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Some Results for $CAT(0)$ Spaces

Mehdi Asadi[a,1], **S. Mansour Vaezpour** [b,2], **Hossein Soleimani**[c,3]

[a] *Department of Mathematics, Zanjan Branch, Islamic Azad University, Zanjan, Iran.*
[b] *Department of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran.*
[c] *Department of Mathematics, Malayer Branch, Islamic Azad University, Malayer, Iran.*

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Abstract

We shall generalize the concept of $z = (1-t)x \oplus ty$ to n times which contains to verify some their properties and inequalities in $CAT(0)$ spaces. In the sequel with introducing of α-nonexpansive mappings, we obtain some fixed points and approximate fixed points theorems.

Keywords: $CAT(0)$ space, Hyperbolic space, Fixed point.
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1 Introduction

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subseteq R$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t_0)) = |t - t_0|$ for all $t, t_0 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every

[1]masadi@azu.ac.ir
[2]vaez@aut.ac.ir
[3]hsoleimani54@gmail.com

geodesic segment joining any two of its points.
A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$). A comparison triangle for a geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{y}_j) = d(x_i, y_j)$ for $i, j \in \{1, 2, 3\}$.
A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom:
"Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})."$$

Definition 1.1. *([1]) A hyperbolic space is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is such that*

(W1) $d(z, W(x, y, t)) \leq (1 - t)d(z, x) + td(z, y)$

(W2) $d(W(x, y, t), W(x, y, s)) = |t - s|d(x, y)$

(W3) $W(x, y, t) = W(y, x, 1 - t)$

(W4) $d(W(x, z, t), W(y, w, t)) \leq (1 - t)d(x, y) + td(z, w)$

for all $x, y, z, w \in X$ and $t, s \in [0, 1]$.

If $x, y \in X$ and $t \in [0, 1]$ then we use the notation $(1 - t)x \oplus ty$ for $W(x, y, t)$. We shall denote by $[x, y]$ the set $\{(1 - t)x \oplus ty : t \in [0, 1]\}$. A nonempty subset $C \subseteq X$ is convex if $[x, y] \subseteq C$ for all $x, y \in C$.
We remark that any normed space $(X, \|.\|)$ is a hyperbolic space, with

$$(1 - t)x \oplus ty := (1 - t)x + ty.$$

Here we recall a couple of lemmas which will be used next.

Lemma 1.2. *([2, Lemma 2.4]) Let (X, d) be a $CAT(0)$ space. Then*

$$d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z) \leq \max\{d(x, z), d(y, z)\},$$

for $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 1.3. *([2, Lemma 2.5]) Let (X, d) be a $CAT(0)$ space. Then*

$$d((1 - t)x \oplus ty, z)^2 \leq (1 - t)d(x, , z)^2 + td(y, z)^2 - t(1 - t)d(x, y)^2,$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

In particular by Lemma 1.3 we have

$$d(z, \frac{1}{2}x \oplus \frac{1}{2}y)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2,$$

for all $x, y, z \in X$, which is called (CN) inequality of Bruhat-Tits, as it was shown in [3]. In fact (cf. [4], p. 163), a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality.

2 Main results

Throughout this section we let $n \in \mathbb{N}$, $z_1 = x$ and $z_n = y$ until Definition 3.2.

Lemma 2.1. *Let (X, d) be a $CAT(0)$ space. Then*

1. *Let $x, y \in X$, $x \neq y$ and $z_i, z_i' \in [x, y]$ such that $d(x, z_i) = d(x, z_i')$ for all $1 \leq i \leq n$. Then $z_i = z_i'$ for $1 \leq i \leq n$.*

2. *Let $x, y \in X$, then for each $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0, 1]^n$ with $\sum_{i=1}^n \alpha_i = 1$ there exist points $z_1, \cdots, z_n \in [x, y]$ and unique point $z \in [x, y]$ such that $d(z, z_i) = \alpha_i d(x, y)$ for $1 \leq i \leq n$.*

Proof. Since $z_i, z_i' \in [x, y]$, there exist $t_i, t_i' \in [0, l]$ such that $c(t_i) = z_i$ and $c(t_i') = z_i'$. Thus $d(x, z_i) = d(c(0), c(t_i)) = t_i$ and similarly $d(x, z_i') = t_i'$. Since $d(x, z_i) = d(x, z_i')$, we have $t_i = t_i'$, and consequentially $z_i = z_i'$ for $1 \leq i \leq n$, which proves (1).
To prove (2), by [2, Lemma 2.1(iv)], this is true for $n = 2$, because for $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 + \alpha_2 = 1$ there exists unique point $z \in [x, y]$ such that $d(x, z) = \alpha_1 l, d(z, y) = \alpha_2 l$ that for convention we had shown with $z = \alpha_1 x \oplus \alpha_2 y$.
Now by induction let it holds for $n - 1$ and choose $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0, 1]^n$ such that $\sum_{i=1}^n \alpha_i = 1$. Put $\beta_i := \frac{\alpha_i}{1-\alpha_n}$ for $1 \leq i \leq n - 1$. Thus $\sum_{i=1}^{n-1} \beta_i = 1$ and by hypothesis of induction there exists unique point $z' \in [z_1, z_{n-1}]$ such that $d(z', z_i) = \beta_i l$ for $1, \leq i \leq n - 1$, now there exists unique point $z \in [z', z_n]$ such that $d(z, z_n) = \alpha_n l, d(z, z') = (1 - \alpha_n)l$.
To prove (2) directly, let $t_i = 1 - \alpha_n - \alpha_i, t = 1 - \alpha_n \in [0, 1]$ for $1 \leq i \leq n$. Put $z_i = c(t_i l)$ and $z = c(tl)$ so $d(z, z_i) = |t - t_i|l = \alpha_i l$, for $1 \leq i \leq n$. For uniqueness, if $d(z, z_i) = d(z', z_i)$ for $1 \leq i \leq n$, then by (1) and $i = 1$, we have $z = z'$. □

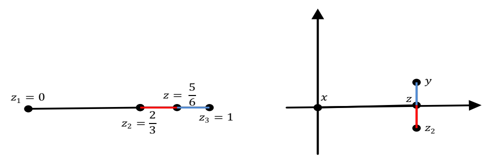
Example 2.2. *Let $X = [0, 1]$ and put*

$$A = \left\{(x, 0) : 0 \leq x \leq \frac{2}{3}\right\} \cup \left\{(\frac{2}{3}, y) : \frac{-1}{6} \leq y \leq \frac{1}{6}\right\}.$$

Define $f : X \to A \subseteq \mathbb{R}^2$ by

$$f(x) = \begin{cases} (x, 0), & 0 \leq x \leq \frac{2}{3}; \\ (\frac{2}{3}, x - \frac{5}{6}), & \frac{2}{3} \leq x \leq \frac{5}{6}; \\ (\frac{2}{3}, x - \frac{5}{6}), & \frac{5}{6} \leq x \leq 1. \end{cases}$$

So f is isometric homeomorphism. For instance let $\alpha_1 = \frac{2}{3}, \alpha_2 = \alpha_3 = \frac{1}{6}$. Therefore $z_1 = x = 0, z_2 = \frac{2}{3}, z_3 = y = 1, z = \frac{5}{6}$ and $l = 1$. Since $t = 1 - \alpha_3 = \frac{5}{6}$ and $t_2 = 1 - \alpha_3 - \alpha_2 = \frac{2}{3}$ so $z_2 = c(t_2) = \frac{2}{3}$, $z = \frac{5}{6}$ and by homeomorphism we have $z_1 = (0, 0), z_2 = (\frac{2}{3}, \frac{-1}{6}), z_3 = (\frac{2}{3}, \frac{1}{6})$ and $z = (\frac{2}{3}, 0)$. And also we have $d(z, z_i) = \alpha_i l$, for $1 \leq i \leq 3$.



Notation: By the point z_α, we mean the unique point

$$z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ such that $\sum_{i=1}^n \alpha_i = 1$ and $z_i \in X$ for $1 \le i \le n$. Also z_α can be written as

$$z_\alpha = (1-\alpha_n) z' \oplus \alpha_n z_n,$$

where $z' = \frac{\alpha_1}{1-\alpha_n} z_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} z_{n-1}$ where $\alpha_n \neq 1$.

Remark 2.3. *Let (X,d) be a $CAT(0)$ space, let $x, y \in X$ such that $x \neq y$ and $\alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n) \in [0,1]^n$ with $\sum_{i=1}^n \alpha_i = 1 = \sum_{i=1}^n \beta_i$. Then*

$$z_\alpha = z_\beta \iff \alpha = \beta.$$

Proof. This is true because,

$$d(z_\alpha, z_i) = d(z_\beta, z_i) \Rightarrow \alpha_i l = \beta_i l \Rightarrow \alpha_i = \beta_i,$$

for $1 \le i \le n$.

Theorem 2.4. *Let (X,d) be a $CAT(0)$ space, let $x, y \in X$ such that $x \neq y$ and $d(x,y) = l$. Then*

1. $[x,y] = \{z_\alpha | \alpha \in [0,1]^n, \ \sum_{i=1}^n \alpha_i = 1\}$.
2. *For all $z \in X$ the following holds:*
   $(\exists z_1, \cdots, z_n \in [x,y]$ *such that* $\sum_{i=1}^n d(z, z_i) = d(x,y)) \iff z \in [x,y]$.
3. *The mapping $f : [0,1]^n \to [x,y]$, $f(\alpha) = z_\alpha$ is continuous and bijective.*

Proof. (1) The case of $n = 2$ is proved in [2, Lemma 2.1]. Now let $z \in [x,y]$. By induction, suppose there exists $\beta \in [0,1]^{n-1}$, such that $\sum_{i=1}^{n-1} \beta_i = 1$ and $z = z_\beta$. Put $\alpha_i = \beta_i$ for $1 \le i \le n-2$ and $\alpha_{n-1} = \alpha_n = \frac{\beta_{n-1}}{2}$ therefore $\sum_{i=1}^n \alpha_i = 1$ and there exists $z' = c(\frac{\beta_{n-1}}{2} l)$ that $d(z', x) = (\sum_{i=1}^{n-2} \beta_i + \frac{\beta_{n-1}}{2}) l$ and $d(z, z') = \frac{\beta_{n-1}}{2} l$. Now $z' = (\sum_{i=1}^{n-2} \beta_i + \frac{\beta_{n-1}}{2}) z_\beta \oplus \frac{\beta_{n-1}}{2} y$ thus $z' \in [x,y]$ and $d(z,z') = \alpha_n l$.
To prove (2) let for every $z \in X$ there exist $z_1, \cdots, z_n \in [x,y]$ such that $\sum_{i=1}^n d(z, z_i) = d(x,y)$. Put $\alpha_i = \frac{d(z,z_i)}{l}$ where $z_i \in [x,y]$ and $1 \le i \le n$, so there exists z_α such that $z_\alpha = z$.

Conversely, if $z \in [x,y]$ then $z = z_\alpha$ for some α and $z_1, \cdots, z_n$ such that $d(z, z_i) = \alpha_i d(x,y)$ so $\sum_{i=1}^n d(z, z_i) = d(x,y)$.
To prove (3) applying (1) and Remark 2.3, we get that f is well defined and bijective. The continuity of f is obvious by induction, because f can be written as $f(\alpha) = g(\beta) \oplus h(\alpha_n)$ where $g(\beta) = z_\beta = \beta_1 z_1 \oplus \cdots \oplus \beta_{n-1} z_{n-1}$, $\beta_i := \frac{\alpha_i}{1-\alpha_n}$ for $1 \le i \le n-1$ and $h(\alpha_n) = \alpha_n z_n$. $\square$

Lemma 2.5. *Let (X,d) be a $CAT(0)$ space. Then*

1. $d(z_\alpha, z) \le \sum_{i=1}^n \alpha_i d(z_i, z) \le \max\{d(z_i, z) : 1 \le i \le n\}$,
2. $d(z_\alpha, z)^2 \le \sum_{i=1}^n \alpha_i d(z_i, z)^2 \le \max\{d(z_i, z)^2 : 1 \le i \le n\}$,
3. $d(z_\alpha, z'_\beta) \le \sum_{i,j=1}^n \alpha_i \beta_j d(z_i, z'_j) \le \max\{d(z_i, z'_j) : 1 \le i, j \le n\}$,

for $\alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n) \in [0,1]^n$ with $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i = 1$ and $z, z_i, z'_i \in X$ for $1 \le i \le n$ which $z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n$, $z'_\beta = \beta_1 z'_1 \oplus \beta_2 z'_2 \oplus \cdots \oplus \beta_n z'_n$.

Proof. By Lemma 1.2 it is true for $n = 2$. So by induction let

$$z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ such that $\sum_{i=1}^n \alpha_i = 1$ and $z_i \in X$ for $1 \le i \le n$. Put $\gamma := \left(\frac{\alpha_1}{1-\alpha_n}, \cdots, \frac{\alpha_{n-1}}{1-\alpha_n}\right)$ that $\sum_{k=1}^{n-1} \frac{\alpha_k}{1-\alpha_n} = 1$ by Theorem 2.1 there exists $v_\gamma \in [x, z_{n-1}]$ such that $v_\gamma = \frac{\alpha_1}{1-\alpha_n} z_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} z_{n-1}$ and we have $z_\alpha = (1-\alpha_n) v_\gamma \oplus \alpha_n z_n$ so

$$\begin{aligned} d(z_\alpha, z) &= d((1-\alpha_n) v_\gamma \oplus \alpha_n z_n, z) \\ &\le (1-\alpha_n) d(v_\gamma, z) + \alpha_n d(z_n, z) \\ &= (1-\alpha_n) d\left(\frac{\alpha_1}{1-\alpha_n} z_1 \oplus \cdots \oplus \frac{\alpha_{n-1}}{1-\alpha_n} z_{n-1}, z\right) + \alpha_n d(z_n, z) \\ &\le \sum_{i=1}^n \alpha_i d(z_i, z) \\ &\le \max\{d(z_i, z) : 1 \le i \le n\}. \square \end{aligned}$$

This proves (1).
(2) can easily proved according to Lemma 1.3 and again by induction on $n \ge 2$.$\square$

Lemma 2.6. *Let (X,d) be a hyperbolic space. Then*

$$d(z_\alpha, z'_\alpha) \le \sum_{i=1}^n \alpha_i d(z_i, z'_i) \le \max\{d(z_i, z'_i) : 1 \le i \le n\},$$

for $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0,1]^n$ with $\sum_{i=1}^n \alpha_i = 1$ and $z_i, z'_i \in X$ for $1 \le i \le n$ which $z_\alpha = \alpha_1 z_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z_n$, $z'_\alpha = \alpha_1 z'_1 \oplus \alpha_2 z_2 \oplus \cdots \oplus \alpha_n z'_n$.

Proof. By the property of $(W4)$ it is true for $n = 2$. The remaining is similar to the proof of the lemma 2.5. $\square$

3 Fixed points and approximate fixed points for T_α maps

In 2008 T. Suzuki [5], defined condition (C) for mappings on a subset of a Banach space, as following: "Let T be a mapping on a subset C of a Banach space E. Then T is said to satisfy condition (C) if

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$."
This condition is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. In that paper, he has presented fixed point theorems and convergence theorems for mappings satisfying condition (C). Also Examples 1 and 2 in the same paper stated that there exists a map T which satisfies condition (C), but T is not nonexpansive, and there exists a map T which is quasi-nonexpansive, but it does not satisfy condition (C).
Recently B. Nanjaras, B. Panyanaka and W. Phuengrattana in [6], A. Razani and H. Salahifard in [7] and other mathematicians has proved some theorems according to single-valued mappings or multi-valued mappings which are satisfying Suzuki's condition (C) in a $CAT(0)$ space.

Some basic properties on condition (C) by [6, Propositions 3.2, 3.3], [7, Theorems 2.3, 2.7 and Corollary 2.8] and [8, Theorem 1.3] are:

$P1$ ([6, Lemma 2.5]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a $CAT(0)$ space X and let $\{\alpha_n\} \subseteq [0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_n \alpha_n < 1$. Suppose that $x_{n+1} = \alpha_n y_n \oplus (1 - \alpha_n)x_n$ and $d(y_{n+1}, y_n) \leq d(x_{n+1}, x_n)$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(y_n, x_n) = 0$.

$P2$ ([6, Proposition 3.2]) Let K be a nonempty subset of a $CAT(0)$ space X. If $T : K \to K$ be a nonexpansive mapping, then T satisfies condition (C).

$P3$ ([6, Proposition 3.3]) Let K be a nonempty subset of a $CAT(0)$ space X. If $T : K \to K$ satisfies condition (C) and has a fixed point, then T is a quasi-nonexpansive mapping.

$P4$ ([7, Theorem 2.3]) Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ satisfies the condition (C) and $F(T) \neq \emptyset$, then $F(T)$ is Δ-closed and convex set.

$P5$ ([7, Theorem 2.7]) Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ satisfies condition (C), then $F(T)$ is nonempty.

$P6$ ([7, Corollary 2.8]) Let K be a bounded closed convex subset of a complete $CAT(0)$ space X. If $T : K \to K$ satisfies condition (C), then $F(T)$ is nonempty, Δ-closed and convex.

$P7$ ([8, Theorem 1.3]) Let (X, d) be a convex subset of a $CAT(0)$ space and $f : X \to X$ a quasi-nonexpansive map whose fixed point set is nonempty. Then $F(f)$ is closed, convex and hence contractible.

And now, we start our results by following definitions.

Definition 3.1. *([5]) Let T be a mapping on a subset K of a $CAT(0)$ space (X, d). Then T is said to satisfy* condition (C) *if*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in K$.

The following we will use this notation $T_\alpha = \alpha_1 T_1 \oplus \cdots \oplus \alpha_n T_n$ where $T_1, \cdots, T_n : X \to [x, y]$ for $1 \leq i \leq n$ and $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0, 1]^n$ a multiindex satisfying $\sum_{i=1}^n \alpha_i = 1$.

Definition 3.2. *([9-10])Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in [0, 1]^n$ be a multiindex satisfying $\sum_{i=1}^n \alpha_i = 1$. The maps $T_1, \cdots, T_n$ on X are said to be α-nonexpansive if*

$$\sum_{i=1}^{n} \alpha_i d(T_i x, T_i y) \leq d(x, y), \tag{3.1}$$

for all $x, y \in X$.

Theorem 3.3. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space (X, d). If $T_\alpha : K \to K$ is defined by $T_\alpha = \alpha_1 T_1 \oplus \cdots \oplus \alpha_n T_n$ which $T_1, \cdots, T_n$ are selfmaps on K, which commute each other and satisfy condition (C), then T_α has a fixed point.*

Proof. By $P5$, $F(T_i) \neq \emptyset$ for $1 \leq i \leq n$. We say $\bigcap_{i=1}^n F(T_i) \neq \emptyset$. By induction we assume that $L := \bigcap_{i=1}^{n-1} F(T_i) \neq \emptyset$. Let $x \in L$ so we have

$$T_n x = T_n(T_i x) = T_i(T_n x),$$

thus $T_n x \in F(T_i)$ for $1 \leq i \leq n - 1$. Therefore $T_n x \in L$ hence $T_n(L) \subseteq L$. By $P6$, $F(T_i)$ nonempty and convex and since T_i satisfy the condition (C) by $P3$, T_i is a quasinonexpansive map and by $P7$, $F(T_i)$ closed and convex, for $(1 \leq i \leq n)$, therefore L and $F(T_n)$ are nonempty, bounded closed convex subsets of a complete

$CAT(0)$ space. Thus $T : L \to L$ satisfies the condition of the $P4$, hence $T_n x$ has a fixed point in L, that is,

$$L \cap F(T_n) = \bigcap_{i=1}^{n} F(T_i) \neq \emptyset.$$

If we let $x \in \bigcap_{i=1}^{n} F(T_i)$, then

$$d(x, T_\alpha x) \leq \sum_{i=1}^{n} \alpha_i d(x, T_i x) = 0,$$

namely $x \in F(T_\alpha)$. □

Theorem 3.4. *Let K be a bounded closed convex subset of a complete $CAT(0)$ space (X, d). If $T_\alpha : K \to K$ defined by $T_\alpha = \alpha_1 T_1 \oplus \cdots \oplus \alpha_n T_n$ which $T_1, \cdots, T_n$ are selfmaps on K, which T_1 satisfies the condition (C) and $d(x, T_n x) \leq d(x, T_1 x)$ for every $x \in K$, then $\inf_{x \in K} d(x, T_\alpha x) = 0$.*

Proof. Let $x_1 \in K$, define sequence $\{x_n\} \subseteq K$ by $x_{n+1} := tT_1 x_n \oplus (1-t)x_n$ for $n \in \mathbb{N}$, where $t \in [\frac{1}{2}, 1)$. Then by the assumption $\frac{1}{2} d(x_n, T_1 x_n) \leq td(x_n, T_1 x_n) = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$ hence $d(T_1 x_{n+1}, T_1 x_n) \leq d(x_{n+1}, x_n)$. So by $P1$ we have $\inf_{x \in K} d(x, T_1 x) = 0$. So

$$\begin{aligned}
d(x, T_\alpha x) &\leq d(x, T_1 x) + d(T_1 x, T_\alpha x), \\
&= d(x, T_1 x) + \alpha_1 d(T_1 x, T_n x), \\
&\leq d(x, T_1 x) + d(T_1 x, x) + d(x, T_n x), \\
&\leq 3d(x, T_1 x),
\end{aligned}$$

therefore there exists $\{x_n\} \subseteq K$ such that $d(x_n, T_1 x_n) \to 0$ as $n \to \infty$ thus $d(x_n, T_\alpha x_n) \to 0$.□

Corollary 3.5. *([7, Lemma 2.5]) Let K be a bounded closed convex subset of a complete $CAT(0)$ space (X, d). If $T : K \to K$ satisfies the condition (C), then there exists an approximate fixed point sequence for T, i.e., $\inf_{x \in K} d(x, Tx) = 0$.*

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