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Reproducing Kernel Hilbert Space(RKHS) method for solving singular perturbed initial value problem

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Abstract

In this paper, a numerical scheme for solving singular initial/boundary value problems presented. By applying the reproducing kernel Hilbert space method (RKHSM) for solving these problems, this method obtained to approximated solution. Numerical examples are given to demonstrate the accuracy of the present method. The result obtained by the method and the exact solution are found to be in good agreement with each other and it is noted that our method is of high significance. We compare our results with other paper. The comparison of the results with exact ones is made to confirm the validity and efficiency.

Key words: Reproducing Kernel Hilbert Space(RKHS), Gram-Schmidt orthogonalization process, Singular initial value problems.

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1 Introduction

In this paper, the RKHSM will be used to investigate the singular initial value problems. In recent years, a lot of attention has been devoted to the study of RKHSM to investigate various scientific models. The RKHSM which accurately computes the series solution is of great interest to applied sciences. The method provides the solution in a rapidly convergent series with components that can be elegantly computed. The efficiency of the method was used by many authors to investigate several scientific applications. Reproducing kernel theory has important applications in numerical analysis, differential equations, probability and statistics, learning theory and so on. The reproducing kernels have been successfully applied to several linear and nonlinear problems.([1],[2],[3],[4])

Singular perturbed impulsive boundary value problems arise very frequently in the fields of fluid mechanics, fluid dynamics, elasticity, reaction-diffusion processes, chemical kinetics and other branches of applied mathematics, which have become an important area of investigation in recent years. Also singular boundary value problems for ordinary differential equations arise in the theory of thermal explosions and in the study of Electro-hydrodynamics. Such problems also occur in the study of generalized axially symmetric potentials after separation of variables has been employed.

There is considerable interest on numerical methods on singular boundary value problems. Some methods are discussed for a class of singularly perturbed singular initial value problems as follows

$$\begin{cases} \varepsilon u'(x) + \dfrac{s}{b(x)} u(x) = f(x), \qquad 0 < x \leq 1, \\ u(\alpha) = 0, \end{cases} \tag{1.1}$$

where b and f are sufficiently regular given functions in $[a, b]$ and $\alpha \in \{0, 1\}$ and s are constants. The equivalent form of Eq. (1.1) is

$$\begin{cases} Lu = f(x), & 0 \leq x \leq 1, \\ Bu = 0, \end{cases} \tag{1.2}$$

where L is a differential operator of the form

$$Lu(x) = \varepsilon u' + \frac{s}{b(x)} u, \tag{1.3}$$

and B is initial condition. There is no loss of generality in considering only homogeneous conditions in Eq.(1.1) because it is always possible to reduce nonhomogeneous problems to the treated cases, by means of suitable transformations. Also, other form of the above equation is

$$\begin{cases} \varepsilon b(x) u'(x) + s u(x) = b(x) f(x), & 0 < x \leq 1. \\ u(\alpha) = 0, \end{cases} \tag{1.4}$$

Here, we will use RKHS method for a class of singular initial/boundary value problem on the assumption that solution is unique. This problem has been well studied and the results can be found in much of the literature [5].

However, this problem solved in [5] but we have some reasons as motivation for publishing this paper that listed below.

Firstly, some formulaes in [5] are false e.g. Gram-Schmidt orthogonalization formula and function f in examples.

Secondly, the number of nodes in [5] is very low ($N = 28$). We solved the problem with $N = 50$, $N = 100$ nodes.

And finally, [5] has no figures to show the behaviour of absolute errors. We presented the logarithmic graph of absolute error.

The paper is organized as follows. Section 2 is devoted to several definitions and theorems for reproducing kernel spaces and a linear operator is introduced. Solution representation in $W^m[a, b]$ has been presented. It provides the main results, the exact and approximate solution of Eq.(1.1) are developed for the kind of problems in the reproducing kernel space. We have proved that the approximate solution converges to the exact solution uniformly. Some numerical experiments are illustrated in Section 3. There are some conclusions in the last section.

2 Preliminaries

For implementation of this method we need some definitions and theorems. For more details and proofs of theorems see [4].

Definition 2.1 *The inner product space* $W_2^m[a,b]$ *is defined as* $W_2^m[a,b] = \{u(x)|u, u', \ldots, u^{(m-1)}$ *are absolutely continuous real valued functions* $u, u', \ldots, u^{(m)} \in L^2[a,b], Bu = 0\}$
The inner product in $W_2^m[a,b]$ *is given by*

$$(u(.), v(.))_{W_2^m} = \sum_{i=0}^{m-1} u^i(a)v^i(a) + \int_a^b u^{(m)}(x)v^{(m)}(x)dx, \qquad (2.1)$$

and the norm $\|u\|_{W_2^m}$ *is denoted by*

$$\|u\|_{W_2^m} = \sqrt{(u,u)_{W_2^m}},$$

where $u, v \in W_2^m[a,b]$.

Theorem 2.1 *The space* $W_2^m[a,b]$ *is a reproducing kernel space. That is, for any* $u(.) \in W_2^m[a,b]$ *and each fixed* $x \in [a,b]$, *there exists* $K(x,.) \in W_2^m[a,b]$, *such that* $(u(.), K(x,.))_{W_2^m} = u(x)$.
The reproducing kernel $K(x,.)$ *can be denoted by*

$$K(x,y) = \begin{cases} \sum_{i=0}^{2m} c_i(y)x^{i-1}, & x \le y, \\ \sum_{i=0}^{2m} d_i(y)x^{i-1}, & x > y. \end{cases} \qquad (2.2)$$

Proof. *The proof can be found in [4].*

Definition 2.2 *($W_2^1[a,b]$). The inner product space $W_2^1[a,b]$ is defined as*

$W_2^1[a,b] = \{u(x)|u$ *is absolutely continuous real function,* $u, u' \in L^2[a,b]\}$,
The inner product in $W_2^1[a,b]$ is given by

$$(u(.), v(.))_{W_2^1} = u(a)v(a) + \int_a^b u^{(1)}(x)v^{(1)}(x)dx, \tag{2.3}$$

and the norm $\|u\|_{W_2^1}$ is denoted by $\|u\|_{W_2^1} = \sqrt{(u,u)_{W_2^1}}$, where $u, v \in W_2^1[a,b]$.
The space $W_2^1[a,b]$ is a reproducing kernel space and its reproducing kernel function $T_x(.)$ is given by

$$T_x(y) = \begin{cases} 1 + x, & x \leq y, \\ 1 + y, & x > y. \end{cases} \tag{2.4}$$

Also, we can simplify to

$$T_x(y) = 1 + \frac{x + y - |x - y|}{2}. \tag{2.5}$$

Theorem 2.2 *The space $W_2^2[0,1]$ is a reproducing kernel Hilbert space. That is $\forall u(y) \in W_2^2[0,1]$ and each fixed $x, y \in [0,1]$, there exists $R_x(y) \in W_2^2[0,1]$ such that $< u(y), R_x(y) >= u(x)$ and $R_x(y)$ is called the reproducing kernel function of space $W_2^2[0,1]$. The reproducing kernel function $R_x(.)$ is given by*

$$R_x(y) = \begin{cases} c_1 + c_2x + c_3x^2 + c_4x^3, & x \leq y, \\ d_1 + d_2x + d_3x^2 + d_4x^3, & x > y. \end{cases} \tag{2.6}$$

Proof. *We computed $R_x(y)$ for two cases of $\alpha = 0, 1$.*
From definition of inner product in $W_2^2[a,b]$ is given by

$$(u(x), R_y(x))_{W_2^2} = \sum_{i=0}^{1} u^{(i)}(a)R_y^{(i)}(a) + \int_a^b u^{(2)}(x)R_y^{(2)}(x)dx. \tag{2.7}$$

Through several integrations by parts for (2.7) we have

$$
\begin{aligned}
(u(x), R_y(x))_{W_2^2} = \sum_{i=0}^{1} u^{(i)}(a)[R_y^{(i)}(a) - (-1)^{1-i} R_y^{3-i}(a)] \\
+ \sum_{i=0}^{1} (-1)^{1-i} u^i(b) R_y^{3-i}(b) + \int_a^b u(x) R_y^{(4)}(x)dx,
\end{aligned} \tag{2.8}
$$

and then, $R_y^{(4)}(x) = \delta(x-y)$.

Case(I): *For* $\alpha = 0$ *from reproducing property and relations between derivatives we obtain following system*

$$
\begin{aligned}
&1) R_x(a) = 0, \\
&2) R_x^{(1)}(a) - R_x^{(2)}(a) = 0, \\
&3) R_x^{(i)}(b) = 0; \quad i = 2, 3, \\
&4) R_x^{(i)}(y^-) = R_x^{(i)}(y^+); \quad i = 0, 1, 2, \\
&5) R_x^{(3)}(y^+) - R_x^{(3)}(y^-) = 1.
\end{aligned} \tag{2.9}
$$

By solving above system, coefficients are as

$$
c_1(y) = 0, \qquad c_2(y) = y, \qquad c_3(y) = \tfrac{1}{2}y, \qquad c_4(y) = -\tfrac{1}{6},
$$

$$
d_1(y) = -\tfrac{1}{6}y^3, \qquad d_2(y) = \tfrac{1}{2}y^2 + y, \qquad d_3(y) = 0, \qquad d_4(y) = 0,
$$

and

$$
R_x(y) = \begin{cases} xy + \dfrac{1}{2}x^2 y - \dfrac{1}{6}x^3, & x \le y, \\ xy + \dfrac{1}{2}y^2 x - \dfrac{1}{6}y^3, & x > y. \end{cases} \tag{2.10}
$$

Case(II): *For* $\alpha = 1$ *from reproducing property and relations between* *derivatives we obtain following system*

$$
\begin{aligned}
&1) R_x(b) = 0, \\
&2) R_x^{(i)}(a) + (-1)^i R_x^{(3-i)}(a) = 0; \quad i = 0, 1, \\
&3) R_x^{(2)}(b) = 0, \\
&4) R_x^{(i)}(y^-) = R_x^{(i)}(y^+); \quad i = 0, 1, 2, \\
&5) R_x^{(3)}(y^+) - R_x^{(3)}(y^-) = 1.
\end{aligned} \tag{2.11}
$$

By solving above system, coefficients are as

$$c_1(y) = \tfrac{1}{14}(8 - 6y - 3y^2 + y^3), \qquad c_2(y) = \tfrac{1}{14}(-6 + 8y - 3y^2 + y^3),$$

$$c_3(y) = \tfrac{1}{28}(-6 + 8y - 3y^2 + y^3), \qquad c_4(y) = \tfrac{1}{84}(-8 + 6y + 3y^2 - y^3),$$

$$d_1(y) = \tfrac{1}{42}(24 - 18y - 9y^2 - 4y^3), \qquad d_2(y) = \tfrac{1}{14}(-6 + 8y + 4y^2 + y^3),$$

$$d_3(y) = \tfrac{1}{28}(-6 - 6y - 3y^2 + y^3), \qquad d_4(y) = \tfrac{1}{84}(6 + 6y + 3y^2 - y^3).$$

Remark 2.1 *In Problem* (1.1) *where* $u \in W_2^m[a,b]$ *and* $f \in W_2^1[a,b]$, *it is clear that* $L : W_2^m[a,b] \to W_2^1[a,b]$ *is a bounded linear operator. For any fixed* $x_i \in [a,b]$, *Let* $\varphi_i(.) = T_{x_i}(.)$, *where* $T_{x_i}(.)$ *is reproducing kernel of* $W_2^1[a,b]$. *Further assume that* $\psi_i(.) = (L^*\varphi_i)(.)$, *where* L^* *is the adjoint operator of* L.

Theorem 2.3 *Let* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[a,b]$, *then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is the complete system of* $W_2^m[a,b]$ *and* $\psi_i(x) = L_y K(x,y)|_{y=x_i}$, *where the subscript* y *of operator* L_y *indicates that the operator* L *applies to functions of* y.

Proof. See [4].

Remark 2.2 *The orthonormal system* $\{\overline{\psi_i}(x)\}_{i=1}^{\infty}$ *of* $W_2^m[a,b]$ *can be de-*

rived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$*, as*

$$\overline{\psi_i}(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \qquad (i = 1, 2, \ldots). \tag{2.12}$$

To orthonormalize the sequence $\{\overline{\psi_i}(x)\}_{i=1}^{\infty}$ *in the reproducing kernel space by Gram-Schmidt process, orthogonal coefficients* β_{ik} *given by*

$$\beta_{11} = \frac{1}{\|\psi_1\|}, \qquad \beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} b_{ik}^2}}, \qquad \beta_{ij} = \frac{-\sum_{k=j}^{i-1} b_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} b_{ik}^2}}, \tag{2.13}$$

where $b_{ik} = (\psi_i, \overline{\psi_k})$.

Theorem 2.4 *If* $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[a, b]$ *and the solution of problem* (1.1) *is unique, then the solution of the problem, can be represented in* $W_2^m[a, b]$ *as follows*

$$u(x) = \sum_{i=1}^{\infty}\sum_{j=1}^{i} \beta_{ik} f(x_j)\overline{\psi_i}(x). \tag{2.14}$$

Now, the approximate solution can be obtained by taking finitely many terms in the series representation of u *and*

$$u_N(x) = \sum_{i=1}^{N}\sum_{j=1}^{i} \beta_{ik} f(x_j)\overline{\psi_i}(x). \tag{2.15}$$

Proof. The proof can be found in [4].

3 Numerical examples

In this section, some numerical examples are given to demonstrate the accuracy of the given method. The examples are computed using Mathematica 10.3. Results obtained by the method are compared with the exact solution of each example and are found to be in good agreement.The numerical results are given with $x_i = \dfrac{i-1}{n-1}, \quad i = 1, 2, \ldots, N$ for two follwing examples.

Example.1. Consider the Initial Value Problem

$$\begin{cases} \varepsilon u'(x) + \dfrac{1}{\sqrt{x}} u(x) = f(x), & 0 < x \leq 1, \\ u(0) = 0, \end{cases} \tag{3.1}$$

where $\varepsilon = 10^{-6}, 10^{-10}$ and $f(x) = \dfrac{(x+x^2)}{\sqrt{x}} + \varepsilon(1+2x)$. The exact solution of the problem is $u(x) = x + x^2$.
The absolute errors of approximate solutions of Example.1 in different nodes for various values of N are reported in Table.1. It can be seen that there is a good agreement between theory and numerical results and also we can increase the accuracy by increasing N. Figure 1 is the graph of $Log_{10}|u - u_N|$.

Table.1. Absolute error for Example.1 with various N and $\varepsilon = 10^{-6}$.

x_i	$N = 25$	$N = 50$	$N = 100$	[5]($N = 28$)
0.2	7.11495×10^{-12}	1.01738×10^{-12}	5.95238×10^{-13}	6.77481×10^{-08}
0.4	1.19460×10^{-11}	2.31616×10^{-12}	1.24607×10^{-13}	2.54592×10^{-11}
0.6	2.45803×10^{-11}	3.35709×10^{-12}	1.93190×10^{-13}	5.13651×10^{-11}
0.8	2.82394×10^{-11}	4.58766×10^{-12}	2.62641×10^{-13}	1.36892×10^{-07}
1.0	2.30920×10^{-08}	1.15480×10^{-08}	5.73539×10^{-09}	2.53664×10^{-08}

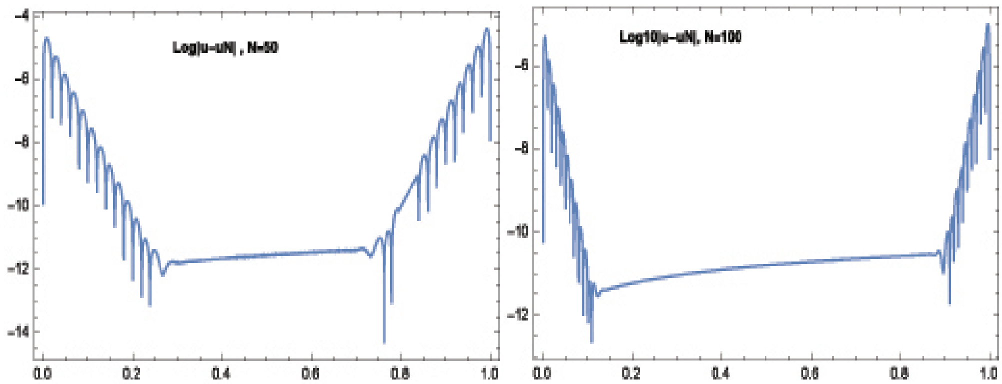

Figure 1. Graph of $Log_{10}|u - u_N|$ for Example1 with $m = 2$, $\varepsilon = 10^{-6}$, N=50(left) and $N = 100$,(Right),

Example.2. Consider the Initial Value Problem

$$\begin{cases} \varepsilon u'(x) + \dfrac{1}{x} u(x) = f(x), & 0 < x \leq 1, \\ u(1) = 0, \end{cases} \tag{3.2}$$

where $\varepsilon = 10^{-6}, 10^{-10}$ and $f(x) = \dfrac{(e^x - e)}{x} + \varepsilon(e^x)$. The exact solution of the problem is $u(x) = e^x - e$. The absolute errors of approximate solutions of Example.2 in different nodes for various values of N are reported in Table.2. It can be seen that there is a good agreement between theory and numerical results and also we can increase the accuracy by increasing N. Figure.2 and Figure.3 are the graph of $Log_{10}|u - u_N|$.

Table.2. Absolute error for Example.2 with various N and $\varepsilon = 10^{-6}$.

x_i	$N = 25$	$N = 50$	$N = 100$	[5]$(N = 28)$
0.0	2.68941×10^{-5}	3.46582×10^{-06}	4.39932×10^{-07}	1.92676×10^{-5}
0.2	3.35509×10^{-8}	7.19425×10^{-11}	1.45883×10^{-13}	1.09778×10^{-8}
0.4	3.19744×10^{-6}	2.88658×10^{-11}	1.90736×10^{-13}	1.88865×10^{-6}
0.6	4.04121×10^{-6}	1.46549×10^{-12}	1.67977×10^{-13}	1.88907×10^{-6}
0.8	5.35683×10^{-9}	7.99361×10^{-11}	5.99520×10^{-14}	8.74630×10^{-9}

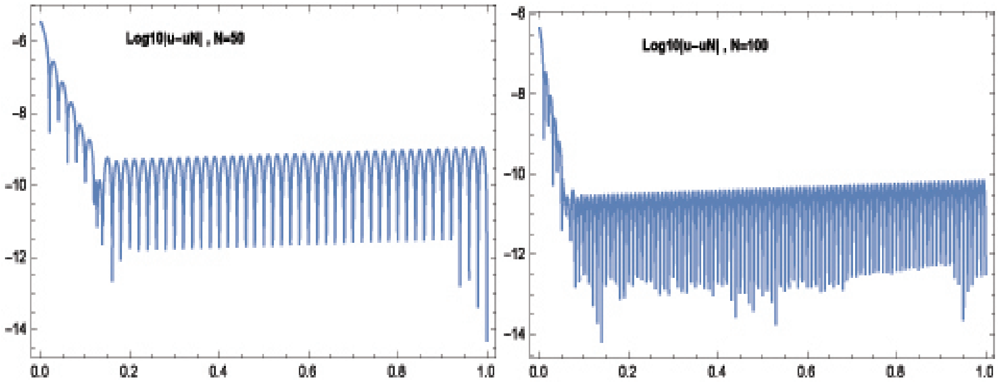

Figure 2. Graph of $Log_{10}|u - u_N|$ for Example.2 with $m = 2$, $\varepsilon = 10^{-6}$, N=50(left) and $N = 100$(Right),

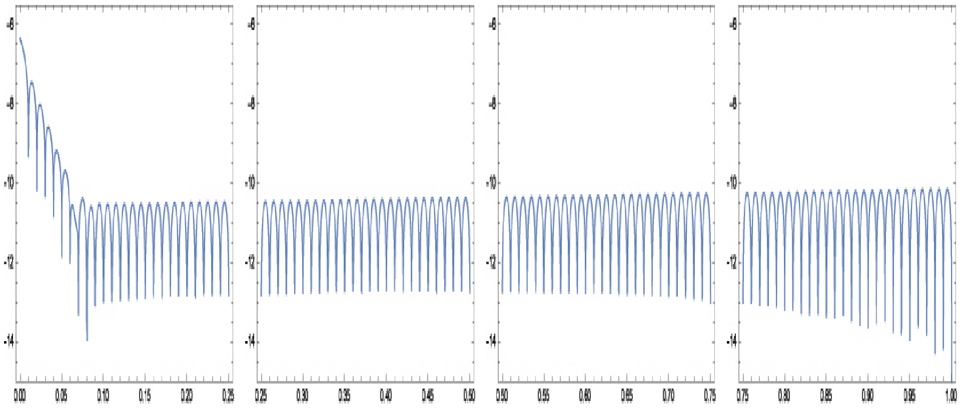

Figure 3. Magnification of graph $Log_{10}|u - u_N|$ for Example.2 with $m = 2$, $\varepsilon = 10^{-6}$, N=100.

4 Conclusions

In this paper we have presented a numerical scheme based on reproducing kernel Hilbert space to solve singular initial value problems. The method has been tested on some illustrative numerical examples. The computational results are found to be in good agreement with the exact solutions. In the current work, to demonstrate the accuracy and usefulness of this method, numerical examples have been presented. As demonstrated by the computational results, it is very easy to implement the proposed method for similar problems.

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