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Tau-Collocation Method for Linear Stochastic Îto-Volterra Integral Equations

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ABSTRACT

In this work, we propose a numerical Tau-collocation method for obtaining approximate solutions of linear stochastic Îto-Volterra integral equations. The method is based on a combination of the successive approximations method, the Gauss quadrature formulas and Îto approximation. The applicability of the present method is investigated through illustrative examples.

1 Introduction

Stochastic integral equations are very important in the study of many phenomena in physics, mechanics, medical, finance, sociology, biology, etc. Many orthogonal functions or polynomials, such as block pulse functions, Walsh functions, Fourier series, Bernstein polynomials, linear spline interpolation, wavelets and Laguerre polynomials, were used to estimate solutions of functional equations [1–16].

In this paper, we consider the following stochastic Îto-Volterra integral equation:

$$u(x) = f(x) + \int_0^x k_1(x,t)u(t)\,dt + \int_0^x k_2(x,t)u(t)\,dB(t), \quad x,t \in [0,T], \tag{1.1}$$

where $f(x)$, $k_1(x,t)$ and $k_2(x,t)$ are known functions, $u(x)$ is unknown function which should be determined, $B(x)$ is Brownian motion process defined on probability space $(\Omega, \mathcal{F}, P)$, and $\int_0^x k_2(x,t)u(t)\,dB(t)$ is the Îto integral.

The purpose of this paper is to solve the linear stochastic Îto-Volterra integral equations using the Tau-collocation method. In this purpose, the linear stochastic Îto-Volterra integral equations are transformed into a system of linear algebraic equations in matrix form, thus by solving this unknown coefficients are obtained.

This paper is structured as follows: In Section 2, stochastic calculus for developing our method are given. In Section 3, we introduce the Tau-collocation approach. In Section 4, the detailed theorem and formulation of the Tau-collocation method for linear stochastic Îto-Volterra integral equations are brought. In Section 5, the Tau-collocation approximation of the linear stochastic Îto-Volterra integral equations. In Section 6, the convergence analysis of this method is proved. In Section 7, some numerical results are given to clarify the method. In Section 8, we will have a conclusion of our study.

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2 Stochastic calculus

Gauss quadrature formulas will be used to compute Brownian motion. For this purpose, we transfer the t-intervals $[0, x]$ into the s-intervals $[0, 1]$ by means of the transformation $t = sx$.

$$\int_0^x f(t)\, dB(t) = \int_0^1 f(sx)\, dB(sx). \tag{2.1}$$

Lemma 2.1. *For any $t > 0$, $B(x + sx)$ or $B(\alpha + \beta s)$ where $\alpha = \beta = x$ is normally distributed with mean 0 and variance $\beta^2 t$. For any $t, s \geq 0$ we have*

$$E[B(\alpha + \beta t)B(\alpha + \beta s)] = \min\{\beta t, \beta s\}. \tag{2.2}$$

Proof. Let

$$B(\alpha + \beta t) = B(\alpha + \beta t) - B(\alpha),$$

that

$$E[B(\alpha + \beta t)B(\alpha + \beta s)] = \min\{\beta t, \beta s\} = \beta s,$$

if $s < t$ then

$$B(\alpha + \beta t) = B(\alpha + \beta s) + B(\alpha + \beta t) - B(\alpha + \beta s)$$

and

$$\begin{aligned} E[B(\alpha + \beta t)B(\alpha + \beta s)] &= E[B^2(\alpha + \beta s)] + E[B(\alpha + \beta s)(B(\alpha + \beta t) - B(\alpha + \beta s))] \\ &= E[B^2(\alpha + \beta s)] = \beta s. \end{aligned}$$

Therefore,

$$E[B(\alpha + \beta t)B(\alpha + \beta s)] = \min\{\beta t, \beta s\}. \qquad \square$$

Lemma 2.2 (Translation invariance)**.** *For fixed t_0, the stochastic process*

$$\widetilde{B}(\alpha + \beta t) = \widetilde{B}(\alpha + \beta t + t_0) - \widetilde{B}(t_0 + \alpha) \tag{2.3}$$

is also a Brownian motion.

Proof. The stochastic process $\widetilde{B}(\alpha + \beta t)$ is a Brownian motion. For any $s < t$,

$$\widetilde{B}(\alpha + \beta t) - \widetilde{B}(\alpha + \beta s) = \widetilde{B}(\alpha + \beta t + t_0) - \widetilde{B}(\alpha + \beta s + t_0),$$

we have $\widetilde{B}(\alpha + \beta t) - \widetilde{B}(\alpha + \beta s)$ is normally distributed with mean 0 and

$$\begin{aligned} \text{var}[\widetilde{B}(\alpha + \beta t) - \widetilde{B}(\alpha + \beta s)] &= \text{var}[\widetilde{B}(\alpha + \beta t + t_0) - \widetilde{B}(\alpha + \beta s + t_0)] \\ &= \text{var}[\alpha + \beta t + t_0 - \alpha - \beta s - t_0] \\ &= \text{var}[\beta t - \beta s] = \text{var}[\beta(t - s)] = \beta^2(t - s). \end{aligned}$$

For $\widetilde{B}(\alpha+\beta t)$ with fixed t_0, for any $0 \leq t_1 < t_2 < \ldots < t_n$, we have

$$0 \leq t_0 < \alpha+\beta t_1+t_0 < \alpha+\beta t_2+t_0 < \ldots < \alpha+\beta t_n+t_0.$$

We have

$$\widetilde{B}(\alpha+\beta t_k+t_0)-\widetilde{B}(\alpha+\beta t_{k-1}+t_0), \quad k=1,2,3,\ldots,n,$$

are independent random variables. Thus by equation (2.3), the random variables

$$\widetilde{B}(\alpha+\beta t_k)-\widetilde{B}(\alpha+\beta t_{k-1}), \quad k=1,2,3,\ldots,n,$$

are independent and so $\widetilde{B}(\alpha+\beta t)$ is a Brownian motion. □

Lemma 2.3 (Scaling invariance)**.** *For any real number $\lambda > 0$ the stochastic process*

$$\widetilde{B}(\alpha+\beta t)=\frac{\widetilde{B}(\alpha+\lambda\beta t)}{\sqrt{\lambda\beta}} \tag{2.4}$$

is also a Brownian motion.

Proof. For any $s < t$,

$$\widetilde{B}(\alpha+\beta t)-\widetilde{B}(\alpha+\beta s)=\frac{1}{\sqrt{\lambda\beta}}[\widetilde{B}(\alpha+\lambda\beta t)-\widetilde{B}(\alpha+\lambda\beta s)],$$

that $\widetilde{B}(\alpha+\beta t)-\widetilde{B}(\alpha+\beta s)$ is normally distributed with mean 0 and

$$\begin{aligned}
\text{var}[\widetilde{B}(\alpha+\beta t)-\widetilde{B}(\alpha+\beta s)] &= \text{var}\left[\frac{1}{\sqrt{\lambda\beta}}(\widetilde{B}(\alpha+\lambda\beta t)-\widetilde{B}(\alpha+\lambda\beta s))\right] \\
&= \text{var}\left[\frac{1}{\sqrt{\lambda\beta}}(\alpha+\lambda\beta t-\alpha-\lambda\beta s)\right] \\
&= \text{var}\left[\frac{1}{\sqrt{\lambda\beta}}(\lambda\beta t-\lambda\beta s)\right] \\
&= \text{var}\left[\frac{1}{\sqrt{\lambda\beta}}(\lambda\beta(t-s))\right] \\
&= \frac{\lambda^2\beta^2}{\lambda\beta}(t-s)=\lambda\beta(t-s). \qquad \square
\end{aligned}$$

3 The Tau-collocation method

In this section, the Tau-collocation method is applied for solving linear stochastic Îto-Volterra integral equation

$$u(x)=f(x)+\int_0^x k_1(x,t)u(t)\,dt+\int_0^x k_2(x,t)u(t)\,dB(t), \quad x,t \in [0,T]. \tag{3.1}$$

We define $u_N(x)$ as an approximation function of the exact solution $u(x)$ as follows [17–20]:

$$u_N(x) = \sum_{i=0}^{N} a_i L_i(x) = aLX, \tag{3.2}$$

where L is a non-singular lower triangular coefficient matrix by

$$LX = [L_0(x), L_1(x), \dots, L_N(x)]^T, \tag{3.3}$$

where $L_i(x)$, $i = 0, 1, \dots, N$ are shifted Legendre polynomials, with a standard basis vector

$$X = [1, x, x^2, \dots, x^N]^T, \tag{3.4}$$

and

$$a = [a_0, a_1, a_2, \dots, a_N]. \tag{3.5}$$

4 Explain the method for linear stochastic Îto-Volterra integral equations

In this section, we derive formulas for numerical solvability of linear stochastic Îto-Volterra integral equations (3.1) based on orthogonal polynomial basis functions.

4.1 Legendre-Gauss nodes and weights

Let $P_{N+1}(x)$ be the Legendre polynomial of order $N+1$ on $[0, 1]$. For any function $f(x) \in C[a, b]$ the Legendre-Gauss quadrature formula follows:

$$\int_a^b f(t)\,dt = (b-a) \sum_{i=0}^{N} w_i f((b-a)x_i + (b+a)), \tag{4.1}$$

where distinct nodes $\{x_i\}_{i=0}^N$ are the zeros of $P_{N+1}(x)$ and $\{w_i\}_{i=0}^N$ are corresponding weights as follows [21]:

$$w_i = \frac{1}{(1-x_i^2)[P'_{N+1}(x_i)]^2}, \quad i = 0, 1, 2, \dots, N. \tag{4.2}$$

4.2 Îto approximation

If a stochastic process $\{X(t)\}_{t \geq 0}$ is measurable on the filtration $\mathcal{F}$ for any $t \geq 0$, then the Îto integral of this process is defined by [22],

$$\int_a^b X(t)\,dB(t) = \lim_{N \to \infty} \sum_{i=0}^{N-1} X(t_i)(B(t_{i+1}) - B(t_i)), \tag{4.3}$$

where $B(t)$ is a Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, P)$. The result of the integral in (4.3) is a random variable, limit is defined in $L^2(\Omega, P)$ space and the approximation of the Îto integral is calculated at the left endpoint of interval $[t_i, t_{i+1}]$.

4.3 Matrix representation for integration term

Now, we present the matrix representation of the integration term for a class of stochastic Îto-Volterra integral equations.

Theorem 4.1. *Let $\{x_i\}_{i=0}^N$ be the set of the $N+1$ Gauss points of the shifted Legendre polynomial in $[0,1]$ and $\{w_i\}_{i=0}^N$ be the corresponding quadrature weights. Let the approximated solution $u_N(x)$ be given by* (3.2)*, then*

$$\int_0^{x_i} k(x_i,t)u_N(t)\,dt, \quad i=0,1,2,\ldots,N. \tag{4.4}$$

Gauss quadrature formulas will be used to compute the integral terms in (4.4)*. For this purpose, we transfer the t-intervals $[0,x_i]$ into the s-interval $[0,1]$ by means of transformations $s=t/x_i$. Using the $(N+1)$-point Gauss quadrature formula relative to the quadrature weights $\{w_j\}_{j=0}^N$, we estimate the integrals and get*

$$\int_0^{x_i} k(x_i,t)u_N(t)\,dt = ax_i\left(\sum_{j=0}^{N} k(x_i,s_jx_i)L(s_jx_i)w_j\right), \quad i=0,1,2,\ldots,N, \tag{4.5}$$

where the points $\{s_j\}_{j=0}^N$ and the collocation points $\{x_i\}_{i=0}^N$ coincide.

Theorem 4.2. *For*

$$\int_0^{x_i} k(x_i,t)u_N(t)\,dB(t), \quad i=0,1,2,\ldots,N, \tag{4.6}$$

to estimate the integral in (4.6)*, by* (4.3)*, it can be approximated as*

$$\int_0^{x_i} k(x_i,t)u_N(t)\,dB(t) = a\sqrt{x_i}\left(\sum_{j=0}^{N} k(x_i,s_jx_i)L(s_jx_i)(B(s_{j+1})-B(s_j))\right), \quad i=0,1,2,\ldots,N. \tag{4.7}$$

5 Tau-collocation approximation of the linear stochastic Îto-Volterra integral equations

Here we apply the previous results for constructing the Tau-collocation approximate solution of the linear stochastic Îto-Volterra integral equations

$$u(x) = f(x) + \int_0^x k_1(x,t)u(t)\,dt + \int_0^x k_2(x,t)u(t)\,dB(t), \quad x,t\in[0,T]. \tag{5.1}$$

Without loss of generality, we can assume that in (5.1), the linear analytic function may be expanded as:

$$u_N(x) = \sum_{i=0}^{N} a_iL_i(x) = aLX. \tag{5.2}$$

Thus, we can obtain

$$u_N(x) = f(x) + \int_0^x k_1(x,t)u_N(t)\,dt + \int_0^x k_2(x,t)u_N(t)\,dB(t), \quad x,t\in[0,T]. \tag{5.3}$$

Following Theorem 1, 2 and (5.2) at collocation points $\{x_i\}_{i=0}^{N}$ in (5.3), we obtain the following linear system of algebraic equations:

$$a\left(LX_i - x_i\left(\sum_{j=0}^{N} k_1(x_i, s_jx_i)L(s_jx_i)w_j\right) - \sqrt{x_i}\left(\sum_{j=0}^{N} k_2(x_i, s_jx_i)L(s_jx_i)(B(s_{j+1}) - B(s_j))\right)\right) = f(x_i),$$
$$i = 0, 1, 2, \ldots, N. \quad (5.4)$$

By solving the linear system in (5.4), we obtain unknown vector a. Then, we obtain an approximate solution for the problem by $u_N(x) \simeq aLX$.

6 Convergence analysis

Theorem 6.1. *Let k_1 and k_2 in (3.1) be continuous on $[a, b]^2 \times L^2(\Omega, C[a, b])$, and for $\mathcal{F}_t$-adapted process $U : [a, b] \to L^2(\Omega, C[a, b])$, the processes k_1 and k_2 are also $\mathcal{F}_t$-adapted. Assume that $u(x)$ is the exact solution of the linear stochastic Îto-Volterra integral equation (3.1). Suppose that the approximate solution $u_N(x)$ is given by the Tau-collocation scheme (5.4) with orthogonal shifted Legendre basis. If k_1 and k_2 satisfy Lipschitz condition in (3.1), then for all sufficiently large N, we have $\|e(x)\| = \|u_N(x) - u(x)\| < \infty$.*

Proof. Let

$$\begin{aligned}
u(x) &= f(x) + \int_a^x k_1(x,t)u(t)\,dt + \int_a^x k_2(x,t)u(t)\,dB(t),\\
u_N(x) &= f(x) + \int_a^x k_1(x,t)u_N(t)\,dt + \int_a^x k_2(x,t)u_N(t)\,dB(t),\\
u_N(x) &= \sum_{i=0}^{N} a_iL_i(x) = aLX.
\end{aligned}$$

Then $\|e(x)\| = \|u(x) - u_N(x)\|$. Now

$$\begin{aligned}
u(x) - u_N(x) &= \int_a^x k_1(x,t)u(t)\,dt + \int_a^x k_2(x,t)u(t)\,dB(t)\\
&\quad - \int_a^x k_1(x,t)u_N(t)\,dt - \int_a^x k_2(x,t)u_N(t)\,dB(t)\\
&= \int_a^x k_1(x,t)[u(t) - u_N(t)]\,dt + \int_a^x k_2(x,t)[u(t) - u_N(t)]\,dB(t).
\end{aligned}$$

By applying the Cauchy-Schwarz inequality, Doob's L^2-inequality and relation $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, where $p = 2$, we have

$$\|e(x)\|^2 = \|u(x) - u_N(x)\|^2 = \left\|\int_a^x [k_1(x,t)u(t) - k_1(x,t)u_N(t)]\,dt + \int_a^x [k_2(x,t)u(t) - k_2(x,t)u_N(t)]\,dB(t)\right\|^2$$

$$\leq 2\left\|\int_a^x [k_1(x,t)u(t) - k_1(x,t)u_N(t)]\,dt\right\|^2 + 2\left\|\int_a^x [k_2(x,t)u(t) - k_2(x,t)u_N(t)]\,dB(t)\right\|^2. \quad (6.1)$$

The first part of equation (6.1) by using the integral operator:

$$2\left\|\int_a^x [k_1(x,t)u(t) - k_1(x,t)u_N(t)]\,dt\right\|^2 \leq 2(b-a)\int_a^x \|k_1(x,t)u(t) - k_1(x,t)u_N(t)\|^2\,dt,$$

which satisfies the Lipschitz condition,

$$2\left\|\int_a^x [k_1(x,t)u(t) - k_1(x,t)u_N(t)]\,dt\right\|^2 \leq 2(b-a)\int_a^x L\|u(t) - u_N(t)\|^2\,dt. \tag{6.2}$$

The second part of equation (6.1) by using the parallelogram rule and isometry property, we have

$$2\left\|\int_a^x [k_2(x,t)u(t) - k_2(x,t)u_N(t)]\,dB(t)\right\|^2,$$

where

$$\begin{aligned} x &= \int_a^x k_2(x,t)u(t)\,dB(t),\\ y &= \int_a^x k_2(x,t)u_N(t)\,dB(t),\\ w &= \int_a^x k_0(x,t)u(t)\,dB(t). \end{aligned}$$

Then

$$\begin{aligned} &4(x+w,y) + 4(x-w,y)\\ &= \left(\left\|\int_a^x k_2(x,t)u(t)\,dt + \int_a^x k_0(x,t)u(t)\,dt + \int_a^x k_2(x,t)u_N(t)\,dt\right\|^2\right.\\ &\quad \left. - \left\|\int_a^x k_2(x,t)u(t)\,dt + \int_a^x k_0(x,t)u(t)\,dt - \int_a^x k_2(x,t)u_N(t)\,dt\right\|^2\right)\\ &\quad + \left(\left\|\int_a^x k_2(x,t)u(t)\,dt - \int_a^x k_0(x,t)u(t)\,dt + \int_a^x k_2(x,t)u_N(t)\,dt\right\|^2\right.\\ &\quad \left. - \left\|\int_a^x k_2(x,t)u(t)\,dt - \int_a^x k_0(x,t)u(t)\,dt - \int_a^x k_2(x,t)u_N(t)\,dt\right\|^2\right)\\ &= 8\int_a^x \|k_2(x,t)u(t) - k_2(x,t)u_N(t)\|^2\;dt. \end{aligned}$$

Which satisfies the Lipschitz condition,

$$2\left\|\int_a^x [k_2(x,t)u(t) - k_2(x,t)u_N(t)]\,dB(t)\right\|^2 = 8\int_a^x L\|u(t) - u_N(t)\|^2\,dt. \tag{6.3}$$

With combination of equations (6.2) and (6.3), we can obtain

$$\begin{aligned}\|e(t)\|^2 &\leq 2(b-a)\int_a^x L\|u(t)-u_N(t)\|^2\,dt + 8\int_a^x L\|u(t)-u_N(t)\|^2\,dt \\ &\leq (2b-2a+8)\int_a^x L\|u(t)-u_N(t)\|^2\,dt \\ &= (2b-2a+8)\int_a^x L\|e(t)\|^2\,dt < \infty. \qquad \square\end{aligned}$$

7 Numerical examples

In this section, we present examples to illustrate the procedure of the Tau-collocation method. Let $u(x)$ denote the exact solution of the given examples, and let $u_N(x)$ be the computed solution by the presented method. The error is defined as:

$$\|E\|_\infty = \max_{1\leq i\leq N} |u(x_i)-u_N(x_i)|. \tag{7.1}$$

Example 1. Consider the following linear stochastic Îto-Volterra integral equation,

$$u(x) = 1 + \int_0^x x^2 u(t)\,dt + \int_0^x t u(t)\,dB(t), \quad x,t\in[0,0.5), \tag{7.2}$$

with the exact solution $u(x) = e^{\frac{x^3}{6}+\int_0^x t\,dB(t)}$, for $0\leq x<0.5$. Here $u(x)$ is an unknown stochastic process defined on the probability space $(\Omega,\mathcal{F},P)$ and $B(x)$ is a Brownian motion process. The numerical results are shown in Table 1.

Table 1. *Numerical results of example 1 with $N=8$.*

x_i	Error	S_E	Lowerbound	Upperbound
0.0	0.018800	0.015336	0.005357	0.032243
0.1	0.010015	0.006304	0.004488	0.015541
0.2	0.064064	0.030806	0.037062	0.091067
0.3	0.084014	0.056669	0.034341	0.133690
0.4	0.171740	0.091861	0.091223	0.252260
0.5	0.295920	0.201060	0.119690	0.472160
0.6	0.235220	0.143950	0.109040	0.361400
0.7	0.311870	0.203790	0.133250	0.490500
0.8	0.381040	0.099407	0.293910	0.468180
0.9	0.350970	0.259300	0.123680	0.578260
1.0	0.398670	0.370050	0.074310	0.723030

Example 2. Consider the following linear stochastic Îto-Volterra integral equation,

$$u(x) = \frac{1}{12} + \int_0^x \cos(x)u(t)\,dt + \int_0^x \sin(x)u(t)\,dB(t), \quad x,t\in[0,0.5), \tag{7.3}$$

with the exact solution

$$u(x) = \frac{1}{12}e^{\frac{x}{4}+\sin(x)+\frac{\sin(2x)}{8}+\int_0^x \sin(t)\,dB(t)},$$

for $0 \leq x < 0.5$. Here $u(x)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$ and $B(x)$ is a Brownian motion process. The numerical results are shown in Table 2.

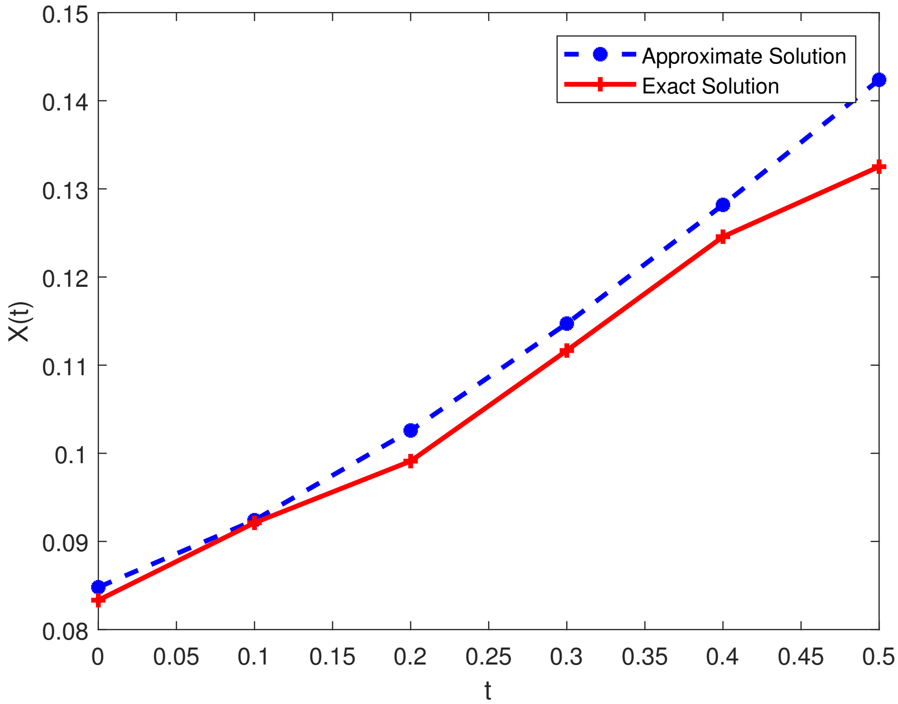

Figure 1: The approximate solution and exact solution of example 1

Table 2. *Numerical results of example 2 with* $N = 4$.

x_i	Error	S_E	Lowerbound	Upperbound
0.0	0.0040000	0.00014907	0.0002693	0.0005306
0.1	0.0021762	0.00070683	0.0015566	0.0027957
0.2	0.0030836	0.00248020	0.0009095	0.0052575
0.3	0.0094344	0.00591730	0.0042476	0.0146210
0.4	0.0187180	0.00635650	0.0131460	0.0242900
0.5	0.0185200	0.00674490	0.0126080	0.0244330
0.6	0.0372890	0.01976300	0.0199660	0.0546110
0.7	0.0247500	0.02786800	0.0003220	0.0491770
0.8	0.0232400	0.01847000	0.0070504	0.0394300
0.9	0.0524220	0.02981300	0.0262890	0.0785540
1.0	0.0782230	0.07504100	0.0124470	0.1440000

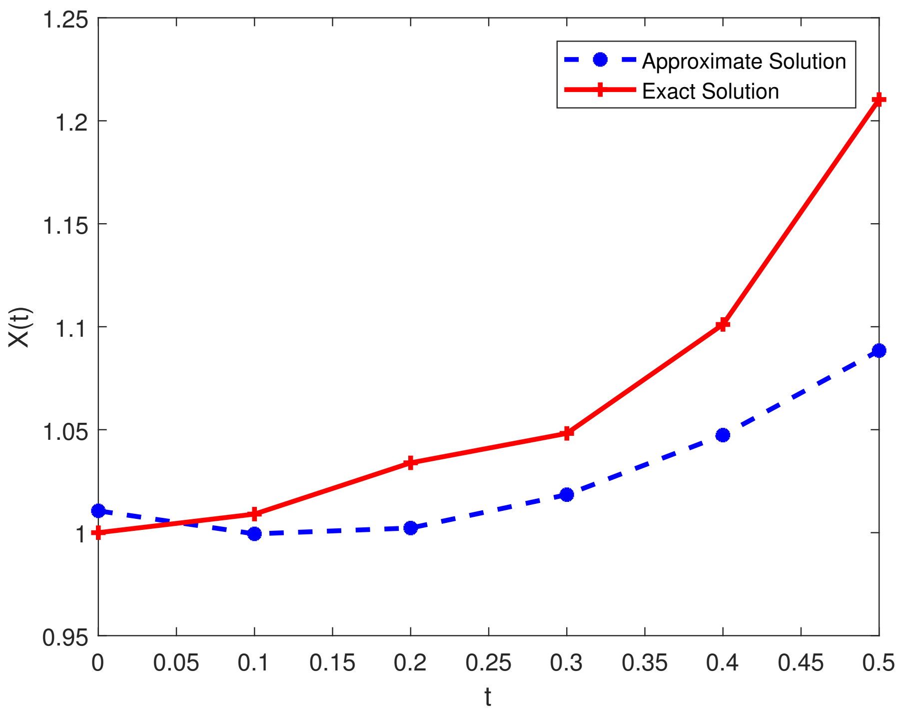

Figure 2: The approximate solution and exact solution of example 2

8 Conclusion

In this paper, a numerical method based on Tau-collocation method is proposed for solving linear stochastic Îto-Volterra integral equations. The main characteristic of this method is that it reduces these linear stochastic Îto-Volterra integral equations to those of solving a linear system of algebraic equations. The numerical results confirmed that Tau-collocation method is a powerful mathematical tool for the exact and numerical solutions of linear equations in terms of accuracy and efficiency.

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