a_1\nu)-(b-a_1)\beta^\delta}{1-\nu}+\tfrac{(c-d_1\nu)-(c-d_1)\beta^\delta}{1-\nu}\right)\\
&(1-\beta)d\beta\\
&\geq \tfrac{1}{1-\nu^{\frac{1}{\delta}}}\int_{\nu^{\frac{1}{\delta}}}^{1} 2a_1(1-\beta)d\beta\\
&= (1-\nu^{\frac{1}{\delta}})a_1.
\end{aligned}$$

$$\begin{aligned}
V_\nu(A,\delta) = &\frac{1}{1-\nu^{\frac{1}{\delta}}}\int_{\nu^{\frac{1}{\delta}}}^{1}\left(\frac{b(1-\beta^\delta)+a_1(\beta^\delta-\nu)}{1-\nu}+\frac{c(1-\beta^\delta)+d_1(\beta^\delta-\nu)}{1-\nu}\right)\\
&(1-\beta)d\beta,\\
= &\frac{1}{1-\nu^{\frac{1}{\delta}}}\int_{\nu^{\frac{1}{\delta}}}^{1}\left(\frac{(b-a_1\nu)-(b-a_1)\beta^\delta}{1-\nu}+\frac{(c-d_1\nu)-(c-d_1)\beta^\delta}{1-\nu}\right)\\
&(1-\beta)d\beta,\\
\leq &\frac{1}{1-\nu^{\frac{1}{\delta}}}\int_{\nu^{\frac{1}{\delta}}}^{1} 2a_1(1-\beta)d\beta = (1-\nu^{\frac{1}{\delta}})d_1.
\end{aligned}$$

Proof is complete.

Definition 5.2 *Let A be a $GIVIFN_B$. Then the ambiguity of the membership function A and the non-membership function A are defined as follows:*

$$G_\mu(A,\delta) = \int_0^{\mu^{\frac{1}{\delta}}}(U(\alpha - L(\alpha)))f(\alpha)d\alpha, \quad f(\alpha) = \frac{2\alpha}{\mu^{\frac{1}{\delta}}},$$

$$G_\nu(A,\delta) = \int_{\nu^{\frac{1}{\delta}}}^{1}(U(\beta - L(\beta)))f(\beta)d\beta, \quad f(\beta) = \frac{2(1-\beta)}{1-\nu^{\frac{1}{\delta}}}.$$

Corollary 5.2 *Let A be a $GIVIFN_B$, then*

$$
\begin{aligned}
G_\mu(A,\delta) &= \int_0^{\mu^{\frac{1}{\delta}}} (U(\alpha)-L(\alpha))f(\alpha)d\alpha,\\
&= \frac{1}{\mu^{\frac{1}{\delta}}}\int_0^{\mu^{\frac{1}{\delta}}} \left(d-\frac{(d-c)\alpha^\delta}{\mu}-a-\frac{(b-a)\alpha^\delta}{\mu}\right)2\alpha d\alpha,\\
&= \mu^{\frac{1}{\delta}}\left(d-a-\frac{2(d-c)+2(b-a)}{\delta+2}\right).
\end{aligned}
$$

And

$$
\begin{aligned}
G_\nu(A,\delta) &= \int_{\nu^{\frac{1}{\delta}}}^1 (U(\beta)-L(\beta))f(\beta)d\beta,\\
&= \frac{2(-b+c)}{(1-\nu)(1-\nu^{\frac{1}{\delta}})}\left[\frac{1}{2}-\frac{1}{(\delta+1)(\delta+2)}-\nu^{\frac{1}{\delta}}+\frac{\nu^{\frac{\delta+1}{\delta}}}{\delta+1}+\frac{\nu^{\frac{2}{\delta}}}{2}-\frac{\nu^{\frac{\delta+2}{\delta}}}{\delta+2}\right]\\
&\quad+\frac{2(-a_1+d_1)}{(1-\nu)(1-\nu^{\frac{1}{\delta}})}\left[\frac{1}{(\delta+1)(\delta+2)}-\frac{\nu}{2}+\frac{\delta\nu^{\frac{\delta+1}{\delta}}}{\delta+1}-\frac{\delta\nu^{\frac{\delta+2}{\delta}}}{2(\delta+1)}\right].
\end{aligned}
$$

It can be easily shown that $G_\mu(A,\delta)=G_\mu(-A,\delta)$ *and* $G_\nu(A,\delta)=G_\nu(-A,\delta)$.

Theorem 5.2 *Let* $A=(a_1,a,b,c,d,d_1,[\underline{\mu},\bar{\mu}],[\underline{\nu},\bar{\nu}],\delta)$, *then*

i. $\mu^{\frac{1}{\delta}}(c-b)\le G_\mu(A,\delta)\le \mu^{\frac{1}{\delta}}(d-a)$,
ii. $(1-\nu^{\frac{1}{\delta}})(c-b)\le G_\nu(A,\delta)\le (1-\nu^{\frac{1}{\delta}})(d_1-a_1)$.

Proof. (i)

$$
\begin{aligned}
G_\mu(A,\delta) &= \int_0^{\mu^{\frac{1}{\delta}}} (U(\alpha)-L(\alpha))f(\alpha)d\alpha,\\
&= \int_0^{\mu^{\frac{1}{\delta}}} \left(d-\frac{(d-c)\alpha^\delta}{\mu}-a-\frac{(b-a)\alpha^\delta}{\mu}\right)f(\alpha)d\alpha,\\
&\ge \frac{1}{\mu^{\frac{1}{\delta}}}\int_0^{\mu^{\frac{1}{\delta}}} 2(c-b)\alpha d\alpha=\mu^{\frac{1}{\delta}}(c-b),
\end{aligned}
$$

and

$$
\begin{aligned}
G_\mu(A,\delta) &= \int_0^{\mu^{\frac{1}{\delta}}} \left(d - \frac{(d-c)\alpha^\delta}{\mu} - a - \frac{(b-a)\alpha^\delta}{\mu} \right) f(\alpha) d\alpha, \\
&\leq \frac{1}{\mu^{\frac{1}{\delta}}} \int_0^{\mu^{\frac{1}{\delta}}} 2(d-a)\alpha d\alpha = \mu^{\frac{1}{\delta}}(d-a).
\end{aligned}
$$

Proof is complete.
(ii)

$$
\begin{aligned}
G_\nu(A,\delta) &= \frac{2}{(1-\nu^{\frac{1}{\delta}})} \int_{\nu^{\frac{1}{\delta}}}^1 \left(\frac{c(1-\beta^\delta)+d_1(\beta^\delta-\nu)}{1-\nu} - \frac{b(1-\beta^\delta)+a_1(\beta^\delta-\nu)}{1-\nu} \right) \\
&\quad (1-\beta) d\beta, \\
&= \frac{2}{(1-\nu^{\frac{1}{\delta}})} \int_{\nu^{\frac{1}{\delta}}}^1 \left(\frac{(c-d_1\nu)-(c-d_1)\beta^\delta}{1-\nu} - \frac{(b-a_1\nu)-(b-a_1)\beta^\delta}{1-\nu} \right) \\
&\quad (1-\beta) d\beta, \\
&\geq \frac{2}{(1-\nu^{\frac{1}{\delta}})} \int_{\nu^{\frac{1}{\delta}}}^1 (c-b)(1-\beta) d\beta = (1-\nu^{\frac{1}{\delta}})(c-b).
\end{aligned}
$$

$$
\begin{aligned}
G_\nu(A,\delta) &= \frac{2}{(1-\nu^{\frac{1}{\delta}})} \int_{\nu^{\frac{1}{\delta}}}^1 \left(\frac{c(1-\beta^\delta)+d_1(\beta^\delta-\nu)}{1-\nu} - \frac{b(1-\beta^\delta)+a_1(\beta^\delta-\nu)}{1-\nu} \right) \\
&\quad (1-\beta) d\beta, \\
&= \frac{2}{(1-\nu^{\frac{1}{\delta}})} \int_{\nu^{\frac{1}{\delta}}}^1 \left(\frac{(c-d_1\nu)-(c-d_1)\beta^\delta}{1-\nu} - \frac{(b-a_1\nu)-(b-a_1)\beta^\delta}{1-\nu} \right) \\
&\quad (1-\beta) d\beta, \\
&\leq \frac{2}{(1-\nu^{\frac{1}{\delta}})} \int_{\nu^{\frac{1}{\delta}}}^1 (d_1-a_1)(1-\beta) d\beta = (1-\nu^{\frac{1}{\delta}})(d_1-a_1).
\end{aligned}
$$

Proof is complete.

Definition 5.3 *Let* $A = (a_1, a, b, c, d, d_1, [\underline{\mu}, \bar{\mu}], [\underline{\nu}, \bar{\nu}], \delta)$. *A value index and an ambiguity index for the* A *are defined as follows:*

$$
V(A,\delta) = \frac{V_\mu(A,\delta) + V_\nu(A,\delta)}{2}, \quad G(A,\delta) = \frac{G_\nu(A,\delta) + G_\mu(A,\delta)}{2}.
$$

Procedure for ranking $GIVIFN_B$.
Let $A = (a_1^{'}, a_1, b_1, c_1, d_1, d_1^{'}, [\underline{\mu}_1, \bar{\mu}_1], [\underline{\nu}_1, \bar{\nu}_1], \delta)$
and $B = (a_2^{'}, a_2, b_2, c_2, d_2, d_2^{'}, [\underline{\mu}_2, \bar{\mu}_2], [\underline{\nu}_2, \bar{\nu}_2], \delta)$ be two $GVIIFN_Bs$, then a ranking function is as follows

$$R(A, \delta) = V(A, \delta) + G(A, \delta),$$

where

i. If $R(A, \delta) > R(B, \delta)$, then $A > B$.
ii. If $R(A, \delta) < R(B, \delta)$, then $A < B$.
iii. If $R(A, \delta) = R(B, \delta)$, then $A = B$.

Example 5.1 *Let* $A = (0.5, 1, 2, 3, 4, 4.5, [0.49, 0.64], [0.25, 0.36], 2)$, $\mu_A = 0.565$, $\nu_A = 0.305$, *and* $B = (0.75, 1.25, 2, 3, 3.75, 4.25, [0.36, 0.49], [0.16, 0.25, 2])$, $\mu_B = 0.425$, $\nu_B = 0.205$, *then*
$V_\mu(A, \delta) = 1.879$, $V_\nu(A, \delta) = 0.9932$, $V(A, \delta) = 1.4361$, $V_\mu(B, \delta) = 1.95$, $V_\nu(B, \delta) = 1.3276$, $V(B, \delta) = 1.6388$, $G_\mu(A, \delta) = 1.503$, $G_\nu(A, \delta) = 0.6327$, $G(A, \delta) = 1.0662$, $G_\mu(B, \delta) = 1.141$, $G_\nu(B, \delta) = 0.8451$, $G(B, \delta) = 0.99305$, $R(A, \delta) = 2.5023$, $R(B, \delta) = 2.63185$. *Since* $R(A, \delta) < R(B, \delta)$, *therefore* $A < B$.

6 Conclusions

We have introduced new generalized interval valued intuitionistic fuzzy numbers. We studied (α_1, α_2)-cut, (β_1, β_2)-cut and $(\vec{\alpha}, \overleftarrow{\beta})$-cut of $GIVIFN_B$. Then, the values and ambiguities of the membership degree and the non-membership degree and the value index and ambiguity index for $GIVIFN_Bs$ are defined. They are used to define ranking function of $GIVIFN_B$.

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