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On The Perimeter of an Ellipse

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Abstract

Let E be the ellipse with major and minor radii a and b respectively, and P be its perimeter, then

$$P = \lim_{n\to\infty} 4\tan\frac{\pi}{n}(a+b+2\sum_{k=2}^{2^{m-2}}\sqrt{a^2\cos^2\frac{(2k-2)\pi}{n}+b^2\sin^2\frac{(2k-2)\pi}{n}}),$$

where $n = 2^m$. So without considering the limit, it gives a reasonable approximation for P, it means that we can choose n large enough such that the amount of error be less than any given small number. On the other hand, the formula satisfies both limit status $b \to a$ and $b \to 0$ which give respectively $P = 2\pi a$ and $P = 4a$.

Keywords: Ellipse, Perimeter, Surrounding polygon.

1 introduction

Let E be the ellipse with cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If $P(E)$ refers to the perimeter of E, then we have

$$P(E) = 4\int_0^a \sqrt{1+(\frac{dy}{dx})^2}\ dx$$

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Unfortunately this integral dose not have an analytic solution, and so there are many approximations $f(a,b)$ for $P(E)$, for example *Ramanujan* II, and *Cantrell* which are as follows (one can find many such approximations in [1])

$$R(a,b)=\pi(a+b)(1+\frac{3h}{10+\sqrt{4-3h}}\ ) \ , \quad C(a,b)=4(a+b)-2(4-\pi)\frac{ab}{H_p}$$

where

$$h=\frac{(a-b)^2}{(a+b)^2} \ , \quad H_p=(\frac{a^p+b^p}{2})^{\frac{1}{p}}$$

in this article we present an approximation based on some surrounding polygons which are the images of surrounding regular polygons of a circle C under a transformation S which maps C onto E. If P_n be the perimeter of a surrounding $n-gon$ of E, we will see that $P_n \to P(E)$ as $n \to \infty$, hence $|P_n - P(E)| \to 0$. Meanwhile if we put $\epsilon = |f(a,b)-P(E)|$, there exist a natural number n_ϵ such that $|P_{n_\epsilon}-P(E)| < \epsilon$.

Maybe the oldest approximation be $f(a,b)=\pi(a+b)$ which is the average of $2\pi a$ and $2\pi b$, on the other hand the renowned lower bound for $P(E)$ is $2\pi\sqrt{ab}$. Clearly we can use *surrounded* polygons instead of surrounding polygons, and then we will obtain an approximation Q_n such that $Q_n \to P(E)$ as $n \to \infty$ and we have $Q_n < P(E) < P_n$, hence for each n, Q_n and P_n are ,respectively, lower and upper bounds for $P(E)$.

2 surrounding polygon approximation

Consider the circle C with radius b, and its surrounding polygons. We will compute the coordinates of the corners using some trigonometry, if $A(x_1,y_1)$ and $B(x_2,y_2)$ be two adjacent corners, then $S(A)(\frac{a}{b}x_1,y_1)$ and $S(B)(\frac{a}{b}x_2,y_2)$ are two adjacent corners of a surrounding polygon of the ellipse $E=S(C)$, where $S: R^2 \to R^2$ is a map defined by $S(x,y)=(\frac{a}{b}x,y)$.

Lemma 2.1. *The image of the circle C under the map S described above, is the ellipse E, moreover if AB is a tangent segment to C at the point T, then $S(A)S(B)$ is a tangent segment to E at the point $S(T)$.*

Proof. The equation of C is $x^2+y^2=b^2$, we have to show that the point $S(A)(\frac{a}{b}x,y)$ satisfies the equation of E, to this end we have $\frac{(\frac{a}{b}x)^2}{a^2}+\frac{y^2}{b^2}=\frac{x^2}{b^2}+\frac{y^2}{b^2}=\frac{1}{b^2}(x^2+y^2)=\frac{b^2}{b^2}=1$. If $A(x_1,y_1)$, $B(x_2,y_2)$ and $T(\alpha,\beta)$ then we know that $y'(\alpha)=\frac{y_2-y_1}{x_2-x_1}$, we have to show that $Y'(\frac{a}{b}\alpha)=\frac{y_2-y_1}{\frac{a}{b}(x_2-x_1)}$, but this is true since $Y(\frac{a}{b}\alpha)=y(\alpha)$, therefore $\frac{a}{b}Y'(\frac{a}{b}\alpha)=y'(\alpha)$. Here y and Y are extracted from the equations of C and E, respectively. □

Now consider the surrounding regular n-gon of C. For simplicity we consider the first quarter of C, we also assume that C is centered at origin. We start with the surrounding square and we make an octagon by drawing four tangent lines passing through the points which are the intersections of C and the segments connecting the center of C to the corners of the square. Each tangent line intersects two adjacent edges of the square and makes two adjacent corners of the surrounding regular octagon of C (Fig.1). We will continue this procedure to make any surrounding regular n-gon of C such that $n = 2^m$. Put $N(0, b)$, $M(b, 0)$ and Let $A_1, A_2, ..., A_p$ where $p = 2^{m-2}$, be the corners of the surrounding regular n-gon of C relevant to its first quarter. The following lemma will help us not only to find P_n, but also to compute the coordinates of A_k , $1 \leq k \leq p$.

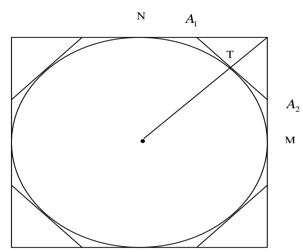

Fig. 1

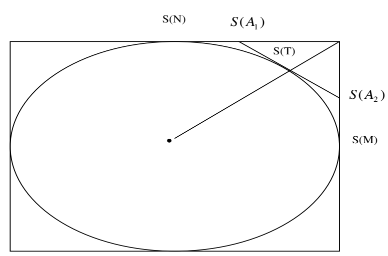

Fig. 2

Lemma 2.2. *If* $A_k(x_k,\ y_k)$ *and* $A_{k+1}(x_{k+1},\ y_{k+1})$ *be two adjacent corners of the* *surrounding regular n-gon of C, then for $1 \le k \le p-1$ we have*

$$x_{k+1} = x_k + 2b\tan\frac{\pi}{n}\cos\frac{2k\pi}{n}\ , \qquad y_{k+1} = y_k - 2b\tan\frac{\pi}{n}\sin\frac{2k\pi}{n}$$

Proof. For $1 \le k \le p-1$, A_{k+1} is obtained by rotating the point A_1 by $\frac{2k\pi}{n}$ clockwise, hence

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} \cos\frac{2k\pi}{n} & \sin\frac{2k\pi}{n} \\ -\sin\frac{2k\pi}{n} & \cos\frac{2k\pi}{n} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

so

$$x_{k+1} = x_1\cos\frac{2k\pi}{n} + y_1\sin\frac{2k\pi}{n}\ ,\ \ y_{k+1} = -x_1\sin\frac{2k\pi}{n} + y_1\cos\frac{2k\pi}{n}$$

The similar equations can be written for x_k and y_k, then we compute $x_{k+1} - x_k$, and $y_{k+1} - y_k$, finally by using some trigonometry and keeping in mind that $x_1 = b\tan\frac{\pi}{n}$, $y_1 = b$ are the coordinates of A_1 , the proof will be completed. □

Now $S(A_k),\ 1 \le k \le p$, are the corners of a surrounding n-gon of E and we have

$$S(N)(0,b)\ ,\ S(M)(\frac{a}{b}b,0) = (a,0),\ \textit{and}\ \ S(A_k)(\frac{a}{b}x_k,\ y_k)\ \ 1 \le k \le p$$

Note that we express the length of the segment AB again by AB, meanwhile it is easy to verify that if AB is a horizontal segment, then $S(A)S(B) = (\frac{a}{b})AB$ and for a vertical segment $AB,\ S(A)S(B) = AB$, therefore we have

$$S(N)S(A_1) = \frac{a}{b}b\tan\frac{\pi}{n} = a\tan\frac{\pi}{n}\ , \quad S(A_p)S(M) = b\tan\frac{\pi}{n}$$

$$S(A_k)S(A_{k+1}) = \sqrt{(\frac{a}{b}(x_{k+1}-x_k))^2 + (y_{k+1}-y_k)^2} =$$

$$\sqrt{(2a\tan\frac{\pi}{n}\cos\frac{2k\pi}{n})^2 + (2b\tan\frac{\pi}{n}\sin\frac{2k\pi}{n})^2} = 2\tan\frac{\pi}{n}\sqrt{a^2\cos^2\frac{2k\pi}{n} + b^2\sin^2\frac{2k\pi}{n}}$$

so, the perimeter of the surrounding n-gon of E is

$$P_n = 4(S(N)S(A_1) + S(A_{2^{m-2}})S(M) + \sum_{k=1}^{2^{m-2}-1} S(A_k)S(A_{k+1})) =$$

$$4(S(N)S(A_1) + S(A_{2^{m-2}})S(M) + \sum_{k=2}^{2^{m-2}} S(A_{k-1})S(A_k)) =$$

$$4(a\tan\frac{\pi}{n} + b\tan\frac{\pi}{n} + 2\tan\frac{\pi}{n}\sum_{k=2}^{2^{m-2}}\sqrt{a^2\cos^2\frac{(2k-2)\pi}{n} + b^2\sin^2\frac{(2k-2)\pi}{n}}\ ) =$$

$$4\tan\frac{\pi}{n}(a + b + 2\sum_{k=2}^{2^{m-2}}\sqrt{a^2\cos^2\frac{(2k-2)\pi}{n} + b^2\sin^2\frac{(2k-2)\pi}{n}}\ )$$

Theorem 2.3. *If $P = \lim_{n\to\infty} P_n$, then*
i. $P = P(E)$, *the perimeter of* E
ii. $\lim_{b\to a} P = 2\pi a$
iii. $\lim_{b\to 0} P = 4a$

Proof. i. First of all, note that for a fixed pair (a,b), P_n is decreasing (proof by triangle inequality; see Fig.2 to compare P_4 and P_8) and bounded bellow ($P_n \geq P(E)$), therefore it is convergent. Now we have

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} 4(a+b)\tan\frac{\pi}{n} +$$

$$\lim_{n\to\infty} 8\tan\frac{\pi}{n}\sum_{k=2}^{\frac{n}{4}}\sqrt{a^2\cos^2\frac{(2k-2)\pi}{n}+b^2\sin^2\frac{(2k-2)\pi}{n}} =$$

$$4\lim_{n\to\infty}\frac{2\pi}{n}(\sum_{k=1}^{\frac{n}{4}}\sqrt{a^2\cos^2\frac{(2k-2)\pi}{n}+b^2\sin^2\frac{(2k-2)\pi}{n}}-a) =$$

$$4\lim_{n\to\infty}\frac{\frac{\pi}{2}}{\frac{n}{4}}\sum_{k=1}^{\frac{n}{4}}\sqrt{a^2\cos^2\frac{(k-1)\frac{\pi}{2}}{\frac{n}{4}}+b^2\sin^2\frac{(k-1)\frac{\pi}{2}}{\frac{n}{4}}} =$$

$$4\int_0^{\frac{\pi}{2}}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\ d\theta =$$

$$4\int_0^a\sqrt{1+\frac{b^2x^2}{a^2(a^2-x^2)}}\ dx = 4\int_0^a\sqrt{1+(\frac{dy}{dx})^2}\ dx = P(E)$$

Here we have used the change of variable $x = a\sin\theta$

ii. Consider P_n and P as functions of b, defined on $[0,a]$, clearly P_n is continuous since it is a finite sum of continuous functions. To prove the continuity of P, we refer the reader to [2, theorem 2.27] with $g(x) = \frac{a}{\sqrt{a^2-x^2}}$. Now we have

1. $[0,a]$ is compact,
2. $P_n \to P$, pointwise on $[0,a]$,
3. $P_n(b) \geq P_{n+1}(b)$ for all $b \in [0,a]$, $n = 1,2,3,\ldots$.

(Note that by an abuse of language, P_{n+1} may be considered as P_{2n}, in other words we can formally define $P_{n+i} = P_n$ for $1 \leq i \leq n-1$). Hence, we conclude that $P_n \to P$ uniformly on $[0,a]$, [3, theorem 7.13], so we are allowed to write

$$\lim_{b\to a} P = \lim_{b\to a}\lim_{n\to\infty} P_n(b) = \lim_{n\to\infty}\lim_{b\to a} P_n(b) = \lim_{n\to\infty} 4\tan\frac{\pi}{n}(2^{m-1}a) =$$

$$lim_{n\to\infty} 4\tan\frac{\pi}{n}(\frac{n}{2}a) = 2\pi a$$

$$iii.\ \lim_{b\to 0} P = \lim_{b\to 0}\lim_{n\to\infty} P_n(b) = \lim_{n\to\infty} 4\tan\frac{\pi}{n}(a + 2\sum_{k=2}^{\frac{n}{4}} a\cos\frac{2(k-1)\pi}{n}) =$$

$$\lim_{n\to\infty} 4\tan\frac{\pi}{n}(-a + 2\sum_{k=1}^{\frac{n}{4}} a\cos\frac{2(k-1)\pi}{n}) = \lim_{n\to\infty}(-4a\tan\frac{\pi}{n}) +$$

$$\lim_{n\to\infty} 8\tan\frac{\pi}{n}\sum_{k=1}^{\frac{n}{4}} a\cos\frac{2(k-1)\pi}{n} = \lim_{n\to\infty} 8\frac{\pi}{n}\sum_{k=1}^{\frac{n}{4}} a\cos\frac{2(k-1)\pi}{n} =$$

$$\lim_{n\to\infty} 4a(\frac{\frac{\pi}{2}}{\frac{n}{4}}\sum_{k=1}^{\frac{n}{4}}\cos\frac{(k-1)\frac{\pi}{2}}{\frac{n}{4}} = 4a\int_o^{\frac{\pi}{2}}\cos x dx = 4a$$

Note that we have used the fact that $\lim_{n\to\infty}\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}} = 1$.

□

We can call P_n an *infinite-type* approximation because it has a limit process, so we may call a formula without limit process, a *finite-type* approximation. Since $P_n \to P(E)$, for any given $\epsilon > 0$ there exist a natural number N such that if $n > N$, then $|P_n - P| < \epsilon$, in other words choosing n large enough one may obtain a good approximation to evaluate the perimeter of E. Let $f(a,b)$ be any finite-type approximation for $P(E)$, there is a number n such that $|P_n - P| < |f(a,b) - P| = \epsilon$.

References

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