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Module contractibility for semigroup algebras

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Abstract

In this paper, we find the relationships between module contractibility of a Banach algebra and its ideals. We also prove that module contractibility of a Banach algebra is equivalent to module contractibility of its module unitization. Finally, we show that when a maximal group homomorphic image of an inverse semigroup S with the set of idempotents E is finite, the module projective tensor product $\ell^1(S)\widehat{\otimes}_{\ell^1(E)}\ell^1(S)$ is $\ell^1(E)$-module contractible.

Key words: Banach module; module contractibility; module derivation; inverse semigroups.

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1 Introduction

A Banach algebra $\mathcal{A}$ is called *contractible* (*super-amenable*) if $H^1(\mathcal{A}, \mathcal{X}) = \{0\}$ for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$, where the left hand side is the *first cohomology group* of $\mathcal{A}$ with coefficient in $\mathcal{X}$ (see [6,?]). A Banach space E has the *approximation property* if there is a net $(T_j)_j$ in $\mathcal{F}(E)$, the space of the bounded finite rank operators on E such that $T_j \longrightarrow id_E$ uniformly on compact subsets on E. It is shown in [15, Theorem 4.1.5] if $\mathcal{A}$ is a contractible Banach algebra and has the approximation property, then $\mathcal{A}$ is finite dimensional. In particular ℓ^1 has the approximation property [6], so ℓ^1-convolution algebra of infinite semigroup is not contractible. Also if S is a regular semigroup with a finite number of idempotents, then contractibilty of $\ell^1(S)$ implies finiteness of S [8, Theorem 3.5]. But for groups, Selivanov showed in [16] that for any locally compact group G, $L^1(G)$ is contractible if and only if G is finite (see also [15, Exercise 4.1.7]).

Amini in [1] introduced the concept of module amenability and showed that for an inverse semigroup S, the semigroup algebra $\ell^1(S)$ is module amenable as a Banach module on $\ell^1(E)$, where E is the set of idempotents of S, if and only if S is amenable. This is the semigroup analog of Johnson's theorem for locally compact groups [10]. Pourmahmood in [12] developed the concept of contractibility for a Banach algebra $\mathcal{A}$ to the case that there is an extra $\mathfrak{A}$-module structure on $\mathcal{A}$, and show that $\ell^1(S)$ is contractible (as $\ell^1(E)$-module) if and only if an appropriate group homomorphic image of S is finite. When S is a group, this is just the Selivanov's theorem (in the discrete case); see [16]. For an finite group G, it follows from the Selivanov's theorem [16] and [15, Excersie 4.1.4] that the projective tensor product $\ell^1(G)\widehat{\otimes}\ell^1(G) = \ell^1(G \times G)$ is contractible. This is not true for any discrete semigroup.

In the current paper, we investigate the hereditary properties of module contractibility for Banach algebras. Among many other things, we study the relationships between module contractibility of a Banach algebra and its ideals. We also find a similar result for module contractibility of the semigroup algebra of an inverse semigroup which is the semigroup analog of Selivanov's theorem for locally compact groups [16] with another method. We employ this result to show that when $\ell^1(S)\widehat{\otimes}_{\ell^1(E)}\ell^1(S)$ is $\ell^1(E)$-module contractible (the module amenability case of this has earlier been proved in [5] by author).

2 Module contractibility

Throughout this paper, $\mathcal{A}$ and $\mathfrak{A}$ are Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \ (ab) \cdot \alpha = a(b \cdot \alpha)$$

where $a, b \in \mathcal{A}, \alpha \in \mathfrak{A}$.

Let $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha \cdot (a \cdot x) = (\alpha \cdot a) \cdot x, \ a \cdot (\alpha \cdot x) = (a \cdot \alpha) \cdot x, \ (\alpha \cdot x) \cdot a = \alpha \cdot (x \cdot a)$$

where $a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in \mathcal{X}$. and similarly for the right or two-sided actions. Then we say that $\mathcal{X}$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. Moreover, if $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}, x \in \mathcal{X}$, then $\mathcal{X}$ is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module.

Consider the module projective tensor product $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ which is isomorphic to the quotient space $(\mathcal{A}\widehat{\otimes}\mathcal{A})/I$, where I is the closed linear span of $\{a \cdot \alpha \otimes b - a \otimes \alpha \cdot b : \alpha \in \mathfrak{A}, a, b \in \mathcal{A}\}$. Also consider the closed ideal J of $\mathcal{A}$ generated by elements of the form $(a \cdot \alpha)b - a(\alpha \cdot b)$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. Then I and J are $\mathcal{A}$-submodules and $\mathfrak{A}$-submodules of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ and $\mathcal{A}$, respectively, and the quotients $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ and $\mathcal{A}/J$ are $\mathcal{A}$-modules and $\mathfrak{A}$-modules. Also, $\mathcal{A}/J$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically. Also, let $\omega : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ be the product map, i.e., $\omega(a \otimes b) = ab$, and let $\tilde{\omega} : \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} = (\mathcal{A}\widehat{\otimes}\mathcal{A})/I \longrightarrow \mathcal{A}/J$ be its induced product map, i.e., $\tilde{\omega}(a \otimes b + I) = ab + J$.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and $\mathcal{X}$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded map $D : \mathcal{A} \longrightarrow \mathcal{X}$ is called a *module derivation* if

$$D(a \pm b) = D(a) \pm D(b), \ D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}),$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Note that $D : \mathcal{A} \longrightarrow \mathcal{X}$ is bounded if there exist $t > 0$ such that $\|D(a)\| \leq t\|a\|$, for each $a \in \mathcal{A}$. Since D preserves subtraction, boundedness of D implies its norm continuity. When $\mathcal{X}$ is commutative, each $x \in \mathcal{X}$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These are called *inner* module derivations.

Definition 2.1 *The Banach algebra $\mathcal{A}$ is called module contractible (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $\mathcal{X}$, each module derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ is inner.*

One should remember that a left Banach $\mathcal{A}$-module $\mathcal{X}$ is called a *left essential* $\mathcal{A}$-module if the linear span of $\mathcal{A} \cdot \mathcal{X} = \{a \cdot x : a \in \mathcal{A},\, x \in \mathcal{X}\}$ is dense in $\mathcal{X}$. Right essential $\mathcal{A}$-modules and (two-sided) essential $\mathcal{A}$-bimodules are defined similarly.

Proposition 2.2 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with one of the following conditions:*

(i) *$\mathfrak{A}$ has an identity for $\mathcal{A}$;*
(ii) *$\mathcal{A}$ is an essential left or right $\mathfrak{A}$-module,*

then every $\mathfrak{A}$-module derivation is also a derivation. In particular, contractibility of $\mathcal{A}$ implies its module contractibility.

Proof. (i) Let $\mathfrak{e} \in \mathfrak{A}$ be a identity for $\mathcal{A}$, that is $\mathfrak{e}.a = a.\mathfrak{e} = a$, for each $a \in \mathcal{A}$, and $\mathcal{X}$ be a commutative $\mathcal{A}$-$\mathfrak{A}$-module. Assume that $D : \mathcal{A} \longrightarrow \mathcal{X}$ is a module derivation, then obviously $D(a \cdot \lambda\mathfrak{e}) = D(\lambda a)$, for each $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. On the other hand,

$$D(a \cdot \lambda\mathfrak{e}) = D(a) \cdot \lambda\mathfrak{e} = \lambda D(a) \cdot \mathfrak{e} = \lambda D(a \cdot \mathfrak{e}) = \lambda D(a).$$

Thus D is $\mathbb{C}$-linear, and so inner.

(ii) Since $\mathcal{A}$ is an essential left $\mathfrak{A}$-module, for each $a \in \mathcal{A}$, there is a sequence $(E_n) \subseteq \mathfrak{A} \cdot \mathcal{A}$ such that $\lim_n E_n = a$. Suppose that $E_n = \sum_{m=1}^{K_n} \alpha_{n,m} \cdot a_{n,m}$ for some finite sequences $(a_{n,m})_{m=1}^{m=K_n} \subseteq \mathcal{A}$ and $(\alpha_{n,m})_{m=1}^{m=K_n} \subseteq \mathfrak{A}$. Let $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} D(\lambda E_n) &= D(\lambda \sum_{m=1}^{K_n} \alpha_{n,m} \cdot a_{n,m}) = \sum_{m=1}^{K_n} D((\lambda\alpha_{n,m}) \cdot a_{n,m}) \\ &= \sum_{m=1}^{K_n} (\lambda\alpha_{n,m}) \cdot D(a_{n,m}) = \sum_{m=1}^{K_n} \lambda D(\alpha_{n,m} \cdot a_{n,m}) = \lambda D(E_n), \end{aligned}$$

and so, by the continuity of D, $D(\lambda a) = \lambda D(a)$. The right case is similarly. $\square$

As we will see later in section 3, there are module contractible Banach algebras that are not contractible, so the converse of the above Proposition is false. It is known that every contractible Banach algebra has an identity. We have a similar result for the module case as follows.

Proposition 2.3 *Let $\mathcal{A}$ be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. If $\mathcal{A}$ is module contractible, then it is unital.*

Proof. Let's consider $\mathcal{X} = \mathcal{A}$ as an $\mathcal{A}$-bimodule, with actions

$$a \cdot b := ab, \quad b \cdot a := 0 \quad (a \in \mathcal{A}, b \in \mathcal{X}).$$

Let $D : \mathcal{A} \longrightarrow \mathcal{X}$ be the identity map, it is clear that D is a module derivation. This means that there is $a_0 \in \mathcal{A}$ such that $aa_0 = a$, for all $a \in \mathcal{A}$. Therefore a_0 is a right identity for $\mathcal{A}$. Similarly, $\mathcal{A}$ has a left identity. The left and right identities now have to coincide. $\square$

The following result is proved in [12, Proposition 3.3] (see [4, Proposition 3.4] for a different proof).

Proposition 2.4 *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules. If $\mathcal{A}$ is $\mathfrak{A}$-module contractible and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous Banach algebra homomorphism which is $\mathfrak{A}$-module with dense range, then $\mathcal{B}$ is also $\mathfrak{A}$-module contractible.*

Corollary 1 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. Then module contractibility of $\mathcal{A}$ implies module contractibility of $\mathcal{A}/\mathcal{I}$. In particular, if $\mathcal{A}$ is module contractible, then so is $\mathcal{A}/J$.*

Proposition 2.5 *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module and $\mathcal{I}$ be a closed ideal and $\mathfrak{A}$-submodule of $\mathcal{A}$. If $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are module contractible, then so is $\mathcal{A}$.*

Proof. Assume that $\mathcal{X}$ be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module with compatible actions and $D : \mathcal{A} \longrightarrow \mathcal{X}$ be a bounded module derivation. Since $\mathcal{I}$ is module amenable, there exists $x_1 \in \mathcal{X}$ such that $D\mid_{\mathcal{I}} = D_{x_1}$. Thus, the map $\widetilde{D} = D - D_{x_1}$ vanishes on $\mathcal{I}$. This map induces a module derivation from $\mathcal{A}/\mathcal{I}$ into $\mathcal{X}$ defined by $\widetilde{D}(a+\mathcal{I}) = \widetilde{D}(a)$. Due to module amenability of $\mathcal{A}/\mathcal{I}$, there is $x_2 \in \mathcal{X}$ such that $\widetilde{D} = D_{x_2}$. Consequently, $D = D_{x_1+x_2}$. $\square$

A Banach algebra is contractible if and only if it has a diagonal [15]. Recall that a *diagonal* for $\mathcal{A}$ is an element $M \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ satisfying

$$a \cdot \omega(M) = a, \;\; a \cdot M = M \cdot a \;\; (a \in \mathcal{A}).$$

Definition 2.6 *An element $M \in \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is called a module diagonal if $a \cdot M = M \cdot a$ and $a \cdot \widetilde{\omega}(M) = a + J$, for all $a \in \mathcal{A}$.*

We have the following theorem which is proved in [12, Theorem 3.5].

Theorem 2.7 *Let $\mathcal{A}$ be a commutative Banach $\mathfrak{A}$-module. Then $\mathcal{A}$ is module contractible if and only if $\mathcal{A}$ has a module diagonal.*

Assume that $\mathcal{A}$ is a unital and module contractible Banach algebra such that $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is commutative $\mathfrak{A}$-module. We wish to show that $\mathcal{A}$ has module diagonal M such that $\widetilde{\omega}(M) = e + J$, where e is an identity for $\mathcal{A}$. Put $T = e \otimes e + I$, we have $\widetilde{\omega}(a \cdot T - T \cdot a) = J$. Hence $\widetilde{\omega}$ vanishes on the range of D_T, and D_T could be regarded as a module derivation into K=ker $\widetilde{\omega}$. Since $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is commutative $\mathfrak{A}$-module, so is K, hence by module contractibility of $\mathcal{A}$, there is $N \in K$ such that $D_T = D_N$. Now it is easy to see that $M = N - T$ is a module diagonal and $\widetilde{\omega}(M) = e + J$.

The next result is a characterization of the closed ideals of a module contractible Banach algebra which are module contractible themselves.

Proposition 2.8 *Let $\mathcal{A}$ be a commutative module contractible Banach $\mathfrak{A}$-module and $\mathcal{I}$ be a closed ideal and $\mathfrak{A}$-submodule of $\mathcal{A}$. Then $\mathcal{I}$ is module contractible if and only if $\mathcal{I}$ has an identity.*

Proof. Assume that e is the identity of $\mathcal{I}$. For each $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$ we have

$$\begin{aligned}((a\cdot\alpha)\otimes b - a\otimes(\alpha\cdot b))(e\otimes e) &= (a\cdot\alpha)e\otimes be - ae\otimes(\alpha\cdot b)e\\ &= (ae\cdot\alpha)\otimes be - ae\otimes\alpha\cdot(be)\in I_{\mathcal{I}},\end{aligned}$$

where $I_{\mathcal{I}}$ is corresponding ideal of $\mathcal{I}\widehat{\otimes}\mathcal{I}$. If $M=\sum_j a_j\otimes b_j + I$ is a module diagonal for $\mathcal{A}$, by using the above equalities, it is not hard to check that $\mathcal{M}=\sum_j a_je\otimes b_je + I_{\mathcal{I}}$ is a module diagonal for $\mathcal{I}$. Now, It follows from Theorem 2.7 that $\mathcal{I}$ is module contractible. By Proposition 2.3, module contractibility of $\mathcal{I}$ implies that it has an identity. $\Box$

Lemma 2.9 *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules. If $\mathcal{B}$ is a right essential $\mathfrak{A}$-module, then so is $\mathcal{A}\widehat{\otimes}\mathcal{B}$.*

Proof. Suppose that $f=\Sigma_{i=1}^n a_i\otimes b_i\in\mathcal{A}\widehat{\otimes}\mathcal{B}$, where $a_i\in\mathcal{A}$ and $b_i\in\mathcal{B}$ for all i. Since $\mathcal{B}$ is an essential right $\mathfrak{A}$-module, we may assume that $b_i=\lim_j(\Sigma_j b_i^j\cdot\alpha^j)$ in which $b_i^j\in\mathcal{B}$ and $\alpha^j\in\mathfrak{A}$. We have

$$f=\Sigma_{i=1}^n a_i\otimes(\lim_j(\Sigma_j b_i^j\cdot\alpha^j)) = \lim_j(\Sigma_{i=1}^n a_i\otimes(\Sigma_j b_i^j\cdot\alpha^j))$$

$$=\lim_j(\Sigma_j\Sigma_{i=1}^n a_i\otimes b_i^j\cdot\alpha^j).$$

The above equalities show that f belongs to closed linear span $(\mathcal{A}\widehat{\otimes}\mathcal{B})\cdot\mathfrak{A}$, i.e., $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is a right essential $\mathfrak{A}$-module. $\Box$
It follows from the above Lemma that if $\mathcal{A}$ and $\mathcal{B}$ are amenable such that $\mathcal{B}$ is an essential right $\mathfrak{A}$-module, then every $\mathfrak{A}$-module derivation on $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is also a derivation. Therefore the contractibility of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ implies its module contractibility.

Theorem 2.10 *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-modules. Then $\mathcal{A}\oplus\mathcal{B}$ is module contractible if and only if $\mathcal{A}$ and $\mathcal{B}$ are module contractible.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be module contractible. Since $\mathcal{A}$, the closed ideal of $\mathcal{A}\oplus\mathcal{B}$ and the quotient algebra $(\mathcal{A}\oplus\mathcal{B})/\mathcal{A} \cong \mathcal{B}$ are module contractible, $\mathcal{A}\oplus\mathcal{B}$ is module contractible by Proposition 2.5.

For the converse, assume that $\mathcal{A}\oplus\mathcal{B}$ is module contractible. By Corollary 1, the Banach algebras $(\mathcal{A}\oplus\mathcal{B})/\mathcal{A} \cong \mathcal{B}$ and $(\mathcal{A}\oplus\mathcal{B})/\mathcal{B} \cong \mathcal{A}$ are module contractible. $\square$

Let $\mathfrak{A}$ be a non-unital Banach algebra. Then the unitization of $\mathfrak{A}$ which is $\mathfrak{A}^{\#} = \mathfrak{A}\oplus\mathbb{C}$ is a unital Banach algebra which contains $\mathfrak{A}$ as a closed ideal. Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions. Then $\mathcal{A}$ is a Banach algebra and a Banach $\mathfrak{A}^{\#}$-bimodule with compatible actions in the obvious way, i.e., the action of $\mathfrak{A}^{\#}$ on $\mathcal{A}$ is as follows:

$$(\alpha,\lambda)\cdot a = \alpha\cdot a + \lambda a, \quad a\cdot(\alpha,\lambda) = a\cdot\alpha + \lambda a \qquad (\lambda\in\mathbb{C},\ \alpha\in\mathfrak{A},\ a\in\mathcal{A}).$$

Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions and let $\mathcal{A}^{\sharp} = (\mathcal{A}\oplus\mathfrak{A}^{\#}, \bullet)$, where the multiplication $\bullet$ is defined via

$$(a,u)\bullet(b,v) = (ab + av + ub, uv) \qquad (a,b\in\mathcal{A},\ u,v\in\mathfrak{A}^{\#}).$$

Then with the actions defined by

$$\alpha\cdot(a,v) = (\alpha\cdot a, \alpha v), \quad (a,v)\cdot\alpha = (a\cdot\alpha, v\alpha) \qquad (a\in\mathcal{A},\ \alpha\in\mathfrak{A}\ v\in\mathfrak{A}^{\#}),$$

we see that $\mathcal{A}^{\sharp}$ is a unital Banach algebra with identity $1_{\mathcal{A}}$ and a Banach $\mathfrak{A}$-bimodule with compatible actions.

It follows from [15, Exercise 4.1.3 (iii)] that contactibility of a Banach algebra $\mathcal{A}$ is equivalent to contractibility of $\mathcal{A}^{\#} = \mathcal{A}\oplus\mathbb{C}$. We generalize this for the module version.

Theorem 2.11 *Let $\mathcal{A}$ be a Banach algebra and a Banach $\mathfrak{A}$-bimodule with compatible actions. Then $\mathcal{A}$ is $\mathfrak{A}$-module contractible if and only if*

$\mathcal{A}^\sharp$ *is* $\mathfrak{A}$*-module contractible.*

Proof. . Let $\mathcal{X}$ be a commutative Banach $\mathcal{A}^\sharp$-$\mathfrak{A}$-module and $D : \mathcal{A}^\sharp \to \mathcal{X}$ be an $\mathfrak{A}$-module derivation. Since $D(1_{\mathcal{A}}) = 0$, we have $D(\alpha) = \alpha \cdot D(1_{\mathcal{A}}) = 0$ for all $\alpha \in \mathfrak{A}$, and thus D reduces to a module derivation $D : \mathcal{A} \to \mathcal{X}$. Since $\mathcal{X}$ is also a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, D is inner.

Conversely, suppose that $\mathcal{X}$ is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module. Then $\mathcal{X}$ is a commutative Banach $\mathcal{A}^\sharp$-$\mathfrak{A}$-module in the usual way. Now every $\mathfrak{A}$-module derivation $D : \mathcal{A} \to \mathcal{X}$ extends to an $\mathfrak{A}$-module derivation $\tilde{D} : \mathcal{A}^\sharp \to \mathcal{X}$ defined by $\tilde{D}(a, u) = D(a)$ for all $a \in \mathcal{A}$, $u \in \mathfrak{A}^\#$. By the hypothesis, $\tilde{D}$ is inner and thus D is inner. Therefore $\mathcal{A}$ is $\mathfrak{A}$-module amenable. $\square$

3 Module contractibility for semigroup algebras

We start this section with the definition of an inverse semigroup.

Definition 3.1 *A discrete semigroup S is called an inverse semigroup if for each $s \in S$ there is a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e = e^* = e^2$. The set of idempotents of S is denoted by E.*

There is a natural order on E, defined by

$$e \leq d \Longleftrightarrow ed = e \quad (e, d \in E).$$

It is easy to see that E is indeed a commutative subsemigroup of S. In particular $\ell^1(E)$ could be regarded as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible canonical actions [1]. However, for technical reasons, here we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s, \ \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E). \tag{3.1}$$

We say that the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left (right) if for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, $\alpha \cdot a = \psi(\alpha)a$ $(a \cdot \alpha = \psi(\alpha)a)$, where ψ is a bounded linear functional on $\mathfrak{A}$.

If ψ is the augmentation character on $\ell^1(E)$, then for each $e \in E$ we have $\psi(\delta_e) = 1$. So for each $f = \sum_{e\in E} f(e)\delta_e \in \ell^1(E)$ and $g = \sum_{s\in S} g(s)\delta_s \in \ell^1(S)$, we have

$$\begin{aligned} f \cdot g &= (\sum_{e\in E} f(e)\delta_e) \cdot (\sum_{s\in S} g(s)\delta_s) = \sum_{s\in S, e\in E} f(e)g(s)\delta_e \cdot \delta_s \\ &= \sum_{s\in S, e\in E} f(e)g(s)\delta_s = (\sum_{e\in E} f(e))(\sum_{s\in S} g(s)\delta_s) = \psi(f)g. \end{aligned}$$

Therefore the left action is trivial. In this case, the ideal J (see section 2) is the closed linear span of $\{\delta_{set} - \delta_{st} \quad s, t \in S, e \in E\}$. We consider an equivalence relation on S as follows:

$$s \approx t \Longleftrightarrow \delta_s - \delta_t \in J \ \ (s, t \in S).$$

For an inverse semigroup S, the quotient $S/\approx$ is a discrete group (see [2] and [12]). Indeed, $S/\approx$ is homomorphic to the maximal group homomorphic image G_S [11] of S [13]. In particular, S is amenable if and only if G_S is amenable [7,**?**]. As in [14, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(G_S)$. With the notations of the previous section, $\ell^1(S)/J$ is a commutative $\ell^1(E)$-bimodule with the following actions:

$$\delta_e \cdot (\delta_s + J) = \delta_s + J, \ (\delta_s + J) \cdot \delta_e = \delta_{se} + J \quad (s \in S, e \in E).$$

The following theorem is a semigroup analog of the Selivanov's theorem [16] for groups, characterizing module contractibility of the semigroup algebra of an inverse semigroup with the set of idempotents which has been proven for the first time in [12] by using this fact that every module diagonal for $\ell^1(G_S)$ is a diagonal. We bring another proof for it.

Theorem 3.2 *Let S be an inverse semigroup with the set of idempotents E. Then $\ell^1(S)$ is module contractible, as an $\ell^1(E)$-module with trivial left action and canonical right action, if and only if G_S is finite.*

Proof. Suppose that $\ell^1(S)$ is module contractible, then $\ell^1(G_S)$ is contractible by [12, Theorem 3.7]. Since G_S is a (discrete) group, it has to be finite by Selivanov's theorem [16]. Conversely, if G_S is finite, then $\ell^1(G_S)$ is contractible [16]. By a similar method of [2, Proposition 3.3], we can prove that $\ell^1(S)$ is contractible. With the actions (3.1) of $\ell^1(E)$ on $\ell^1(S)$, the semigroup algebra $\ell^1(S)$ is always a right essential $\ell^1(E)$-module. Indeed, if $f \in \ell^1(S)$, we have

$$f = \sum_{s\in S} f(s)\delta_s = \sum_{s\in S} f(s)\delta_s * \delta_{s^*s} = \sum_{s\in S} f(s)\delta_s \cdot \delta_{s^*s}.$$

The above inequalities show that f belongs to the closed linear span of $\ell^1(S)\cdot \ell^1(E) = \{\delta_s \cdot \delta_e : e \in E, s \in S\}$. Now the result follows from Proposition 2.2. $\square$

The author in [3] has proven that if $G_S = S/\approx$ is finite and E is an upward direct set, then $\ell^1(S)\widehat{\otimes}\ell^1(S)$ is module contractible. The upward directed condition for E is strong and in fact in the next theorem we show that is redundant. Therefore, the hypothesis on E being upward directed can be eliminated and $\ell^1(S)\widehat{\otimes}\ell^1(S)$ is module contractible when G_S is finite.

Theorem 3.3 *Let S be an inverse semigroup with the set of idempotents E. Consider the following assertions:*

(i) *$\ell^1(S)\widehat{\otimes}\ell^1(S)$ is module contractible;*
(ii) *$\ell^1(G_S)\widehat{\otimes}\ell^1(G_S)$ is module contractible;*
(iii) *$\ell^1(G_S)\widehat{\otimes}\ell^1(G_S)$ is contractible;*
(iv) *$\ell^1(S)\widehat{\otimes}\ell^1(S)$ is contractible.*

Then $(iv) \Longrightarrow (i) \Longleftrightarrow (ii) \Longleftrightarrow (iii)$.

Proof. Similare to the proof of [5, Theorem 5], we can show that the parts $(i), (ii)$ and (iii) are equivalent. Now, assume that $\ell^1(S)\widehat{\otimes}\ell^1(S)$ is contractible. Then $\ell^1(S)$ is a right essential $\ell^1(E)$-module, and so $\ell^1(S)\widehat{\otimes}\ell^1(S)$ is a right essential $\ell^1(E)$-module by Lemma 2.9. Therefore $\ell^1(S)\widehat{\otimes}\ell^1(S)$ is module contractible. $\square$

It has been shown in [1, Lemma 3.1] that if $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, then $\ell^1(S)\widehat{\otimes}_{\ell^1(E)}\ell^1(S) \cong$

$\ell^1(S \times S)/I$, where I is the closed ideal of $\ell^1(S \times S)$ generated by the set of elements of the form $\delta_{(set,u)} - \delta_{(st,u)}$, where $s, t, u \in S$ and $e \in E$.

Corollary 2 *If G_S is finite, then $\ell^1(S)\widehat{\otimes}_{\ell^1(E)}\ell^1(S)$ is module contractible.*

Proof. Since G_S finite, $\ell^1(G_S)$ is contractible by Selivanov's theorem. It follows from the Johnson's theorem that $\ell^1(G_S)\widehat{\otimes}\ell^1(G_S)$ is contractible (see also [15, Exercise 4.1.3 (iv)]). Now, the result is a consequence of Theorem 3.3 and Corollary 1. $\square$

We close this paper by some examples of module contractible Banach algebras. Let $\mathfrak{G}$ be a commutative unital Banach algebra with unit element $\mathfrak{e}$. Consider $\mathbb{A} = M_n(\mathfrak{G})$, the Banach algebra of $n \times n$ matrices with entries from $\mathfrak{G}$. Then $\mathbb{A}$ is a unital commutative $\mathfrak{G}$-bimodule with the following natural actions

$$\alpha \cdot [\beta_{ij}] = [\alpha\beta_{ij}], \ \ [\beta_{ij}] \cdot \alpha = [\beta_{ij}\alpha] \ \ (\alpha \in \mathfrak{G}, [\beta_{ij}] \in \mathbb{A}).$$

Consider the set of matrix units $\{E_{ij}; i, j = 1, ..., n\}$, where E_{ij} is the matrix having $\mathfrak{e}$ at the i^{th} row and j^{th} column, and zero elsewhere. The identity matrix E, which is the unit element of $\mathbb{A}$, is the matrix whose diagonal entries are $\mathfrak{e}$ and has zero entries elsewhere. Let I, J be the corresponding closed ideals, as in section 2. If we put $M = \sum_{i,j=1}^{n} \frac{1}{n} E_{ij} \otimes E_{ji} + I$, then we have $\tilde{\omega}(M) = \sum_{i=1}^{n} E_{ii} + J = E + J$. Also

$$E_{lk} \cdot M = \sum_{i,j=1}^{n} E_{lk} \frac{1}{n} E_{ij} \otimes E_{ji} + I = \sum_{i=1}^{n} \frac{1}{n} E_{li} \otimes E_{ik} + I = \sum_{i,j=1}^{n} \frac{1}{n} E_{ij} \otimes E_{ji} E_{lk} + I$$
$$= M \cdot E_{lk},$$

for each $1 \leq l, k \leq n$. Hence for each $A \in \mathbb{A}$, we have $A \cdot M = M \cdot A$. It follow that M is a module diagonal for $\mathbb{A}$, therefore $\mathbb{A}$ is module contractible by Theorem 2.7. Observe that in this case, $J = \{0\}$, but yet $\mathbb{A}$ is not necessarily contractible. This shows that the assumption that the action is trivial from one side could not be dropped from Theorem 3.2. As a concrete example, consider $\mathfrak{G} = \ell^1(S)$, where $S = [0, 1]$ is a unital commutative semigroup with multiplication $st = min\{s + t, 1\}$, for $s, t \in S$, then $\mathfrak{G} = \ell^1(S)$ and $\mathbb{A} = M_n(\mathfrak{G})$ are not even weakly amenable [9], but still $\mathbb{A}$ is $\mathfrak{G}$-module contractible with $J = \{0\}$.

The last example shows that there is an inverse semigroup S for which $\ell^1(S)$ is module contractible but not contractible. Let $(\mathbb{N}, \vee)$ be the commutative semigroup of positive integers with maximum operation $m \vee n = max(m,n)$, then each element of $\mathbb{N}$ is an idempotent, that is $E_{\mathbb{N}} = \mathbb{N}$. Hence $\mathbb{N}/\approx$ is the trivial group with one element. Therefore $\ell^1(\mathbb{N})$ is module contractible, as an $\ell^1(\mathbb{N})$-module. If $\ell^1(\mathbb{N})$ has a diagonal $M = \sum_{n=1}^{\infty} f_n \otimes g_n$, it should be $M = \delta_1 \otimes \delta_1$. In this case, we have $\delta_p \cdot M = M \cdot \delta_p$ $(p \in \mathbb{N})$, but this equality holds if and only if, $\delta_p \otimes \delta_1 = \delta_1 \otimes \delta_p$, for each $p \in \mathbb{N}$, which is absurd. Therefore $\ell^1(\mathbb{N})$ is not contractible by [15, Exercise 4.1.3]. Note that however, in this case, $\ell^1(\mathbb{N})$ has an identity.

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