



Best approximation by closed unit balls

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Abstract

We obtain a sufficient and necessary theorems simple for compactness and weakly compactness of the best approximate sets by closed unit balls. Also we consider relations Kadec-Klee property and Shur property with this objects. These theorems are extend of papers mohebi and Narayana.

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1 Introduction

Let W be a non-empty subset of a normed linear space X . For any $x \in X$, the (possibly empty) set of best approximations x from M is defined by

$$P_W(x) = \{y \in W : \|x - y\| = d(x, W)\},$$

where $d(x, W) = \inf\{\|x - y\| : y \in W\}$.

The subset W is said to be proximal if the set $P_W(x)$ is non-empty for every $x \in X$.

We denote by B_X the closed unit ball of X , that is

$$B_X = \{x \in X : \|x\| \leq 1\}.$$

Theorem 1.1. [7] *Let X be a normed linear space, W a subspace of X . For $x \in X$*

$$P_W(x) = W \cap (x + d(x, W))B_X.$$

Example 1.2. Consider $X = R^2$ with two norm

$$\|(x, y)\|_1 = \max(|x|, |y|), \quad \|(x, y)\|_2 = |x| + |y|,$$

also $W = \{(x, y) : y = 0, x \in R\}$.

With $\|\cdot\|_1$

$B_{R^2} = \{(x, y) : \max\{|x|, |y|\} \leq 1\}$ and $d((0, 1), W) = \inf\{\|(x, 1)\|_1 : x \in R\} = 1$. Therefore

$$((0, 1) + d((0, 1), W)B_{R^2}) \cap W = \{(x, 0) : \max\{|x|, 1\} \leq 1\} = \{(x, 0) : |x| \leq 1\}.$$

With $\|\cdot\|_2$

$B_{R^2} = \{(x, y) : |x| + |y| \leq 1\}$ and $d((0, 1), W) = \inf\{\|(x, 1)\|_2 : x \in R\} = \inf\{|x| + 1 : x \in R\} = 1$. Also

$$((0, 1) + d((0, 1), W)B_{R^2}) \cap W = \{(0, 0)\}.$$

Example 1.3. The space $X = C[0, 1]$ with norm $\|f\| = \sup_{x \in [0, 1]} |f(x)|$ is a normed linear space, we consider the subspace $W = \{f \in X : f(0) = 0\}$ of X . If $f \in X$, then $g = f - f(0)$ is a best approximation of f in W and

$$B_X = \{f \in X : \|f\| \leq 1\} = \{f \in X : |f(x)| \leq 1 \forall x \in [0, 1]\},$$

also $d(f, W) = |f(0)|$. Then

$$P_W(f) = W \cap \{f + |f(0)|B_X\} = \{g \in X : g(0) = 0, \frac{|g(x) - f(x)|}{|f(0)|} \leq 1 \forall x \in [0, 1]\}.$$

Also it follows that $g = f + f(0)$ is a best approximation of f .

2 Best approximation by closed unit balls

In this section we will consider the best approximation and compactness and weakly compactness by closed unit balls. The following Theorems are extends of Lemma 2.1 and theorem 2.1, of [5].

Theorem 2.1. *Let X be a normed linear space, W a linear subspace of X .*

(a) *If W is proximal, then for $r > 0$ there exists $z \in rB_X$ such that $d(z, W) = r$.*

(b) *If for $r > 0$ there exists $z \in rB_X$ such that $d(z, W) = r$ and $\text{codim}(W) = 1$, then W is proximal.*

Proof. a) Suppose W is proximal and $r > 0$ be given. For $x \in X \setminus W$ consider $g_0 \in P_W(x)$. Put $y = \frac{x - g_0}{d(x, W)}$. Then $y \in B_X$ and $d(y, W) = 1$. If $z = ry$, we have $z \in rB_X$ and $d(z, W) = r$.

b) If for $r > 0$, there exists a $z \in rB_X$ such that $d(z, W) = r$. Since $\text{codim}(W) = 1$, we have $X = W \oplus \langle z \rangle$. For arbitrary $x \in X \setminus W$. There exists the element $h \in W$ and the scalar α such that $x = h + \frac{\alpha}{r}z$. In this case $h \in P_W(x)$, and the set $P_W(x) \neq \emptyset$. ■

Conclusion 2.2. *Let X be a normed linear space, W a linear subspace of X with codimension n . Then W is proximal if and only if for $r > 0$ there exists $z_1, z_2, \dots, z_n \in rB_X$ such that $d(z_i, W) = r$ for every $1 \leq i \leq n$.*

Theorem 2.3. Let X be a normed linear space, W a closed linear subspace of X . Then the following statements are equivalent:

(a) W is proximal.

(b) For every $x \in X$ in the subspace of X to form $F_x = W \oplus \langle x \rangle$ there exists z such that $d(z, W) = 1$.

Proof. From Theorem 2.1 and $\text{codim}_{F_x} W = 1$, we have $a) \Rightarrow b)$. Also $X = \bigcup_{x \in X} F_x$, then we have $b) \Rightarrow a)$.

Theorem 2.4. Let X be a normed linear space, W a linear subspace of X and $r > 0$.

(a) If there exists a unique $z \in rB_X$ such that $d(z, W) = r$. Then W is Chebyshev.

(b) If W is Chebyshev and $\text{codim}(W) = 1$, Then there exists a $z \in rB_X$ such that $d(z, W) = r$ and for any $z' \in rB_X$ such that $d(z', W) = r$. Then $z = \alpha z'$ where $|\alpha| = 1$.

Proof. (a) If there exists a $z \in rB_X$, such that $d(z, W) = r$. Then by Lemma 1.4, W is proximal. For $x \in X$ suppose $g_1, g_2 \in P_W(x)$, consider $z_i = r \frac{x - g_i}{d(x, W)}$, for $i = 1, 2$. Then $z_i \in rB_X$ and $d(z_i, W) = r$, for $i = 1, 2$. Therefore $z_1 = z_2$ and it follows that $g_1 = g_2$.

(b) Since W is proximal, by Lemma 1.3, there exists a $z \in rB_X$ such that $d(z, W) = r$. If $z' \in X$ and $z' \in rB_X$ such that $d(z', W) = r$. Since $\text{codim}(W) = 1$, $X = W \oplus \langle z \rangle = W \oplus \langle z' \rangle$, therefore there exists a $g \in W$ such that $z = g + \alpha z'$. Also

$$\|z\| = d(z, W) = r = \|z - g\| = d(z, W).$$

It follows that $0, g \in P_W(z)$. Because W is Chebyshev. Then $g = 0$ and $z = \alpha z'$. Also $\|z\| = \|z'\| = r$, therefore $|\alpha| = 1$.

Example 2.5. Suppose $X = R^2$ and $W = \{(x, 0) : x \in R\}$. Then

$\text{codimen}(W) = 1$, if $z = (0, 1)$. Then $d(z, W) = 1$ and $\|Z\|_2 = 1$. by Theorem 2.1, W is proximal. If there exists a $z' = (x, y)$ such that $d(z', W) = \|z'\|_2 = 1$. Then $|x| + |y| = 1$ and $\min\{|x - x_0|, |y| = 1\}$. Theorem $(x, y) = y(0, 1)$ and $|y| = 1$, by Theorem 2.4, W is Chebyshev.

Theorem 2.6. *Let X be a normed linear space, W a linear subspace of X and $r > 0$.*

(a) *If for every sequence $\{x_n\} \subseteq rB_X$ by $d(x_n, W) = r$, has a convergent subsequence (weakly convergent subsequence). Then the set $P_W(x)$ is compact (weakly compact) for every $x \in X \setminus W$.*

(b) *If the set $P_W(x)$ is compact (weakly compact) for every $x \in X \setminus W$ and $\text{codimen}(W) = 1$. Then for $r > 0$ and every sequence $\{x_n\} \subseteq rB_X$, by $d(x_n, W) = r$, has a convergent subsequence (weakly convergent subsequence).*

Proof. (a) Suppose $x \in X \setminus W$ and $\{g_n\} \subseteq P_W(x)$. Hence $x_n = r \frac{x - g_n}{d(x, W)} \in rB_X$ and $d(x_n, W) = r$. There exists a subsequence $\{x_{n_k}\}$ and $x_0 \in X$, such that $x_n \rightarrow x_0$ ($x_{n_k} \rightarrow x_0$). Therefore

$$g_{n_k} \rightarrow x - \frac{1}{r}x_0d(x, W) \quad (g_{n_k} \rightarrow x - \frac{1}{r}x_0d(x, W)).$$

Since W is closed (weakly closed) $g_0 = x - \frac{1}{r}x_0d(x, W) \in W$. Also $\frac{1}{r}x_0d(x, W) \in d(x, W)B_X$, Therefore $g_0 \in P_W(x)$.

(b) Since $\text{codim}(W) = 1$, there exists $z \in X$ such that $X = W \oplus \langle z \rangle$. Choose $r > 0$, Suppose $\{x_n\} \subseteq rB_X$ and $d(x_n, W) = r$. Therefore

$$x_n = g_n + k_n z$$

for some $g_n \in W$ and scalars k_n . Thus $\frac{g_n}{k_n} \in P_W(z)$ and $|k_n| = \frac{r}{d(z, W)}$. Since $\{k_n\}$ is a bounded sequence of scalars, has a convergent subsequence and $\{g_n\}$ has a convergent sequence (weakly convergent subsequence). Therefore $\{x_n\}$ has a convergent sequence (weakly convergent subsequence).

If we omit the condition $\text{codimen}(W) = 1$ at (b). Then the theorem is

not true.

Example 2.7. Let $W = W_0 = \ell^1$ with the standard basis $\{e_n\}_{n \geq 1}$. Put $X = W \oplus W_0$ and define a norm on X by

$$\|x + y\| = \sum_{n=1}^{\infty} [|c_n| \vee (2^{-n} \|y\|)] < +\infty,$$

where $\sum_{n=1}^{\infty} c_n e_n = x \in W$, $y \in W_0$ and $a \vee b = \max(a, b)$.

Then by [3], the set $P_W(x)$ is compact for every $x \in X \setminus W$. Let $x_n = (0, \dots, 0, 1, 0, \dots)$, $n = 1, 2, \dots$; where the term 1 is in the n th place. Then $d(x_n, W) = 1$ and $\|x_n\| = 1$. But the sequence $\{x_n\}$ hasn't any convergent sequence.

Definition 2.8. let X be a normed space. The set X is said to have the sequential Kadec-Klee property if weak and norm sequential convergence coincide on $S_X = \{x \in X : \|x\| = 1\}$.

Theorem 2.9. *Let X be a normed linear space, X is a reflexive space and has the Kadec-Klee property. Then in every closed linear subspace of W of X , the set $P_W(x)$ is compact for every $x \in X \setminus W$.*

Proof. Since X is reflexive, the closed unit ball B_X is weakly compact. Choose $x \in X \setminus W$, consider the sequence $\{g_n\} \subseteq P_W(x)$. We define

$$x_n = \frac{x - g_n}{d(x, W)}.$$

Then $x_n \in B_X$ and $d(x_n, W) = 1$. Therefore there exists a subsequence $\{x_{n_k}\}$ and $x_0 \in B_X$ such that $x_{n_k} \rightharpoonup x_0$. Since X has Kadec-Klee property, $x_{n_k} \rightarrow x_0$. Thus

$$g_{n_k} \rightarrow x - x_0 d(x, W).$$

Then the set $P_W(x)$ is compact.

Conclusion 2.10. *Let X be a reflexive normed linear space, closed linear subspace of W of X has Kadec-Klee property, Then the set $P_W(x)$ is compact for every $x \in X \setminus W$.*

Remark 2.11. *Let X be a normed linear space, W a linear subspace of X and $x \in X \setminus W$. then*

$$x + d(x, W)B_X = \{g \in X : \|x - g\| = d(x, W)\}.$$

Theorem 2.12. *Let X be a normed linear space, W a linear subspace of X and $x \in X \setminus W$. If the set $x + d(x, W)B_X$ is compact (weakly compact). Then for $r > 0$ and every sequence $\{x_n\} \subseteq rB_X$ by $d(x_n, W) = r$, has a convergent subsequence (weakly convergent subsequence).*

Proof. For $r > 0$, we consider the sequence $\{x_n\} \subseteq rB_X$ by $d(x_n, W) = r$. Define the sequence $\{g_n\}$ by $g_n = x - \frac{x_n}{r}d(x, W)$. Then $\|g_n - x\| = d(x, W)$ and $\{g_n\} \subseteq x + d(x, W)B_X$ and $\{g_n\}$ has a convergent subsequence (weakly convergent subsequence). Therefore $\{x_n\}$ has a convergent subsequence (weakly convergent subsequence).

Definition 2.13. A Banach space X has the Schur property if every weakly null sequence in X is norm null.

Theorem 2.14. *Let X be a reflexive normed linear space. If X has the Schur property. Then for $r > 0$ and every sequence $\{x_n\} \subseteq rB_X$ by $d(x_n, W) = r$, has a convergent subsequence (weakly convergent subsequence).*

Proof. The set $x + d(x, W)B_X$ is compact for every $x \in X \setminus W$. Form Theorem 2.8, for $r > 0$ and every sequence $\{x_n\} \subseteq rB_X$ by $d(x_n, W) = r$, has a convergent subsequence (weakly convergent subsequence).

3 The orthogonality by closed unit balls

Definition 3.1. Let X be a normed linear space, W a linear subspace of X and $x \in X$. We say that x is Birkhoff orthogonality with W and denoted by $x \perp^B W$ if and only if $\|x\| \leq \|x + \alpha y\|$ for every $y \in W$ and for every scalar α . (see [7]) If $y \in X$, then $x \perp^B y$ if and only if $x \perp^B \langle y \rangle$.

Theorem 3.2. Let X be a normed linear space, W a linear subspace of X and $x \in X$. Then $x \perp^B W$ if and only if $x \in d(x, W)B_X$.

Proof. We have

$$\begin{aligned} x \perp^B W &\iff \|x\| \leq \|x + \alpha y\| \quad \forall y \in W, \quad \forall \alpha \text{ scalar} \\ &\iff \|x\| \leq d(x, W) \\ &\iff x \in d(x, W)B_X. \end{aligned}$$

Definition 3.3. Let X be a normed linear space, W a linear subspace of X , $x \in X$ and $\epsilon > 0$. We say that $x \perp_\epsilon^B W$ if and only if $\|x\| \leq \|x + \alpha y\| + \epsilon$ for every $y \in W$ and for every scalar α , (see [7]).

Corollary 3.4. Let X be a normed linear space, W a linear subspace of X , $x \in X$ and $\epsilon > 0$. Then $x \perp_\epsilon^B W$ if and only if $x \in (d(x, W) + \epsilon)B_X$.

Lemma 3.5. Let X be a normed linear space, W a linear subspace of X and $x \in X \setminus W$. Define

$$W_x = \{f \in X^* : f|_W = 0, f(x) = d(x, W)\}.$$

Then W_x is closed and convex.

Proof. The Proof is trivial.

Theorem 3.6. [7] *Let X be a normed linear space, W a linear subspace of X , $x \in X \setminus W$ and $M \subseteq W$. Then $M \subseteq P_W(x)$ if and only if there exists $f \in X^*$ such that*

$$f|_W = 0, \|f\| = 1, f(x - g_0) = \|x - g_0\| \text{ for all } g_0 \in M.$$

For every $g_0 \in P_W(x)$ such $f \in X^$ is denoted by f_{g_0} .*

Theorem 3.7. *Let X be a normed linear space, W a linear subspace of X and $x \in X \setminus W$. Then the map*

$$\Psi : P_W(x) \longrightarrow B_{X^*} \cap W_x,$$

where for $g_0 \in P_W(x)$ we have $\Psi(g_0) = f_{g_0}$, is well define and onto.

Corollary 3.8. *Let X be a normed linear space, W a linear subspace of X and $x, y \in X \setminus W$. Then*

a) $x \perp^B y$ if and only if there exists $f \in X^*$ such that

$$\|f\| = 1, f(x) = \|x\| \text{ and } f(y) = 0;$$

b) $x \perp^B W$ if and only if there exists $f \in X^*$ such that

$$\|f\| = 1, f(x) = d(x, W) \text{ and } f|_W = 0;$$

relatively low numbers of data points.

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