The behavior of homological dimensions

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Abstract

Let R be a commutative noetherian ring. We study the behavior of injective and flat dimension of R-modules under the functors $\mathbf{Hom}_R$(-,-) and -$\otimes_R$-.

Keywords: Injective cogenerator, Injective dimension, Flat dimension
2000 AMS Subject Classification: *18G05, 18G20*

1 Introduction

Throughout this note, rings are commutative noetherian with identity, modules are unitary and $\mathcal{A}$ is an abelian category. Homological dimensions are most important invariants of modules in commutative algebra. The necessary condition to the definition of dimensions (injective, flat, projective,...) is the existence of the homological resolutions. Homological algebra, as a connected system of notions and results, was first develop for categories of modules by Cartan and Eilenberg in [2] and was immediately generalized by Buchsbaum in [1] to exact categories. The generalization was an important since it was covered to large classes of Grothendieck categories other than the category of modules, for example the category of sheaves, the category of quasi-coherent sheaves, and the category of chain complexes of a given Grothendieck category, for more details see, [3], [4], [5] and [7]. This generalization is not sufficient to cover all the potential, or even all the actual, applications of homological algebra. This first became apparent to the author in connection with studies in the homotopy category of chain complexes of modules. However such categories as the category of Banach spaces and continuous linear maps, or the category of abelian varieties over a field of non-zero characteristic, ought clearly to have homological algebra. But they are not exact category. The notion

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of an abelian category is grounded on an additive category, i.e, a category in which maps can be added. In such category exactness can be defined, as well as the notions of kernel, image and cokernel.

2   Notations and Recollections

First we want to fix some notations and recall the most important definitions and facts.

Definition 2.1. An object P of $\mathcal{A}$ is called projective if it satisfies the following universal lifting property:
Let $g : B \longrightarrow C$ be a surjection and $h : P \longrightarrow C$ be a morphism, then there is at least one morphism $f : P \longrightarrow B$ such that $h = g \circ f$.

Let $\mathcal{A} = R$-Mod be the category of all R-module. We can see that a projective module is a projective object of $\mathcal{A}$. The notion of projective module first appeared by Cartan and Eilenberg in [2]. It is easy to see that free R-modules are projective(lift a basis) and direct summand of free R-modules are also projective modules. So a R-module is projective if and only if it is a direct summand of a free module. The category $\mathcal{A}$ has enough projective if for every object A of $\mathcal{A}$ there is a surjection $P \longrightarrow A$, where P is a projective object of $\mathcal{A}$. There is an another characterization of projective objects in the category $\mathcal{A}$.
An object M is a projective object if and only if the covariant functor $Hom_R(M,\text{-})$ is an exact functor, i.e, for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ of objects of $\mathcal{A}$ the following sequence of abelian groups is exact.

$$0 \longrightarrow Hom_{\mathcal{A}}(M, A) \longrightarrow Hom_{\mathcal{A}}(M, B) \longrightarrow Hom_{\mathcal{A}}(M, C) \longrightarrow 0$$

Definition 2.2. Let M be an object of $\mathcal{A}$. A projective resolution of M is a chain complex $\mathbf{P_\bullet}$ with $P_i = 0$ for each $i < 0$, together a morphism $P_0 \longrightarrow M$ such that the augmented complex

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is an acyclic complex.

The dual of projective object in the opposite category $\mathcal{A}^{\mathrm{op}}$ is called injective object in the category $\mathcal{A}$. So the concept of injectivity in $\mathcal{A}$ can be defined as follows.

Definition 2.3. An object I of $\mathcal{A}$ is called injective if it satisfies the following universal lifting property:
Let $g : B \longrightarrow C$ be an injection and $h : B \longrightarrow I$ be a morphism, then there is at least one morphism $f : C \longrightarrow I$ such that $h = f \circ g$.

The category $\mathcal{A}$ has enough injective if for any object A of $\mathcal{A}$ there is an injection $A \longrightarrow I$ with I is injective. the notion of injective module was invented by R. Bear in 1940, long before projective modules were introduced. A R-module I is called injective

if and only if for every ideal J of R, every homomorphism $J \longrightarrow I$ can be extended to a homomorphism $R \longrightarrow I$. We know that the definition of injective R-modules is dual to the definition of projective R-modules. So one can deduce the following result.

Proposition 2.4. *[8, Lemma 2.3.4] Let I be an object of $\mathcal{A}$. Then the following are equivalent*
(i) I is an injective object of $\mathcal{A}$.
(ii) I is a projective object of $\mathcal{A}^{\mathrm{op}}$.
(iii) The contravariant functor $Hom_{\mathcal{A}}(\text{-},I)$ is exact, that is, it takes short exact sequence in $\mathcal{A}$ to short exact sequence of abelian groups.

Definition 2.5. Let M be an object of $\mathcal{A}$. An injective resolution of M is a cochain complex $\mathbf{I_\bullet}$ with $I_i = 0$ for each $i < 0$, together a morphism $M \longrightarrow I_0$ such that the augmented complex

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

is an acyclic complex.

It is a well-known that any Grothendieck category has enough injective. For example the category R-Mod is a Grothendieck category and so it has enough injective.

Proposition 2.6. *[8, Lemma 2.3.6] If $\mathcal{A}$ has enough injective, then every object of $\mathcal{A}$ has injective resolution.*

3   Left and Right Derived Functors

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a covariant right exact functor of abelian categories. If $\mathcal{A}$ has enough projective, we can construct the left derived functors $\mathrm{L}_i F (i > 0)$ of F as follows. If A is an object of $\mathcal{A}$, choose (once and for all) a projective resolution $\mathbf{P_\bullet} \longrightarrow A$ and define

$$\mathrm{L}_i F = \mathrm{H}_i(F(\mathbf{P_\bullet})).$$

Note that since $F(P_1) \longrightarrow F(P_0) \longrightarrow F(A) \longrightarrow 0$ is exact, we have $\mathrm{L}_0 F(A) \cong F(A)$. It is well-known that for each $i \geq 0$, $\mathrm{L}_i F$ is an additive functor of abelian categories. For given R-modules A and B, let $\mathbf{P_\bullet} \longrightarrow B$ be a projective resolution of B. Consider the covariant right exact functor $A \otimes_R \text{-}$, its right derived functors are called Tor groups. So

$$\mathrm{Tor}_i^R(A, B) = \mathrm{L}_i(A \otimes_R B) = \mathrm{H}_i(A \otimes_R \mathbf{P_\bullet}).$$

We know that the notion of flat R-module can be characterized by Tor groups via the following properties.

Proposition 3.1. *[8, Proposition 3.2.1] Let A be a R-module. Then the following are equivalent.*
(i) A is flat.
(ii) $\mathrm{Tor}_i^R(A, B)$ vanishes for all $i \neq 0$ and all B.
(iii) $\mathrm{Tor}_1^R(A, B)$ vanishes for all B.

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor of abelian categories. If $\mathcal{A}$ has enough injective then we can construct the right derive d functors $\mathrm{R}^i F(A)(i \geq 0)$ of F as follows. For this purpose choose an injective resolution $A \longrightarrow \mathbf{I}^\bullet$ and define

$$\mathrm{R}^i F(A) = \mathrm{H}^i(F(\mathbf{I}^\bullet)).$$

The sequence $0 \longrightarrow F(A) \longrightarrow F(I^0) \longrightarrow F(I^1)$ is exact so we have $\mathrm{R}^0 F(A) \cong F(A)$. We know that the category of R-modules has enough injective. Therefore the following definition is well-defined.

Definition 3.2. Let A and B be R-modules. The right derived functors of covariant left exact functor $Hom_R(A,$-$)$ are called Ext groups, i.e

$$\mathbf{Ext}^i_R(A, B) = \mathrm{R}^i Hom_R(A,)(B).$$

In particular, $\mathbf{Ext}^0_R(A, B) = Hom_R(A, B)$. It is well-known the injective and projective objects can be characterized by Ext groups.

Proposition 3.3. *[8, Proposition 2.5.1] The following conditions are equivalent;.*
(i) B is an injective R-module.
(ii) $Hom_R($-$,B)$ is an exact functor.
(iii) $\mathbf{Ext}^i_R(A, B)$ vanishes for all $i > 0$ and all R-module A.
(iv) $\mathbf{Ext}^1_R(A, B)$ vanishes for all R-module A.

The vanishing of Ext group with respect to the first variable characterizes projective R-modules.

Proposition 3.4. *[8, Proposition 2.5.2] The following conditions are equivalent;.*
(i) A is an projective R-module.
(ii) $Hom_R(A,$-$)$ is an exact functor.
(iii) $\mathbf{Ext}^i_R(A, B)$ vanishes for all $i > 0$ and all R-module B.
(iv) $\mathbf{Ext}^1_R(A, B)$ vanishes for all R-module B.

4   Homological dimensions in the category of R-modules

An R-module F is called flat if the functor $F \otimes_R$- is an exact functor. We know that all projective R-modules are flat. So every R-module is a homomorphic image of some flat R-modules. Thus for a given R-module M, there is a flat resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Now we recall the definition of important homological invariants of R-modules. This invariants can be defined by the length of homological resolutions.

Definition 4.1. Let A be a R-module.
(i) The projective dimension $\mathrm{pd}(A)$ is the minimum of integers n (if it exists) such that

there is a projective resolution of length n.

(ii) The injective dimension $\mathrm{id}(A)$ is the minimum of integers n (if it exists) such that there is an injective resolution of length n.

(iii) The flat dimension $\mathrm{fd}(A)$ is the minimum of integers n (if it exists) such that there is a flat resolution of length n.

If there exists no finite resolution, we put $\mathrm{pd}(A)$, $\mathrm{id}(A)$ or $\mathrm{fd}(A)$ equals to $+\infty$.

Lemma 4.2. *Let S be a flat R-algebra and M, N be R-modules. If M is finitely generated, then*

$$Hom_R(M, N) \otimes_R S \cong Hom_R(M \otimes_R S, N \otimes_R S).$$

Proof. The R-module M is finitely generated, so there is an exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of R-modules with F_1 and F_0 are finitely generated free R-modules. Therefore we have the following commutative diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Hom_R(M,N) & \longrightarrow & Hom_R(F_0,N) \otimes_R S & \longrightarrow & Hom_R(F_0,N) \otimes_R S \\
 & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} \\
0 & \longrightarrow & Hom_R(M \otimes_R S, N \otimes_R S) & \longrightarrow & Hom_R(M \otimes_R S, N \otimes_R S) & \longrightarrow & Hom_R(M \otimes_R S, N \otimes_R S)
\end{array}
$$

where the maps φ are given by $\varphi(f \otimes s)(x \otimes t) = s(f(x) \otimes t)$. But the last two vertical morphisms are isomorphisms since F_0 and F_1 are finitely generated free R-modules. Hence the first φ is also an isomorphism. $\qquad\square$

Theorem 4.3. *Let S be a flat R-algebra and M, N be R-modules. If M is finitely generated then*

$$\mathbf{Ext}^i_R(M, N) \otimes_R S \cong \mathbf{Ext}^i_R(M \otimes_R S, N \otimes_R S).$$

Proof. The R-module M has a projective resolution

$$\mathbf{P}_\bullet : \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with each P_i is finitely generated. In the other hand

$$\mathbf{P}_\bullet \otimes_R S : \cdots \longrightarrow P_2 \otimes_R S \longrightarrow P_1 \otimes_R S \longrightarrow P_0 \otimes_R S \longrightarrow M \otimes_R S \longrightarrow 0$$

is a projective resolution of the S-modules $M \otimes_R S$ and hence by Lemma 4.2 we have the following isomorphisms

$$
\begin{aligned}
\mathbf{Ext}^i_R(M, N) &\cong \mathrm{H}^i(Hom_R(\mathbf{P}_\bullet, N)) \otimes_R S \\
&\cong \mathrm{H}^i(Hom_R(\mathbf{P}_\bullet \otimes_R S, N \otimes_R S)) \\
&\cong \mathbf{Ext}^i_R(M \otimes_R S, N \otimes_R S).
\end{aligned}
$$

$$\square$$

Definition 4.4. An injective R-module E is said to be an injective cogenerator for R-modules if for each R-module M and non-zero element $x \in M$, there is $f \in Hom_R(M, E)$ such that $f(x) \neq 0$.

Equivalently E is an injective cogenerator if and only if $Hom_R(M, E) \neq 0$ for any non-zero R-module M. The group $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator for abelian groups. So that $M^+ = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ for any non-zero R-module M. Moreover, if M is any R-module then $Hom_{\mathbb{Z}}(M, R^+) \cong M^+$. Hence R^+ is an injective cogenerator for R-modules since R^+ is injective. Thus the category R-Mod has an injective cogenerator.

Theorem 4.5. *Let F be an R-module. Then the following are equivalent.*
(i) F is a flat R-module.
(ii) $Hom_R(F, E)$ is an injective R-module for any injective R-module E.
(iii) $Hom_R(F, E)$ is an injective R-module for any injective cogenerator E.

Proof. $(i) \implies (ii)$ Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of R-modules, since F is flat we have the exact sequence of R-modules

$$0 \longrightarrow F \otimes_R A \longrightarrow F \otimes_R B \longrightarrow F \otimes_R C \longrightarrow 0$$

. Let E be an injective R-module then we deduce the following exact sequence

$$0 \longrightarrow Hom_R(F \otimes_R A, E) \longrightarrow Hom_R(F \otimes_R B, E) \longrightarrow Hom_R(F \otimes_R C, E) \longrightarrow 0 \ .$$

So by adjointness of $Hom(\text{-},\text{-})$ and $\text{-} \otimes_R \text{-}$ we have the following short exact sequence

$$0 \longrightarrow Hom_R(A, Hom_R(F, E)) \longrightarrow Hom_R(B, Hom_R(F, E)) \longrightarrow Hom_R(C, Hom_R(F, E)) \longrightarrow 0 \ .$$

Then $Hom_R(F, E)$ is an injective R-module.
$(ii) \implies (iii)$ is trivial.
$(iii) \implies (i)$ Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules, since $Hom_R(F, E)$ is an injective R-module we have the exact sequence of R-modules

$$0 \longrightarrow Hom_R(A, Hom_R(F, E)) \longrightarrow Hom_R(B, Hom_R(F, E)) \longrightarrow Hom_R(C, Hom_R(F, E)) \longrightarrow 0 \ .$$

So by adjointness of $Hom(\text{-},\text{-})$ and $\text{-} \otimes_R \text{-}$ we have the following short exact sequence

$$0 \longrightarrow Hom_R(F \otimes_R A, E) \longrightarrow Hom_R(F \otimes_R B, E) \longrightarrow \mathbf{Hom}_R(F \otimes_R C, E) \longrightarrow 0 \ .$$

Since E is an injective cogenerator we deduce the following exact sequence

$$0 \longrightarrow F \otimes_R A \longrightarrow F \otimes_R B \longrightarrow F \otimes_R C \longrightarrow 0 \ .$$

Then F is flat. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Corollary 4.6. *Let F be a flat R-module. Then the following are equivalent.*
(i) F is flat.
(ii) The character module F^+ is an injective R-module.

Proof. Note that R^+ is an injective cogenerator.   □

Theorem 4.7. *Let R be a ring, A and B be R-module, C an injective R-module. Then there is a natural homomorphism*

$$A \otimes_R Hom_R(B,C) \longrightarrow \mathbf{Hom}_R(\mathbf{Hom}_R(A,B),C)$$

defined by $\tau(a \otimes f)(g) = f(g(a))$, where $a \in A$, $f \in Hom_R(B,C)$, and $g \in \mathbf{Hom}_R(A,B)$. If A is a finitely generated R-module then τ is an isomorphism.

Proof. Since A is finitely generated R-module, then there is an exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of R-modules with F_1 and F_0 are finitely generated free R-modules. Therefore we have the following commutative diagram with exact rows

$$
\begin{array}{ccccccc}
F_1 \otimes_R Hom_R(B,C) & \longrightarrow & F_1 \otimes_R Hom_R(B,C) & \longrightarrow & F_1 \otimes_R Hom_R(B,C) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbf{Hom}_R(\mathbf{Hom}_R(F_1,B),C) & \longrightarrow & \mathbf{Hom}_R(\mathbf{Hom}_R(F_0,B),C) & \longrightarrow & \mathbf{Hom}_R(\mathbf{Hom}_R(A,B),C) & \longrightarrow & 0.
\end{array}
$$

But the first two vertical morphisms are isomorphisms. So τ is an isomorphism.
                                                                                      □

Theorem 4.8. *Let R be a ring, A and B be R-module, C an injective R-module. Then the natural homomorphism*

$$\mathrm{Tor}_i^R(A, \mathbf{Hom}_R(B,C)) \cong \mathbf{Hom}_R(\mathbf{Ext}_R^i(A,B),C).$$

defined by $\tau(a \otimes f)(g) = f(g(a))$ is an isomorphism where $a \in A$, $f \in \mathbf{Hom}_R(B,C)$, and $g \in \mathbf{Hom}_R(A,B)$. If A is a finitely generated R-module then τ is an isomorphism.

Proof. The result followed by the fact that the R-module A has a projective resolution by finitely generated projective R-module.   □

Theorem 4.9. *Let A be a finitely generated R-module and B be a R-module, C be a flat R-module. Then the natural homomorphism*

$$\mathbf{Hom}_R(A,B) \otimes_R C \longrightarrow \mathbf{Hom}_R(A,B) \otimes_R C).$$

defined by $\tau(f \otimes c)(a) = f(a) \otimes c$ is an isomorphism.

Proof. The proof is similar to the proof of Theorem 4.8.   □

Theorem 4.10. *Let A be a finitely generated R-module and B be a R-module, C be a flat R-module. Then the natural homomorphism*

$$\mathbf{Ext}_R^i(A, B) \otimes_R C \cong \mathbf{Ext}_R^i(A, B) \otimes_R C).$$

Proof. The proof is similar to the proof of Theorem 4.9. □

Theorem 4.11. *Let E be a R-module. Then the following are equivalent.*
(i) E is an injective R-module.
(ii) $\mathbf{Hom}_R(E, E')$ is a flat R-module for all injective R-module E'.
(iii) $\mathbf{Hom}_R(E, E')$ is a flat R-module for all injective cogenerator E'.
(iv) $E \otimes_R F$ is an injective R-module for all flat R-module F.
(v) $E \otimes_R F$ is an injective R-module for all faithfully flat R-module F.

Proof. $(i) \implies (ii)$ Let I be an ideal of R. Then I is finitely generated since R is noetherian. But E is injective so we have the following exact sequence

$$0 \longrightarrow \mathbf{Hom}_R(\mathbf{Hom}_R(I, E), E') \longrightarrow \mathbf{Hom}_R(\mathbf{Hom}_R(R, E), E'),$$

id and hence
$$0 \longrightarrow \mathbf{Hom}_R(E, E') \otimes_R I \longrightarrow \mathbf{Hom}_R(E, E').$$

Therefore $\mathbf{Hom}_R(E, E')$ is a flat R-module.
$(ii) \implies (iii)$ and $(iii) \implies (iv)$ are trivial.
$(iii) \implies (i)$ Follows by reversing the proof $(i) \implies (ii)$ and using the fact that E is an injective cogenerator.
$(i) \implies (iv)$ Let I be an ideal of R. Then we have the following exact sequence

$$\mathbf{Hom}_R(R, E) \otimes_R F \longrightarrow \mathbf{Hom}_R(I, E) \otimes_R F \longrightarrow 0.$$

Now let F be a flat R-module then by Theorem 4.9 we have the following short exact sequence
$$\mathbf{Hom}_R(R, E \otimes_R F) \longrightarrow \mathbf{Hom}_R(I, E \otimes_R F) \longrightarrow 0.$$

Hence (iv) follows.
$(v) \implies (i)$ Follows by reversing the proof $(i) \implies (iv)$ above and using the fact that F is faithfully flat R-module. □

Corollary 4.12. *E is an injective R-module if and only if the character module E^+ is a flat R-module.*

Now using the above theorems and we get the main results of this note.

Theorem 4.13. *Let M be a R-module and E be a injective cogenerator. Then $\mathrm{fd}(M) = \mathrm{id}(\mathbf{Hom}_R(M, E))$ and $\mathrm{id}(M) = \mathrm{fd}(\mathbf{Hom}_R(M, E))$.*

Proof. Let M be an R-module and

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a flat resolution of M. Thus we get an injective resolution

$$0 \longrightarrow \mathbf{Hom}_R(M, E) \longrightarrow \mathbf{Hom}_R(F_0, E) \longrightarrow \mathbf{Hom}_R(F_1, E) \longrightarrow \mathbf{Hom}_R(F_2, E) \longrightarrow \cdots$$

of $\mathbf{Hom}_R(M, E)$. So one can deduce that $\mathrm{fd}(M) = \mathrm{id}(\mathbf{Hom}_R(M, E))$. Now consider the injective resolution

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

of M. By applying the functor $\mathbf{Hom}_R$(-,E) we get a flat resolution

$$\cdots \longrightarrow \mathbf{Hom}_R(I_2, E) \longrightarrow \mathbf{Hom}_R(I_1, E) \longrightarrow \mathbf{Hom}_R(I_0, E) \longrightarrow \mathbf{Hom}_R(M, E) \longrightarrow 0$$

of $\mathbf{Hom}_R(M, E)$. So it is easy to see $\mathrm{id}(M) = \mathrm{fd}(\mathbf{Hom}_R(M, E))$. $\square$

Theorem 4.14. *Let M be a R-module and F be a faithfully flat R-module. Then* $\mathrm{id}(M) = \mathrm{id}(M \otimes_R F)$.

Proof. Consider the injective resolution

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots.$$

Thus consider the injective resolution

$$0 \longrightarrow M \otimes_R F \longrightarrow I^0 \otimes_R F \longrightarrow I^1 \otimes_R F \longrightarrow I^2 \otimes_R F \longrightarrow \cdots$$

is an injective resolution of $M \otimes_R F$. So the result follows. $\square$

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