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Bifurcation Problem for Biharmonic Asymptotically Linear Elliptic Equations

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Abstract

In this paper, we investigate the existence of positive solutions for the elliptic equation $\Delta^2 u + c(x)u = \lambda f(u)$ on a bounded smooth domain Ω of $\mathbb{R}^n$, $n \geq 2$, with Navier boundary conditions. We show that there exists an extremal parameter $\lambda^* > 0$ such that for $\lambda < \lambda^*$, the above problem has a regular solution but for $\lambda > \lambda^*$, the problem has no solution even in the week sense. We also show that $\lambda^* = \frac{\lambda_1}{a}$ if $\lim_{t\to\infty} f(t) - at = l \geq 0$ and for $\lambda < \lambda^*$, the solution is unique but for $l < 0$ and $\frac{\lambda_1}{a} < \lambda < \lambda^*$, the problem has two branches of solutions, where λ_1 is the first eigenvalue associated to the problem.

Key words: asymptotically linear, extremal solution, stable minimal solution, regularity.

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1 Introduction and statement of main results

Consider the problem

$$(P_\lambda) \quad \begin{cases} \Delta^2 u + c\, u = \lambda f(u) \text{ in } \quad \Omega \\ \quad u = \Delta u = 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Δ^2 is the biharmonic operator, Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$, $\lambda > 0$ and $c = c(x)$ a positive continuous function in $\overline{\Omega}$ and the conditions imposed on f are as follows:

$$\begin{gathered} H_1 : \quad \text{is a function defined in} [0, \infty) \\ H_2 : f \text{ is } C^1, \text{positive, nondecreasing and convex} \end{gathered} \tag{1.1}$$

and

$$H_3 : \lim_{t \longrightarrow \infty} \frac{f(t)}{t} = a \in (0, +\infty). \tag{1.2}$$

By a solution of (P_λ) we mean a function $u \in C^4(\overline{\Omega})$ satisfying (P_λ). In the sequel we are interested only in nonnegative solutions and for which we have considered only $\lambda > 0$. From maximum principle, if u is a nonnegative solution then $u(x) > 0$ for a.e.

Problems of the form (P_λ) occur in a variety of situations. They generate equations that arise in models of combustion [7,8], thermal explosions [7], nonlinear heat generation [11], and the gravitational equilibrium of polytropic stars [5,10]. In particular, the Helmholtz problem occur in the study of electromagnetic radiation, seismology, acoustics [2,6,15]...

For $c = 0$, Abid *et al.* have studied in [1], the following problem

$$\begin{cases} \quad \Delta^2 u = \lambda f(u) \text{ in } \quad \Omega \\ u = \Delta u = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Before that, in 1996 Radulescu and Minorescu have considered the following harmonic problem

$$\begin{cases} -\Delta u = \lambda f(u) \text{ in } \quad \Omega \\ \quad u = 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

and proved in [13], that there exists $0 < \lambda^* < \infty$, a critical value of the parameter λ, such that (1.4) has a minimal, positive, classical solution u_λ for $0 < \lambda < \lambda^*$ and does not have solutions for $\lambda > \lambda^*$. The value of a was crucial in the study of (1.4) and of the behavior of u_λ when λ approaches λ^*. In the case when $a = +\infty$, it is proved in [4] that a minimal weak solution u^* exists for $\lambda = \lambda^*$. In [12], Martel proves that in this case u^* is the unique weak solution of (E_{λ^*}). Recently, Sanchon in [14] generalizes these results for the p-Laplacian.

In this paper, we study the existence of the critical bifurcation parameter λ^*, the regularity of the eventual solutions and the existence of extremal solution, this means solution for (P_{λ^*}). We give a new proof to show that every weak solution is a classical one. We begin by introducing the following definition.

Definition 1.1 *A weak solution of* (P_λ) *is a function* $u \in L^1(\Omega), u \geq 0$ *such that* $f(u) \in L^1(\Omega)$ *and*

$$\int_\Omega u\Delta^2\zeta + \int_\Omega cu\zeta = \lambda \int_\Omega f(u)\zeta \tag{1.5}$$

for all $\zeta \in C^4(\overline{\Omega})$ *with* $\Delta\zeta = \zeta = 0$ *on* $\partial\Omega$.

We say that u *is a weak super-solution of* (P_λ) *if* " $=$ " *is replaced by* " $\geq$ " *for all* $\zeta \in C^4(\overline{\Omega})$, $\zeta \geq 0$ *and* $\Delta\zeta = \zeta = 0$ *on* $\partial\Omega$.

Remark 1.1 *If* u *is a weak solution of* (P_λ) *and* $u \in L^\infty(\Omega)$, *we say that* u *is regular solution. By elliptic regularity, we know that regular solutions are smooth and solve* (P_λ) *in the classical sense.*

Throughout the paper, we denote $\|.\|_p$ the $L^p(\Omega)$-norm for $1 \leq p \leq \infty$ and $\|.\|$ the H^2-norm given by

$$\|u\|^2 = \int_\Omega |\Delta u|^2.$$

For regular solutions, we introduce a notion of stability.

Definition 1.2 *A regular solution u of (P_λ) is said to be stable if the first eigenvalue $\eta_1(c, \lambda, u)$ of the linearized operator $L_{c,\lambda,u} = -\Delta + c - \lambda f'(u)$ given by*

$$\eta_1(c, \lambda, u) := \inf_{\varphi \in H^2(\Omega)\cap H_0^1(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\Delta\varphi|^2 + \int_\Omega c\varphi^2 - \lambda \int_\Omega f'(u)\varphi^2}{\|\varphi\|_2^2},$$

is positive in $H_0^1(\Omega)$. In other words,

$$\lambda \int_\Omega f'(u)\varphi^2 \leqslant \int_\Omega |\Delta\varphi|^2 + \int_\Omega c\varphi^2 \qquad \text{for any} \quad \varphi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{1.6}$$

If $\eta_1(c, \lambda, u) < 0$, the solution u is said to be unstable.

We denote by λ_1 the first eigenvalue of $L = \Delta^2 + c$ in Ω with Navier boundary condition and φ_1 a positive normalized eigenfunction associated, that is, such that

$$\begin{cases} \Delta^2\varphi_1 + c\varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega \\ \varphi_1 > 0 & \text{in } \Omega \\ \varphi_1 = \Delta\varphi_1 = 0 & \text{on } \partial\Omega \\ \|\varphi_1\|_2 = 1 & \end{cases} \tag{1.7}$$

Next, we let

$$\Lambda := \{\lambda > 0 \text{ such that } (P_\lambda) \text{ admits a solution}\} \text{ and } \lambda^* := \sup \Lambda \leq +\infty.$$

We denote

$$r_0 := \inf_{t>0} \frac{f(t)}{t}\cdot \tag{1.8}$$

Our first main statement asserts the existence of the critical value λ^*.

Theorem 1.1 *Let f a positive function satisfying (1.1) and (1.2). Then there exists a critical value $\lambda^* \in (0, \infty)$ such that the following properties hold true.*

(i) For any $\lambda \in (0, \lambda^*)$, problem (P_λ) has a minimal solution u_λ, which is the unique stable solution of (P_λ) and the mapping $\lambda \mapsto u_\lambda$ is increasing.
(ii) For any $\lambda \in (0, \frac{\lambda_1}{a})$, u_λ is the unique solution of problem (P_λ).
(iii) If problem (P_{λ^*}) has a solution u, then

$$u = u^* = \lim_{\lambda \to \lambda^*} u_\lambda,$$

and $\eta_1(c, \lambda^*, u^*) = 0$.
(iv) For $\lambda > \lambda^*$, the problem (P_λ) has no weak solution.

For the next results, let

$$l := \lim_{t \to \infty} \Big(f(t) - at \Big). \tag{1.9}$$

We distinguish two different situations strongly depending on the sign of l.

Theorem 1.2 *Assume that $l \geq 0$. The following results hold.*

(i) $\lambda^* = \frac{\lambda_1}{a}$.
(ii) Problem (P_{λ^*}) has no solution.
(iii) $\lim_{\lambda \to \lambda^*} u_\lambda = \infty$ uniformly on compact subsets of Ω.

Theorem 1.3 *Assume that $l < 0$ and the function c is nonnegative. Then we have.*

(i) *The critical value* λ^* *belongs to* $(\frac{\lambda_1}{a}, \frac{\lambda_1}{r_0})$.

(ii) (P_{λ^*}) *has a unique solution* u^*.

(iii) *The problem* (P_λ) *has an unstable solution* v_λ *for any* $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$ *and the sequence* $(v_\lambda)_\lambda$ *satisfies*

(a) $\lim\limits_{\lambda \to \frac{\lambda_1}{a}} v_\lambda = \infty$ *uniformly on compact subsets of* Ω,

(b) $\lim\limits_{\lambda \to \lambda^*} v_\lambda = u^*$ *uniformly in* Ω.

2 Proof of Theorem 1.1

In the proof of this Theorem we shall make use of the following auxiliary results.

Lemma 2.1 *Given* $g \in L^1(\Omega)$, *there exists an unique* $v \in L^1(\Omega)$ *which is a weak solution of*

$$\begin{cases} \Delta^2 v + c\, v = g \ \ in \ \ \Omega \\ \quad v = \Delta v = 0 \ \ on \ \partial\Omega, \end{cases} \tag{2.1}$$

in the sense that

$$\int_\Omega v \Delta^2 \zeta + \int_\Omega c\ v\, \zeta = \int_\Omega g\, \zeta \tag{2.2}$$

for all $\zeta \in C^4(\overline{\Omega})$ *with* $\Delta\zeta = \zeta = 0$ *on* $\partial\Omega$.
Moreover, there exists a constant c_0 *independents of* g *such that*

$$\|v\|_1 \leqslant c_0\, \|g\|_1.$$

In addition, if $g \geq 0$ *a.e in* Ω, *then* $v \geq 0$ *a.e in* Ω.

Proof. For the uniqueness, let v_1 and v_2 be two solutions of (2.1). Then $v = v_1 - v_2$ satisfies

$$\int_\Omega v(\Delta^2\zeta + c\zeta) = 0$$

for all $\zeta \in C^4(\overline{\Omega})$ with $\Delta\zeta = \zeta = 0$. Given $\varphi \in \mathcal{D}(\Omega)$, there exist a ζ solution of

$$\begin{cases} \Delta^2 \zeta + c\,\zeta = \varphi \text{ in } \quad \Omega \\ \quad \zeta = \Delta\zeta = 0 \text{ on } \partial\Omega, \end{cases}$$

it follows that

$$\int_\Omega v\varphi = 0.$$

Since φ is arbitrary, we deduce that $v = 0$.
for the existence, since $f = f^+ - f^-$, we can assume that $f \geq 0$. Let $f_n(x) = \min\{f(x), n\}$, then the family $(f_n)_n$ converge to f in $L^1(\Omega)$.
Now let v_n the solution of

$$\begin{cases} \Delta^2 v_n + c\, v_n = f_n \text{ in } \quad \Omega \\ \quad v_n = \Delta v_n = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

The sequence $(v_n)_n$ is monotone nondecreasing.
On the other hand, we have

$$\int_\Omega (v_k - v_l) = \int_\Omega (f_k - f_l)\zeta_0,$$

where ζ_0 is defined by

$$\begin{cases} \Delta^2\zeta_0 + c\zeta_0 = 1 \text{ in } \quad \Omega \\ \quad \zeta_0 = \Delta\zeta_0 = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.4}$$

So

$$\int_\Omega |v_k - v_l| \leqslant c_0 \int_\Omega |f_k - f_l| dx,$$

and $(v_n)_n$ is a Cauchy sequence in $L^1(\Omega)$. Passing to the limit in (2.3), after multiplication by ζ, we have that $v = \lim v_n$ is a weak solution of equation (2.1). If we take $\zeta = \zeta_0$ in (2.2), we obtain

$$\|v\|_1 = \int_\Omega v = \int_\Omega f\zeta_0 \leqslant c_0\|f\|_1.$$

□

Lemma 2.2 *If* (P_λ) *has a weak super solution* $\overline{u}$*, then there exists a weak solution* u *of* (P_λ) *such that* $0 \le u \le \overline{u}$ *and* u *does not depend on* $\overline{u}$*.*

Proof. We use a standard monotone iteration argument and maximum principle for the operator $-\Delta + c$. Let $u_0 = 0$ and u_{n+1} the solution of

$$\begin{cases} \Delta^2 u_{n+1} + c\, u_{n+1} = \lambda f(u_n) \text{ in } \quad \Omega, \\ \quad \Delta u_{n+1} = u_{n+1} = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases}$$

which exists by Lemma 1. We prove that $0 = u_0 \leqslant u_1 \leqslant ... \leqslant u_n \leqslant ... \leqslant \overline{u}$ and $(u_n)_n$ converge to $u \in L^1(\Omega)$ which is a weak solution of (P_λ). Moreover u is independent of $\overline{u}$ by construction. □

The existence of the critical value λ^* is a consequence of the following auxiliary result.

Lemma 2.3 *Problem* (P_λ) *has no solution for any* $\lambda > \lambda_1/r_0$*, but has at least one solution provided* λ *is positive and small enough.*

Proof. To show that (P_λ) has a solution, we use the Lemma 2. To this aim, let $\zeta_0 \in C^4(\overline{\Omega}))$ given by (2.4). The choice of ζ_0 implies that ζ_0 is a super solution of (P_λ) for $\lambda \leqslant 1/f(\|\zeta_0\|_\infty)$. By lemma 2, there exist a weak solution u of (P_λ) such that $0 \le u \le \zeta_0$. Because $\zeta_0 \in C^4(\overline{\Omega})$, $u \in L^\infty(\Omega)$ (u is a regular solution) and then $u \in C^4(\overline{\Omega})$. It follows that problem (P_λ) has a solution for $\lambda \leqslant 1/f(\|\zeta_0\|_\infty)$.

Assume now that u is a solution of (P_λ) for some $\lambda > 0$. Using φ_1 given by (1.7) as a test function, we get

$$\int_\Omega u\Delta^2\varphi_1 + \int_\Omega cu\varphi_1 = \lambda \int_\Omega f(u)\varphi_1$$

This yields

$$(\lambda_1 - \lambda r_0) \int_\Omega \varphi_1 u \geq 0.$$

Since $\varphi_1 > 0$ and $u > 0$, we conclude that the parameter λ should belong to $(0, \lambda_1/r_0)$.
This completes our proof. □

Lemma 2.4 *Assume that the problem* (P_λ) *has a solution for some* $\lambda \in (0, \lambda^*)$*. Then there exists a minimal solution denoted by* u_λ *for the problem* (P_λ)*. Moreover, for any* $\lambda' \in (0, \lambda)$*, the problem* $(P_{\lambda'})$ *has a solution.*

Proof. Fix $\lambda \in (0, \lambda^*)$ and let u be a solution of (P_λ). As above, we use the Lemma 2 to obtain a solution of (P_λ), u_λ which is independent of u used as super solution (as mentioned in the proof of Lemma 2).
Since u_λ is independent of the choice of u, then it is a minimal solution.
Now, if u is a solution of (P_λ), then u is a super solution of the problem $(P_{\lambda'})$ for any λ' in $(0, \lambda)$ and Lemma 2 completes the proof. □

Proof of Theorem 1 (i)

First, we claim that u_λ is stable. Indeed, arguing by contradiction, we deduce that the first eigenvalue
$\eta_1 = \eta_1(c, \lambda, u_\lambda)$ is non positive. Then, there exists an eigenfunction

$$\psi \in C^4(\overline{\Omega}) \quad \text{and} \quad \Delta\psi = \psi = 0 \quad \text{on} \quad \partial\Omega,$$

such that

$$\Delta^2\psi + c\psi - \lambda f'(u_\lambda)\psi = \eta_1\psi \quad \text{in} \quad \Omega \quad \text{and} \quad \psi > 0 \quad \text{in} \quad \Omega.$$

Consider $u^\varepsilon := u_\lambda - \varepsilon\psi$. Hence

$$\Delta^2 u^\varepsilon + c\,u^\varepsilon - \lambda f(u^\varepsilon) = -\eta_1\varepsilon\psi + \lambda\Big[f(u_\lambda) - f(u_\lambda - \varepsilon\psi) - \varepsilon f'(u_\lambda)\psi\Big] = \varepsilon\psi(-\eta_1 + o_\varepsilon(1)).$$

Since $\eta_1 \leq 0$ for $\varepsilon > 0$ small enough, we have

$$\Delta^2 u^\varepsilon + c\,u^\varepsilon - \lambda f(u^\varepsilon) \geq 0 \qquad \text{in} \quad \Omega.$$

Then, for $\varepsilon > 0$ small enough, we use the strong maximum principle (Hopf's Lemma) to deduce that $u^\varepsilon \geq 0$. u^ε is a super solution of (P_λ), so by Lemma 2 we obtain a solution u such that $u \leq u^\varepsilon$ and since $u^\varepsilon < u_\lambda$, then we contradict the minimality of u_λ.

Now, we show that (P_λ) has at most one stable solution. Assume the existence of another stable solution $v \neq u_\lambda$ of problem (P_λ). Let $\varphi := v - u_\lambda$, then by maximum principle $\varphi > 0$ and from (1.6) taking φ as a test function, we have

$$\lambda \int_\Omega f'(v)\,\varphi^2 \leq \int_\Omega |\Delta\varphi|^2 + \int_\Omega c\varphi^2 = \int_\Omega \varphi\Delta^2\varphi + \int_\Omega c\varphi^2 = \lambda \int_\Omega \Big[f(v) - f(u_\lambda)\Big]\varphi.$$

Therefore

$$\int_\Omega \Big[f(v) - f(u_\lambda) - f'(v)(v - u_\lambda)\Big]\varphi \geq 0.$$

Thanks to the convexity of f, the term in the brackets is non positive, hence

$$f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) = 0 \ \text{ in } \Omega,$$

which implies that f is affine over $[u_\lambda, v]$ in Ω. So, there exists two real numbers $\bar{a}$ and b such that

$$f(x) = \bar{a}x + b \quad \text{ in } [0, \max_\Omega v].$$

Finally, since u_λ and v are two solutions to $\Delta^2 w + cw = \bar{a}w + b$, we obtain that

$$0 = \int_\Omega (u_\lambda \Delta v - v\Delta u_\lambda) = b\int_\Omega (v - u_\lambda) = b\int_\Omega \varphi.$$

This is impossible since $b = f(0) > 0$ and φ is positive in Ω.
Finally, by Lemma 4 and the definition of u_λ, we have that the function $\lambda \to u_\lambda$ is an increasing mapping.

Proof of Theorem 1 (ii)

In this stage, we need the following results.

Proposition 2.1 *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Assume that f is a function satisfying* (1.1) *and* (1.2). *If (P_λ) has a weak solution u, then u is a regular solution and hence a classical solution.*

Proof. By convexity of f, we have $a = \sup_{t\geq 0} f'(t)$ and

$$f(t) \leqslant at + f(0) \ \text{ for all } \ t \geq 0. \tag{2.5}$$

Let u a weak solution of (P_λ), $f(u) \in L^1(\Omega)$.
By elliptic regularity, $u \in L^p(\Omega)$, for all $p \geq 1$ such that

$$p < \frac{n}{n-4} \qquad (p \leq \infty \,\text{if}\, n = 2, 3 \,\text{and}\, p < \infty \,\text{if}\, n = 4) \tag{2.6}$$

Again by (2.5), $f(u) \in L^p$ for all p satisfying (2.6) so $u \in W^{4,r}(\Omega)$ for all $r \geq 1$ such that

$$r < \frac{n}{n-8} \qquad (r \leq \infty \,\text{if}\, n = 2, 3, 4, 5, 6, 7 \,\text{and}\, r < \infty \,\text{if}\, n = 8) \tag{2.7}$$

By iteration and after $k(n) = [\frac{n}{4}] + 1$ operations, the solution u belongs to $L^\infty(\Omega)$.
By elliptic regularity and standard bootstrap argument, $u \in C^4(\overline{\Omega})$.
□

Proposition 2.2 *Let $\Omega \subset \mathbb{R}^n$ a smooth bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Assume that $f(t) = f_0(t) = at + b$, where $a, b > 0$. Then*

(i) $\lambda^* = \frac{\lambda_1}{a}$

(ii) The problem (P_λ) has no weak solution for $\lambda = \lambda^*$

Proof. Let $0 < \lambda < \dfrac{\lambda_1}{a}$, the problem (P_λ), given by

$$\begin{cases} \Delta^2 u + (c - \lambda a)u = \lambda b \ \text{ in } \ \Omega \\ \qquad \Delta u = u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.8}$$

has a unique solution in $C^4(\overline{\Omega})$.
Since $\lambda a < \lambda_1$, by Maximum principle $u > 0$. Now let $\lambda = \dfrac{\lambda_1}{a}$.
If the problem (2.8) has a solution u, then by multiplication (2.8) by φ_1 a positive function associated to λ_1 and introduced by (1.7) and integration by parts, it follows that $\int_\Omega \varphi_1 = 0$ which is impossible since $\varphi_1 > 0$ in Ω. So for $f_0(t) = at + b$, a and $b > 0$, we have $\lambda^* = \dfrac{\lambda_1}{a}$ and the equation (P_{λ^*}) has no solution. □

For the proof of Theorem 1 **(ii)**, let $\lambda \in (0, \frac{\lambda_1}{a})$, $b = f(0)$ and w a solution for the problem (2.8) when
$f_0(t) = at + b$. Since we have for the function f in Theorem 1, $f(w) \leq aw + f(0)$, then w is a super-solution of (P_λ) and hence by Lemma 2 and Proposition 1, the equation (P_λ) has a solution.
For the uniqueness, let u a solution of (P_λ) for a reel $\lambda \in (0, \frac{\lambda_1}{a})$. We denote $\lambda_1(L)$ the first eigenvalue of an operator L, that is $\lambda_1(\Delta^2 + c) = \lambda_1$.
Because $a = \sup_{t \geq 0} f'(t)$, we have $\Delta^2 + c - \lambda f'(u) \geq \Delta^2 + c - \lambda a$ and so

$$\lambda_1(\Delta^2 + c - \lambda f'(u)) \geq \lambda_1(\Delta^2 + c - \lambda a)$$

that is

$$\eta_1(c, \lambda, u) \geq \lambda_1 - \lambda a > 0.$$

The solution u is stable then, by Theorem 1 **(i)**, we obtain $u = u_\lambda$.
□

Proof of Theorem 1 (iii)

Suppose that (P_λ) has a solution u. then, for every $\lambda \in (0, \lambda^*)$, we have $u_\lambda \leq u$ and so $u^* = \lim\limits_{\lambda \to \lambda^*} u_\lambda$ is well defined in $L^1(\Omega)$ and furthermore u^* is a weak then classical solution for (P_{λ^*}).
Since $0 \leq u^* \leq u$, u^* is a minimal solution and also satisfies (1.6) for $\lambda = \lambda^*$ so $\eta_1(c, \lambda^*, u^*) \geq 0$.
Now, consider the nonlinear operator

$$\begin{array}{rcl} G : (0, +\infty) \times C^{4,\alpha}(\bar{\Omega}) \cap E & \longrightarrow & C^{0,\alpha}(\bar{\Omega}) \\ (\lambda, u) & \longmapsto & \Delta^2 u + cu - \lambda f(u), \end{array}$$

where $\alpha \in (0,1)$ and E the function space defined by

$$E := \{u \in H^4(\Omega) / \Delta u = u = 0 \ \text{ on } \ \partial\Omega\} \qquad (2.9)$$

Assuming that the first eigenvalue $\eta_1(c, \lambda^*, u^*)$ is positive. By the implicit function theorem applied to the operator G, it follows that problem (P_λ) has a solution for λ in a neighborhood of λ^*. But this contradicts the definition of λ^* so $\eta_1(c, \lambda^*, u^*) = 0$.
Furthermore, u^* is a the unique solution for (P_{λ^*}) and we can proceed as in the proof of of Theorem 1.1(ii). □

Proof of Theorem 1 (iv)

If the problem (P_λ) has a weak solution u for $\lambda > \lambda^*$, then by Proposition 2, u is a classical solution for (P_λ) and this contradicts the definition of λ^*.

3 Proof of Theorem 1.2

In the proof of Theorem 1.2, we shall use the following auxiliary result which is a reformulation of Theorem due to Hörmander [9] and maximum principle.

Lemma 3.1 *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$ with smooth boundary. Let (u_n) be a sequence of nonnegative functions defined on Ω and satisfying $\Delta^2 u_n + cu_n \geq 0$ for a positive continues function c. Then the following alternative holds.*

(i) $\lim_{n\to\infty} u_n = \infty$ *uniformly on compact subsets of* Ω,

or

(ii) (u_n) *contains a subsequence which converges in* $L^1_{loc}(\Omega)$ *to some function* u.

We first prove the following result.

Proposition 3.1 *Let f be a positive function satisfying* (1.1) *and* (1.2)*. Then the following assertions are equivalent.*

(i) $\lambda^* = \dfrac{\lambda_1}{a}\cdot$

(ii) (P_{λ^*}) *has no solution.*

(iii) $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of* Ω.

Proof.

(i)⇒(ii). By contradiction. Assume that (P_{λ^*}) has a solution u. By (ii) of Theorem 1.1, $u = u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ and $\eta_1(c, \lambda^*, u^*) = 0$. Thus there exists $\psi \in C^4(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta^2\psi + c\psi - \lambda^* f'(u^*)\psi = 0 \ \text{ in } \ \Omega \\ \psi > 0 \ \text{ in } \ \Omega \\ \Delta\psi = \psi = 0 \text{ on } \ \partial\Omega. \end{cases} \tag{3.1}$$

Using φ_1 given by (1.7) as a test function, we obtain

$$\int_\Omega (\lambda_1 - \lambda^* f'(u^*))\varphi_1 \psi = 0 \tag{3.2}$$

Since $\varphi_1 > 0$, $\psi > 0$, $\lambda^* = \dfrac{\lambda_1}{a}$, and $a = \sup\limits_{t>0} f'(t)$, we have $\lambda_1 - \lambda^* f'(u^*) \geq 0$.
Then equality (3.2) gives $f'(u^*) = a$ in Ω.
This implies that $f(t) = at + b$ in $[0, max_\Omega u^*]$ for some scalar $b > 0$ and this impossible by Proposition 2.2. Hence (P_{λ^*}) has no solution.

(ii)$\Rightarrow$(iii). By contradiction, suppose that (iii) doesn't hold. By Lemma 3.1 and up to subsequence,
u_λ converges locally in $L^1(\Omega)$ to the function u^* as $\lambda \to \lambda^*$.
Claim: u_λ is bounded in $L^2(\Omega)$.
Indeed, if not, we may assume that

$$u_\lambda = k_\lambda w_\lambda$$

with

$$\int_\Omega {w_\lambda}^2 dx = 1 \qquad \text{and} \qquad \lim_{\lambda \to \lambda^*} k_\lambda = \infty \tag{3.3}$$

We have

$$\frac{\lambda}{k_\lambda} f(u_\lambda) \to 0 \qquad \text{in} \qquad L^1_{loc}(\Omega) \qquad \text{as} \qquad \lambda \to \lambda^*.$$

and then

$$\Delta^2 w_\lambda + c w_\lambda \to 0 \qquad \text{in} \qquad L^1_{loc}(\Omega). \tag{3.4}$$

We have

$$\begin{aligned} \int_\Omega |\Delta w_\lambda|^2 &= \int_\Omega \Delta^2 w_\lambda \, w_\lambda \\ &= \int_\Omega (\frac{\lambda f(u_\lambda)}{k_\lambda} - c w_\lambda) w_\lambda, \end{aligned}$$

then

$$
\begin{aligned}
\int_\Omega |\Delta w_\lambda|^2 &\leqslant \int_\Omega \frac{\lambda f(u_\lambda)}{k_\lambda} w_\lambda \\
&\leqslant \lambda^* \int_\Omega a\, w_\lambda^2 + \frac{f(0)}{k_\lambda} w_\lambda \\
&\leqslant a\lambda^* + c_0 \int_\Omega w_\lambda \\
&\leqslant a\lambda^* + c_0\sqrt{|\Omega|},
\end{aligned}
$$

for some $c_0 > 0$ independent of λ.
Then (w_λ) is bounded in $H^4(\Omega)$ and up to a subsequence, we obtain

$$
\begin{aligned}
w_\lambda \rightharpoonup w \qquad \text{weakly in} \qquad H^4(\Omega) \qquad \text{and} \\
w_\lambda \to w \qquad \text{strongly in} \qquad L^2(\Omega) \qquad \text{as} \qquad \lambda \to \lambda^*.
\end{aligned}
\tag{3.5}
$$

Moreover, by the trace Theorem

$$\Delta w_\lambda = w_\lambda = 0 \qquad \text{on} \qquad \partial\Omega$$

It follows by (3.5), that $w = 0$ in Ω and this contradicts (3.3).
This complete the proof of the claim.
Thus u_λ is bounded in $L^2(\Omega)$ and with the same argument above, u_λ is bounded in $H^4(\Omega)$ and up to a subsequence, we have

$$
\begin{aligned}
&u_\lambda \rightharpoonup u^* \quad \text{weakly in} \quad H^4(\Omega) \qquad \text{and} \\
&u_\lambda \to u^* \qquad \text{in} \qquad L^2(\Omega) \qquad \text{as} \qquad \lambda \to \lambda^*
\end{aligned}
$$

and

$$
\begin{cases}
\Delta^2 u^* + cu^* = \lambda^* f(u^*) & \text{in } \Omega \\
\Delta u^* = u^* = 0 & \text{on } \partial\Omega
\end{cases}
$$

and this impossible by the hypothesis (ii).
It's obvious that (iii)$\Rightarrow$(ii) and hence (ii)$\Leftrightarrow$(iii).

(iii)$\Rightarrow$(i). If (iii) occurs, that (ii) also is true and we have $\lim\limits_{\lambda\to\lambda^*} \|u_\lambda\|_2 = \infty$. Let

$$u_\lambda = k_\lambda w_\lambda \qquad \text{with} \qquad \|w_\lambda\|_2 = 1. \tag{3.6}$$

Up to subsequence, we obtain

$$w_\lambda \rightharpoonup w \qquad \text{weakly in} \qquad H^4(\Omega) \text{ and}$$

$$w_\lambda \to w \qquad \text{in} \qquad L^2(\Omega) \qquad \text{as} \qquad \lambda \to \lambda^*. \tag{3.7}$$

We have also

$$\frac{\lambda}{k_\lambda} f(u_\lambda) \to \lambda^* a w \qquad \text{as} \qquad \lambda \to \lambda^* \tag{3.8}$$

and

$$-\Delta w_\lambda + c w_\lambda \to -\Delta w + c w \qquad \text{in} \qquad L^2(\Omega)$$

and then

$$\begin{cases} -\Delta w + c w = a \lambda^* w & \text{in } \Omega \\ \qquad\qquad w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.9}$$

Taking φ_1 as a test function in (3.9), we obtain

$$\lambda_1 \int_\Omega w \varphi_1 = \int_\Omega w(-\Delta \varphi_1 + c \varphi_1)) = \int_\Omega a \lambda^* w \varphi_1$$

Since $\varphi_1 > 0$ and $w > 0$ in Ω, we have $\lambda^* = \dfrac{\lambda_1}{a}$ and this complete the proof of Proposition 3.1.

To finish the proof of Theorem 1.2, we need only to show that $(P_{\frac{\lambda_1}{a}})$ has no solution. Assume that u is a solution of $(P_{\frac{\lambda_1}{a}})$. Since

$$l := \lim_{t \to \infty} \Big(f(t) - at \Big) \geq 0 \quad \text{and} \quad a = \sup_{t \geq 0} f'(t),$$

we have $l \in (0, \infty)$ and $f(t) - at \geq 0$ and

$$\Delta^2 u + c u = \frac{\lambda_1}{a} f(u) \qquad \text{in} \quad \Omega. \tag{3.10}$$

Taking φ_1 as a test function in (3.10), we get $f(u) = a\, u$ in Ω, which contradicts $f(0) > 0$. This concludes the proof of Theorem 1.2. □

4 Proof of Theorem 1.3

(i) We have shown that

$$\frac{\lambda_1}{a} \leqslant \lambda^* \leqslant \frac{\lambda_1}{r_0}$$

Suppose that $\lambda^* = \frac{\lambda_1}{a}$. By Proposition 3.1, we have

$$\lim_{\lambda \to \lambda^*} u_\lambda = \infty \text{ uniformly on compact subsets of } \Omega.$$

Let u_λ be the minimal solution of (P_λ) for $\frac{\lambda_1}{a} < \lambda < \lambda^*$. Then, multiplying (P_λ) by φ_1 and integrating by parts, we obtain

$$\int_\Omega \varphi_1\Big(\lambda_1 u_\lambda - \lambda f(u_\lambda)\Big) = \int_\Omega \varphi_1\Big((\lambda_1 - a\lambda)u_\lambda - \lambda(f(u_\lambda) - au_\lambda)\Big) = 0 \tag{4.1}$$

and then

$$\lambda \int_\Omega \varphi_1\Big(f(u_\lambda) - au_\lambda\Big) \geq 0 \tag{4.2}$$

Passing to the limit in the inequality (4.2) as λ tends to λ^*, we find

$$0 \leqslant l\lambda^* \int_\Omega \varphi_1 < 0,$$

which is impossible and then $\lambda^* \neq \frac{\lambda_1}{a}$.

If $\lambda^* = \frac{\lambda_1}{r_0}$, let u be a solution of problem (P_{λ^*}) which exists by Proposition 3.1. Multiplying (P_{λ^*}) by φ_1 and integrating by parts, we obtain

$$\lambda_1 \int_\Omega u\varphi_1 = \frac{\lambda_1}{r_0} \int_\Omega f(u)\varphi_1$$

that is

$$\int_\Omega (f(u) - r_0 u)\varphi_1 = 0$$

then $f(u) = r_0\, u$ in Ω, and this contradicts the fact that $f(0) > 0$.

(ii) Since $\lambda^* > \frac{\lambda_1}{a}$, the existence of a solution to (P_{λ^*}) is assured by Proposition 3.1 and the uniqueness is given by Theorem 1.1.

(iii) In this stage, we will use the mountain pass Theorem of Ambrosetti and Rabinowitz.

Theorem 4.1 *[3] Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$. Assume that J satisfies the Palais-Smale condition and the following geometric assumptions.*

(1) There exist positive constants R and ρ such that

$$J(u) \geq J(u_0) + \rho, \text{ for all } u \in E \text{ with } \|u - u_0\| = R.$$

(2) there exists $v_0 \in E$ such that $\|v_0 - u_0\| > R$ and $J(v_0) \leq J(u_0)$.
Then the functional J possesses at least a critical point. The critical value is characterized by

$$\alpha := \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u),$$

where

$$\Gamma := \Big\{ g \in C([0,1], E) \,|\, g(0) = u_0,\ g(1) = v_0 \Big\}$$

and satisfies

$$\alpha \geq J(u_0) + \rho.$$

Let

$$\begin{aligned} J : E &\longrightarrow \mathbb{R} \\ u &\longmapsto \frac{1}{2}\int_\Omega |\Delta u|^2 + \frac{1}{2}\int_\Omega cu^2 - \int_\Omega F(u), \end{aligned}$$

where E is the function space defined by (2.9) and

$$F(t) = \lambda \int_0^t f(s)ds, \text{ for all } t \geq 0.$$

We take u_0 as the stable solution u_λ for each $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$.

The energy functional J belongs to $C^1(E, \mathbb{R})$ and

$$\langle J'(u), v\rangle = \int_\Omega \Delta u \cdot \Delta v + \int_\Omega cuv - \lambda \int_\Omega f(u)v,$$

for all $u, v \in E$.

Since $\eta_1(c, \lambda, u_\lambda) \geq 0$, the function u_λ is a local minimum for J. In order to transform it into a local strict minimum, consider the perturbed functional J_ε defined by

$$\begin{aligned} J_\varepsilon : E &\longrightarrow \mathbb{R} \\ u &\longmapsto J(u) + \frac{\varepsilon}{2}\int_\Omega |\Delta(u - u_\lambda)|^2 + \frac{\varepsilon}{2}\int_\Omega c|u - u_\lambda|^2, \end{aligned} \tag{4.3}$$

for all $\varepsilon \in [0,\ \varepsilon_0]$, where

$$\varepsilon_0 := \frac{3}{4}\ \frac{\lambda a - \lambda_1}{\lambda_1}.$$

We observe that J_ε is also in $C^1(E, \mathbb{R})$ and

$$\langle J'_\varepsilon(u), v\rangle = \int_\Omega \Delta u \Delta v + \int_\Omega cuv - \lambda \int_\Omega f(u)v + \varepsilon \int_\Omega \Delta(u - u_\lambda)\Delta v + \varepsilon \int_\Omega c(u - u_\lambda)v$$

for all $u, v \in E$. Using the same arguments of Mironescu and Rădulescu in [13, Lemma 9], we show that J_ε satisfies the Palais-Smale condition and so we have the next lemma.

Lemma 4.1 *Let $(u_n) \subset E$ be a Palais-Smale sequence, that is,*

$$\sup_{n \in \mathbb{N}} |J_\varepsilon(u_n)| < +\infty, \tag{4.4}$$

$$\|J'_\varepsilon(u_n)\|_{E^*} \to 0 \text{ as } n \to \infty. \tag{4.5}$$

Then (u_n) is relatively compact in E.

Now, we need only to check that the two geometric assumptions are fulfilled. First, since u_λ is a local minimum of J, there exists

$R > 0$ such that for all $u \in E$ satisfying $\|u - u_\lambda\| = R$, we have $J(u) \geq J(u_\lambda)$. Then

$$J_\varepsilon(u) \geq J_\varepsilon(u_\lambda) + \frac{\varepsilon}{2}\int_\Omega |\Delta(u-u_\lambda)|^2.$$

Since $u - u_\lambda$ is not harmonic, we can choose

$$\rho := \frac{\varepsilon\, R^2}{4} > 0$$

and u_λ becomes a strict local minimal for J_ε, which proves (1).
Also, we have

$$J_\varepsilon(t\varphi_1) = \frac{\lambda_1}{2}t^2 + \frac{\varepsilon}{2}\lambda_1 t^2 - \varepsilon\lambda_1 t\int_\Omega \varphi_1 u_\lambda + \frac{\varepsilon}{2}\lambda\int_\Omega f(u_\lambda)u_\lambda - \int_\Omega F(t\varphi_1),\ \forall\, t > 0. \tag{4.6}$$

Recall that $\lim_{t\mapsto+\infty}(f(t) - a\,t)$ is finite, then there exists $\beta \in \mathbb{R}$ such that

$$f(t) \geq a\,t + \beta,\ \forall\, t > 0.$$

Hence

$$F(t) \geq \frac{a\,\lambda}{2}t^2 + \beta\,\lambda\,t,\ \forall\, t > 0.$$

This yields

$$\frac{J_\varepsilon(t\varphi_1)}{t^2} \leqslant \left(\frac{\lambda_1}{2} + \frac{\varepsilon\lambda_1}{2} - \frac{a\,\lambda}{2}\right) + \frac{\varepsilon}{2t^2}\int_\Omega f(u_\lambda)u_\lambda,$$

which implies that

$$\limsup_{t\to+\infty} \frac{1}{t^2}\, J_\varepsilon(t\varphi_1) \leqslant \left(\frac{\lambda_1}{2} + \frac{\varepsilon_0\lambda_1}{2} - \frac{a\,\lambda}{2}\right) < 0,\ \forall\, \varepsilon \in [0,\ \varepsilon_0].$$

Therefore

$$\lim_{t\to+\infty}\ J_\varepsilon(t\varphi_1) = -\infty$$

and so, $\forall\, \varepsilon \in [0,\ \varepsilon_0]$, there exists $v_0 \in E$ such that

$$J_\varepsilon(v_0) \leqslant J_\varepsilon(u_\lambda)$$

and (2) is proved. Finally, let v_ε (respectively. c_ε) be the critical point (respectively. critical value) of J_ε.

Remark 4.1 *The fact that J_ε increases with ε implies that for all $\varepsilon \in [0, \varepsilon_0]$, $c_\varepsilon \in [c_0, c_{\varepsilon_0}[$. Then, c_ε is uniformly bounded. Thus, for all $\varepsilon \in [0,\ \varepsilon_0]$, the critical point v_ε satisfies $\|v_\varepsilon - u_\lambda\| \geq R$.*

Recall that for any $\varepsilon \in [0, \varepsilon_0]$, the function v_ε belongs to E and satisfies

$$\Delta^2 v_\varepsilon + c v_\varepsilon = \frac{\lambda}{1+\varepsilon} f(v_\varepsilon) + \frac{\lambda\varepsilon}{1+\varepsilon} f(u_\lambda) \text{ in } \Omega, \tag{4.7}$$

and

$$J_\varepsilon(v_\varepsilon) = c_\varepsilon. \tag{4.8}$$

By Lemma 3.1, Remark 2, (4.7) and (4.8), there exists $v \in E$ such that

$$v_\varepsilon \to v \text{ in } E, \text{ as } \varepsilon \to 0,$$

satisfying

$$\Delta^2 v + cv = \lambda f(v) \text{ in } \Omega.$$

From Remark 4.1, we see that $v \neq u_\lambda$.

Proof of (a). By contradiction, suppose that (a) doesn't hold. By Lemma 3.1 there is a sequence of positives scalars (μ_n) and a sequence (v_n) of unstable solutions to (P_{μ_n}) such that $v_n \to v$ in $L^1_{loc}(\Omega)$ as $\mu_n \to \lambda_1/a$ for some function v.

We first claim that (v_n) cannot be bounded in E. Otherwise, let $w \in E$ be such that, up to a subsequence,

$$v_n \rightharpoonup w \text{ weakly in } E \quad \text{ and } \quad v_n \to w \text{ strongly in } L^2(\Omega).$$

Therefore,

$$\Delta^2 v_n + c v_n \to \Delta^2 w + cw \text{ in } \mathcal{D}'(\Omega) \quad \text{ and } \quad f(v_n) \to f(w) \text{ in } L^2(\Omega),$$

which implies that $\Delta^2 w + cw = \frac{\lambda_1}{a} f(w)$ in Ω. It follows that $w \in E$ and solves $(P_{\lambda_1/a})$. From Lemma 4.1, we deduce that

$$\eta_1\left(c, \frac{\lambda_1}{a}, w\right) \leqslant 0. \tag{4.9}$$

Relation (4.9) shows that $w \neq u_{\lambda_1/a}$ which contradicts the fact that $(P_{\lambda_1/a})$ has a unique solution. Now, since $\Delta^2 v_n + cv_n = \mu_n f(v_n)$, the unboundedness of (v_n) in E implies that this sequence is unbounded in $L^2(\Omega)$, too. To see this, let

$$v_n = k_n\, w_n, \quad \text{where} \quad k_n > 0, \quad \|w_n\|_2 = 1 \quad \text{and} \quad k_n \to \infty.$$

Then

$$\Delta^2 w_n + cw_n = \frac{\mu_n}{k_n} f(v_n) \to 0 \qquad \text{in} \quad L^1_{loc}(\Omega).$$

So, we have convergence also in the sense of distributions and (w_n) is seen to be bounded in E with standard arguments. We obtain

$$\Delta^2 w + cw = 0 \quad \text{and} \quad \|w\|_2 = 1.$$

The desired contradiction is obtained since $w \in E$.
Proof of (b). As before, it is enough to prove the $L^2(\Omega)$ boundless of v_λ near λ^* and to use the uniqueness property of u^*. Assume that $\|v_n\|_2 \to \infty$ as $\mu_n \to \lambda^*$, where v_n is a solution to (P_{μ_n}). We write again $v_n = l_n\, w_n$. Then,

$$\Delta^2 w_n + cw_n = \frac{\mu_n}{l_n} f(v_n). \tag{4.10}$$

The fact that the right-hand side of (4.10) is bounded in $L^2(\Omega)$ implies that (w_n) is bounded in E. Let (w_n) be such that (up to a subsequence)

$$w_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad w_n \to w \text{ strongly in } L^2(\Omega).$$

A computation already done shows that

$$\Delta^2 w + cw = \lambda^* aw, \quad w \geq 0 \quad \text{and } \|w\|_2 = 1,$$

which forces λ^* to be λ_1/a. This contradiction concludes the proof.
□

References

[1] I. Abid, M. Jleli and N. Trabelsi, *Weak solutions of quasilinear biarmonic problems with positive increasing and convex nonlinearities*, Analysis and Applications **6** (2008), 213-227.

[2] H. Alzubaidi, X. Antoine and C. Chniti, *Formulation and accuracy of On-Surface Radiation Conditions for acoustic multiple scattering problems*, Applied Mathematics and Computation (Elsevier) Volume 277 (2016), 82-100.

[3] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications,* J. Funct. Anal. **14** (1973), 349-381.

[4] H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, *Blow up for $u_t - \Delta u = g(u)$ revisited*, Adv. Diff. Eq. **1** (1996), 73-90.

[5] S. Chandrasekar, *An introduction to the Theory of Stellar Structures* , Dover, New York 1957.

[6] C. Chniti, S. Alhazmi, S. H. Altum and M. Toujani, *Improved DtN and NtD maps for OSRCs techniques: derivation and numerical validation*, Applied Numerical Mathematics (Elsevier) Volume 101, March 2016, 53-70.

[7] D.A. Frank-Kamenetskii, *Diffusion and Heat Transfer in Chemical Kinetics*, New York: Plenum Press 1069.

[8] I.M. Gelfand, *Some problems in the theory of quasilinear equations*, Amer. Math. Soc. Translations Heidelberg, 1(2) **29** (1963), 295-381.

[9] L. Hörmander, *The Analysis of Linear Differential Operators I*, Springer-Verlag, Berlin 1983.

[10] D.D. Joseph and T.S. Lundgren, *Quasilinear Dirichlet driven by positive sources* , Arch. Rachinal Mech. Anal. **49** (1973), 241-269.

[11] D.D. Joseph and E.M. Sparrow, *Nonlinear diffusion induced by nonlinear sources*, Quar. J. Appl. Math. **28** (1970), 327-342 .

[12] Y. Martel, *Uniqueness of weak solution for nonlinear elliptic problems*, Houston J. Math. **23** (1997), 161-168.

[13] P. Mironescu and V. Rădulescu,*The study of a bifurcation problem associated to an asymtotically linear function*, Nonlinear Anal. **26** (1996), 857-875.

[14] M. Sanchón, *Boundedness of the extremal solution of some p-Laplacian problems*, Nonlinear Anal. **67** (2007), 281-294.

[15] B. Thierry, X. Antoine, C. Chniti and H. Alzubaidi, *Computing multiple scattering problems by disks using the nu-diff Matlab toolbox*, Computer Physics Communications (Elsevier) 192 (2015), 348-362.