



## Some notes on convergence of homotopy based methods for functional equations

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### Abstract

Although homotopy-based methods, namely homotopy analysis method and homotopy perturbation method, have largely been used to solve functional equations, there are still serious questions on the convergence issue of these methods. Some authors have tried to prove convergence of these methods, but the researchers in this article indicate that some of those discussions are faulty. Here, after criticizing previous works, a sufficient condition for convergence of homotopy methods is presented. Finally, examples are given to show that even if the homotopy method leads to a convergent series, it may not converge to the exact solution of the equation under consideration.

*Key words:* Homotopy analysis method; Homotopy perturbation method; Convergence theorem; Banach fixed point theorem; Series solution.

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## 1 Introduction

The last two decades of analytic solution methods for solving functional equations have been dominated by homotopy approaches. Homotopy-based methods which are generally known to be homotopy analysis method (HAM) [1] and homotopy perturbation method (HPM) [2], are powerful methods which have proved their efficiency through solving vast variety of functional equations. Ordinary and partial differential equations, fractional differential equations, integral and integro-differential equations are the areas which homotopy methods have been successfully applied. Especially, HAM has been employed to solve some types of differential equations, for the solution of which neither analytic methods nor numerical techniques have been useful [3,4]. These facts prove the great potential behind homotopy approaches.

Other than the known applications of homotopy methods in solving functional equations, there have been some interesting innovations, for example Abbasbandy and Liao have proposed a new *Newton-homotopy method*, which is a generalization of the famous Newton-Raphson method [5]. Also Liao has recently introduced a transform called *homotopy-transform*, which he proves that Euler transform is a special case of it [6]. The homotopy idea has also been used for computing Laplace transform [7], Sumudu transform [8] and Fourier transform [9]. Some other applications of HPM and HAM could be found in [10,11,12] and the references therein.

Despite of these novel applications, there are still serious questions concerning convergence of these methods. Attempts have been done in this direction, but none of them have a universal result and unfortunately some are mathematically wrong.

In almost all analytic and semi-analytic approaches, used for solving functional equations (of which homotopy-based ones are special cases), there are two main concerns about the convergence of the obtained solution, i.e. the convergence of the solution series in its own right, and the convergence to the *exact* solution.

In order to establish the convergence problem more rigorously, a common terminology for HAM and HPM has been used in this paper. Of course

HAM has more auxiliary elements and is more flexible in comparison with HPM, but this common framework is a simple one that allows us to concentrate on the convergence issue and is, hence, sufficient for our discussion.

Consider a nonlinear equation such as  $A(u) = 0$ , where  $A$  is an operator and  $u$  is an unknown function to be determined. Suitably choosing operator  $L$  (usually a linear operator), we construct the "homotopy equation" as follows:

$$H(u, p) = (1 - p)L(u) + pA(u) = 0. \quad (1.1)$$

Assume that the solution of the homotopy equation could be presented as  $u(p) = u_0 + u_1p + u_2p^2 + \dots$ . For  $p = 0$  this equation is equivalent to  $L(u) = 0$ . This equation should be an easy-to-solve one, so this fact must be considered in choosing  $L$ . For  $p = 1$  the homotopy equation is  $A(u) = 0$ , i.e. the original equation.

As  $p \rightarrow 1$ , the homotopy equation (1) converges to the original equation, so we expect  $u(p) = u_0 + u_1p + u_2p^2 + \dots$  to converge to the solution of the original equation when  $p \rightarrow 1$ . However there are issues which should be mentioned about this convergence.

The current paper generally discusses the convergence issue of homotopy methods. In so doing, the previous works in this regard are discussed and criticized in section 2. In the rest of the article the reader will come across a couple of counterexamples on previous convergence results, which modifies our classic thoughts on convergence of homotopy-based methods. Later on, a theorem is presented which gives sufficient conditions for convergence in special class of functional equations.

## 2 Comments on previous results

Biazar and Ghazvini in [13] and Biazar and Aminikhah in [14] have tried to prove the convergence of homotopy method, in the special case of HPM, through discussions based on Banach fixed point theorem.

In [13], page 2634, the authors asserted that in the sequence obtained

from HPM the  $n$ th order approximation, i.e.  $V_n$  is dependent upon the  $(n - 1)$ th order approximation  $V_{n-1}$ . This is not true, because when we use HPM for solving a nonlinear equation  $u_n$ , the  $n$ th term of the iteration method is dependent upon previous terms  $u_0, u_1, \dots, u_{n-1}$  and not necessarily their sums. Thus, it is obvious that the  $n$ th order approximation  $V_n$  is also dependent upon previous terms and not necessarily their sum. This is the fact that can be easily seen in Example 1 and 3 in the same paper, where the authors have given the governing equation for  $V_n$  in pages 2636 and 2639. In fact every nonlinear equation can be a counterexample for their assertion.

The same assumption is used in paper [14], so from the above discussion it is obvious that the proposed assumption in those papers are incorrect, therefore their results are invalid and useless.

In both papers, the presented examples are such that they satisfy a contraction property that ensures convergence of the resulted series, if it is not the case we have no guarantee for convergence of HPM. Actually both papers have a misuse of Banach fixed point theorem.

In [15], Odibat uses an application of contraction mapping theorem to prove convergence of HAM. In page 784, it is not clear why the author has presented the initial guess in the special form (20). The convergence criteria of Theorem 1, which is a famous result of Banach's fixed point theorem, is hard to check specially in the cases where there is no general form for the series' terms. Thus the criteria is not practical for a method like HPM.

In page 785 the discussions on  $\beta_i$ 's are faulty, one can simply check the series  $\sum \frac{x}{i}$ . Here we have  $\phi_i(x) = \frac{x}{i}$ , so  $\beta_i = \frac{\|\phi_{i+1}\|}{\|\phi_i\|} = \frac{1/(\sqrt{3}(i+1))}{1/(\sqrt{3}i)} = \frac{i}{i+1} < 1$ , but  $\sum \phi_i(x)$ , is not convergent for any  $x$  but 0.

In both examples, which are given to assure discussed claims, the author has restricted the domain of the problem to  $[0, 1]$ , this interval simplifies the user to check the convergence criteria, however one may fail to obtain efficient results for the original domain, namely  $[0, \infty]$ .

In HAM case there is a well known theorem by Liao [16] which reduces the convergence problem to the convergence of the resulted series. His

claim is as follows:

**Theorem 2.1:** [9, Theorem 2.1 and Theorem 3.3] *Whenever the solution series obtained by the homotopy analysis method is convergent, it would converge to the exact solution of the equation under study.*

However there is a mathematical problem with his proof. Liao has an incorrect assumption that whenever  $\Sigma u_n(t)$  are convergent then  $\Sigma u'_n(t)$  would also converge. For rejecting this claim consider the series of functions  $\Sigma u_n(t)$ , where  $u_n(t) = (t^n)/n^2$ , this series converges at  $t = 1$ , but  $\Sigma u'_n(t)$  is not convergent at the same point (as another example  $\Sigma u_n(x)$ , with  $u_n(x) = \frac{\sin(nx)}{n^2}$  is convergent on  $\mathbb{R}$ , but  $\Sigma u'_n(t)$  is not convergent at any point  $x$ ). So the following discussion in those theorems are useless.

On the other side in HPM or HAM we replace a nonlinear equation with infinitely linear equations, where in the case of differential equation, the order of  $L$  is not necessarily the order of the original equation [17,18]. So in using HPM or HAM, for solving differential equations, we may not apply all initial boundary conditions of the equation under study for its resulted sub equations. This guides us to the fact that, sometimes, the final solution of HPM or HAM is a solution which does not satisfy the initial boundary conditions, so is not the solution of the main equation.

Next section gives counterexamples to reject Theorem 2.1, then a modification of the theorem is presented.

### 3 Counterexamples and a modified convergence theorem

The following examples show that the solution of homotopy method is not always the solution of the main equation under study, even if the convergence radius of homotopy solution is more than 1.

**Example 3.1:** Consider the Laplace equation  $u_{xx} + u_{yy} = 0$ , subject to

Neumann boundary conditions:

$$\begin{aligned} u_y(x, 0) &= 0, & u_y(x, \pi) &= 0, \\ u_x(0, y) &= \cos(y), & u_x(\pi, y) &= \cosh(\pi)\cos(y). \end{aligned}$$

The exact solution of the equation is  $u(x, y) = \sinh(x)\cos(y)$ . We illustrate the HPM case which is more simple. Let  $L(u) = u_{yy}$ , so the homotopy equation will be:

$$H(u, p) = (1 - p)L(u) + pN(u) = 0,$$

or

$$H(u, p) = u_{yy} + pu_{xx} = 0.$$

By substituting  $u = u_0 + u_1p + u_2p^2 + \dots$  in the homotopy equation and doing homotopy solution process we would have:

$$\begin{aligned} u_{0yy} &= 0, & u_{0y}(x, 0) &= 0, & u_{0y}(x, \pi) &= 0, \\ u_{1yy} &= -u_{0xx}, & u_{1y}(x, 0) &= 0, & u_{1y}(x, \pi) &= 0, \\ u_{2yy} &= -u_{1xx}, & u_{2y}(x, 0) &= 0, & u_{2y}(x, \pi) &= 0, \\ & & \vdots & & & \end{aligned}$$

Solving the above equations, considering boundary conditions (BC), we have:

$$\begin{aligned} u_0 = f_0(x)y + g_0(x) & \xrightarrow{\text{BC}} f_0(x) = 0 \\ & \implies u_0 = g_0(x) \\ u_1 = g_0(x)y^2/2 + f_1(x)y + g_1(x) & \xrightarrow{\text{BC}} g_0''(x) = 0, f_1(x) = 0, \\ & \implies g_0(x) = a_0x + b_0, u_1 = g_1(x). \end{aligned}$$

Solving other equations as well we get  $u_i = a_i x + b_i$ , for  $i = 0, 1, 2, \dots$ . Now we have  $u = u_0 + u_1 + u_2 + \dots = (\sum a_i)x + (\sum b_i)$ . It is obvious that even if  $\sum a_i, \sum b_i$ , are convergent, still this is not the solution of the main equation.

**Example 3.2:** Consider the heat equation:

$$u_t - 9u_{xx} = 0,$$

$$u(x, 0) = \sin(x), \quad u(0, t) = 0, \quad u\left(\frac{\pi}{2}, t\right) = 1.$$

The exact solution of this equation is  $u(x, t) = \frac{2}{\pi}x + \sum_{n=1}^{\infty} B_n e^{-36n^2t} \sin(2nx)$ , where  $B_n$ 's are the Fourier coefficients for the  $\sin(x) - \frac{2}{\pi}x$ , i.e.

$$B_n = 2 \int_0^1 (\sin(x) - \frac{2}{\pi}x) \sin(2nx) dx$$

In this equation we have two choices for  $L$ , namely  $Lu = u_t$ ,  $Lu = -9u_{xx}$ .

We will show that in both cases, the solution of homotopy method is convergent, but does not satisfy initial and boundary conditions simultaneously, so the convergence of the series does not imply that it is the exact solution of the equation.

*Case 1:*  $Lu = u_t$ .

In this case the homotopy equation will be as follows:

$$H(u, p) = (1 - p)u_t + p(u_t - 9u_{xx}) = 0,$$

or

$$H(u, p) = u_t - 9pu_{xx} = 0.$$

By the assumption  $u = u_0 + u_1p + u_2p^2 + \dots$ , we have the following sub equations:

$$\begin{aligned} u_{0t} &= 0, & u_0(x, 0) &= \sin(x), \\ u_{1t} &= 9u_{0xx}, & u_1(x, 0) &= 0, \\ u_{2t} &= 9u_{1xx}, & u_2(x, 0) &= 0, \\ &\vdots & &\vdots \end{aligned}$$

solving the above equations we have:

$$\begin{aligned} u_0 &= \sin(x), \\ u_1 &= -\sin(x)(9t), \\ u_2 &= \sin(x)\frac{(9t)^2}{2}, \\ u_3 &= -\sin(x)\frac{(9t)^3}{3!}, \\ &\vdots \end{aligned}$$

so the final solution is

$$\begin{aligned} u &= u_0 + u_1 + u_2 + \dots \\ &= \sin(x)\left(1 - 9t + \frac{(9t)^2}{2!} + \frac{(9t)^3}{3!} + \dots\right) \\ &= \sin(x)e^{-9t}. \end{aligned}$$

It is obvious that  $\sin(x)e^{-9t}$  is not the exact solution of the above heat equation.

*Case 2:*  $Lu = -9u_{xx}$ .

In this case the homotopy equation will be as follows:

$$H(u, p) = (1 - p)(-9u_{xx}) + p(u_t - 9u_{xx}) = 0,$$

or

$$H(u, p) = pu_t - 9u_{xx} = 0.$$

By the assumption  $u = u_0 + u_1p + u_2p^2 + \dots$ , we have the following sub equations:

$$\begin{aligned} u_{0xx} &= 0, & u_0(0, t) &= 0, u_0\left(\frac{\pi}{2}, t\right) &= 1, \\ u_{1xx} &= \frac{1}{9}u_{0t}, & u_1(0, t) &= u_1\left(\frac{\pi}{2}, t\right) &= 0, \\ u_{2xx} &= \frac{1}{9}u_{1t}, & u_2(0, t) &= u_2\left(\frac{\pi}{2}, t\right) &= 0, \\ & \vdots & & & \end{aligned}$$

solving the above equations we have:

$$\begin{aligned} u_0 &= \frac{2}{\pi}x, \\ u_1 &= 0, \\ u_2 &= 0, \\ u_3 &= 0, \\ & \vdots \end{aligned}$$

so the final solution would be  $u = u_0 + u_1 + u_2 + \dots = \frac{2}{\pi}x$  which is not the exact solution.



Now we can modify the convergence theorem as follows:

**Theorem 3.3:** Consider the equation  $A(u(t)) = 0$ , for  $t \in [a, b]$ . Here  $A$  is an integral operator. If the solution of the homotopy method,  $u(t, p) = u_0 + u_1(t)p + u_2(t)p^2 + \dots$ , ( $u_i(t)$ 's are continuous in  $[a, b]$ ), is convergent at  $p = 1$  and this convergence is uniform on  $[a, b]$ , then  $u(t, 1)$  is the solution to the main equation.

**Proof.** Consider that  $A = L + N$ , so equation (1) can be rewritten as:

$$H(u, p) = L(u) + pN(u) = 0, \quad (3.1)$$

where  $L$  is linear. Let  $u(p) = u_0 + u_1p + u_2p^2 + \dots$  represent solution to (2), as we know  $u_i$ 's are obtained from the sub equations of the homotopy methods (both HPM and HAM) as follows:

$$\begin{aligned} Lu_0 &= 0 \\ Lu_1 &= -N(u) \quad \text{at } p = 0 \\ Lu_2 &= -\frac{\partial N(u)}{\partial p} \quad \text{at } p = 0 \\ Lu_3 &= -\frac{\partial^2 N(u)}{2!\partial p^2} \quad \text{at } p = 0 \\ &\vdots \end{aligned} \quad (3.2)$$

Let  $N_k = \frac{\partial^k N(u)}{k!\partial p^k}$ , at  $p = 0$ . Taking the sum of the above equations we have:

$$\sum_{k=0}^{\infty} L(u_k) = -\sum_{k=0}^{\infty} N_k. \quad (3.3)$$

Now because of uniform convergent of  $u_i$ 's, we can change the order of integration and sigma i.e.  $\sum_{k=0}^{\infty} L(u_k) = L(\sum_{k=0}^{\infty} u_k) = L(u(1))$ . Moreover from the Taylor expansion we have,  $N(u(p)) = \sum_{k=0}^{\infty} N_k p^k$ , and for  $p = 0$ , we have  $N(u(1)) = \sum_{k=0}^{\infty} N_k$ . Now the equation (3) is  $L(u(1)) + N(u(1)) = 0$ , or  $A(u(1)) = 0$ .  $\square$  For operators like differentiation we need more assumption to guarantee the convergence. As we know HPM is a special case of HAM for  $h = -1$ , so results of convergence on HAM must be valid for HPM too.

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