



Integrating Factor, First Integral and Λ -Symmetry for Vector Odes of Second Orders

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ABSTRACT

The goal of this paper is to calculate an integrating factor, λ -symmetry and a first integral for ODEs of second order $\ddot{u} = F(x, u, \dot{u})$ and for a vector ODEs of second order $\ddot{\hat{r}} = \hat{f}$, by the λ -symmetry method.

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1 Introduction

The problem of finding first integrals of ordinary differential equations (ODE) has a long and interesting history which may be traced to the seminal works of Darboux and Lie in the latter half of the nineteenth century. A first integral, for an n th-order ODE, is an expression involving the independent variable, the dependent variable and its derivatives to order $n - 1$. The techniques involved in finding first integrals of systems of one or more ODEs generally make use of integrating factors, which are functions multiplying each of the ODEs to yield a first integral.

In 2001, Muriel and Romero introduced λ -symmetry to find general solutions for examples that have trivial Lie symmetries. They[8] presented techniques to obtain first integral, integrating factor, λ -symmetry of second-order ODEs $\ddot{u} = F(x, u, \dot{u})$ and the relationship between them. In 2006, Muriel, Romero and Olver have expanded the concept of variational problem and conservation law in the case of symmetries to the case of λ -symmetries of ODEs. They have presented an adapted formulation of the Nother's theorem for λ -symmetry of ODEs.

In 2004, Gaeta and Morando expanded this approach to the PDE frame with p independent variables $x = (x^1, \dots, x^p)$ and q dependent variables $u = (u^1, \dots, u^q)$, in order to do this, the central object is a horizontal one-form $\mu = \lambda_i dx^i$ on first order jet space $(J^{(1)}M, \pi, M)$, where μ is a compatible, i.e. $D_i \lambda_j - D_j \lambda_i = 0$, and thus one speaks of μ -symmetries[4].

The types of symmetry are Lie symmetry[11], hidden symmetry[1], λ -symmetry[5], μ -symmetry[4] and C^∞ -symmetry[6]. One of the applications of symmetries is to reduce of order ODEs[5, 6, 7, 8] and PDEs[4, 11]. Also, λ -symmetry is used to calculate an integrating factor and consequently a first integral for ODEs[7, 8, 9, 10].

In 2005, Chandrasekar, Senthilvelan and Lakshmanan obtained a first integral for n th order ODEs and two coupled second order equations by the Prolle-Singer method[2, 3]. The PS method is method that requires longer and more difficult calculations to obtain the first integral compared to the λ -symmetry method. Therefore, it has been tried to obtain integrating factor and the first integral with simple calculations.

In this article, we introduce a procedure to find an integrating factor, λ -symmetry and first integral for vector ODEs of second order $\ddot{\hat{r}} = \hat{f}$ where $\hat{r} = u\hat{i} + w\hat{j}$ and $\hat{f} = F_1(x, u, w, \dot{u}, \dot{w})\hat{i} + F_2(x, u, w, \dot{u}, \dot{w})\hat{j}$, indeed, we obtain an integrating factor $\mu_1(x, u, w, \dot{u}, \dot{w})$ of $\ddot{u} = F_1(x, u, w, \dot{u}, \dot{w})$, an integrating factor $\mu_2(x, u, w, \dot{u}, \dot{w})$ of $\ddot{w} = F_2(x, u, w, \dot{u}, \dot{w})$ and consequently a first integral $I(x, u, w, \dot{u}, \dot{w})$ for $\ddot{\hat{r}} = \hat{f}$, by the λ -symmetry method, which is a simpler method compared to the PS method.

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The outline of this paper, section 2, we calculate an integrating factor and consequently a first integral for n th order ODEs and in a special cases for second order by λ -symmetry method, in section 3. Section 4, we obtain an integrating factor, λ -symmetry and first integral for vector ODEs of second order, by the λ -symmetry method.

2 Integrating factor and λ -symmetry for n th-order ODEs

In this section, we obtain some of the main results about λ -symmetry, first integral and integrating factor for n th-order ordinary differential equation $u^{(n)} = F(x, u^{(n-1)})$. We present a procedure to find a first integral and consequently an integrating factor of this differential equation. We denote by $M^{(n)}$ the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, for $n \in \mathbb{N}$. Their elements are $(x, u^{(n)}) = (x, u, u^{(1)}, \dots, u^{(n)})$, where, for $i = 1, \dots, n$, $u^{(i)}$ denote the derivative of order i of u with respect to x . An integrating factor of an n th-order ordinary differential equation (ODE) $\Delta(x, u^{(n)}) = 0$ is a function $\mu(x, u^{(n-1)})$ such that $\mu\Delta = 0$ is an exact equation:

$$\mu(x, u^{(n-1)})\Delta(x, u^{(n)}) = D_x(G(x, u^{(n-1)})), \quad (2.1)$$

function $G(x, u^{(n-1)})$ in (2.1) is called a first integral of the equation $\Delta(x, u^{(n)}) = 0$ and $D_x(G(x, u^{(n-1)})) = 0$ is a conserved form of the equation $\Delta(x, u^{(n)}) = 0$. Let

$$u^{(n)} = F(x, u^{(n-1)}), \quad (2.2)$$

be a n th-order ordinary differential equation, where F is an analytic function of its arguments. Let $v = \xi(x, u)\partial_x + \eta(x, u)\partial_u$ be a vector field defined on open subset $M \subset X \times U$. The λ -prolongation of order n of v , denoted by $v^{[\lambda, (n)]}$, is the vector field defined on $M^{(n)}$ by

$$v^{[\lambda, (n)]} = \xi(x, u)\partial_x + \sum_{i=0}^{n-1} \eta^{[\lambda, (i)]}(x, u^{(i)})\partial_{u_i},$$

where $\eta^{[\lambda, (i)]}(x, u^{(i)}) = (D_x + \lambda)(\eta^{[\lambda, (i-1)]}(x, u^{(i-1)})) - D_x(\xi)u_i - \lambda\xi u_i$ and $\eta^{[\lambda, (0)]}(x, u) = \eta(x, u)$, for $i = 1, 2, 3, \dots, n$.

A vector field v is a λ -symmetry of (2.2) if there exists function $\lambda(x, u^{(k)}) \in C^\infty(M^{(1)})$, for $k < n$, such that $v^{[\lambda, (n)]}(u^{(n)} - F(x, u^{(n-1)})) = 0$, when $u^{(n)} = F(x, u^{(n-1)})$ [5].

We denote by $A = \partial_x + u^{(1)}\partial_u + u^{(2)}\partial_{u^{(1)}} + \dots + F(x, u^{(n-1)})\partial_{u^{(n-1)}}$ the vector field associated with (2.2).

The vector field $v = \xi(x, u)\partial_x + \eta(x, u)\partial_u$ is a λ -symmetry of (2.2) if and only if

$$[v^{[\lambda, (n-1)]}, A] = \lambda.v^{[\lambda, (n-1)]} + \tau.A, \quad (2.3)$$

for some $\lambda \in C^\infty(M^{(1)})$, $\tau = -(A + \lambda)(\xi(x, u))$ and $v^{[\lambda, (n-1)]} = \sum_{i=0}^{n-1} (D_x + \lambda)^i(1)\partial_{u_i}$ [5].

Function $I(x, u^{(n-1)})$ is a first integral of (2.2), such that $A(I) = 0$ and an integrating factor of (2.2), is any function $\mu(x, u^{(n-1)})$ such that

$$\mu(x, u^{(n-1)})(u^{(n)} - F(x, u^{(n-1)})) = D_x I(x, u^{(n-1)}).$$

[7]

3 λ -Symmetry, integrating factor and first integral for second order ODEs

In this section, we obtain some of the main results about λ -symmetry, first integral and integrating factor for second-order ordinary differential equation as the following

$$\ddot{u} = F(x, u, \dot{u}). \quad (3.1)$$

Let $I(x, u, \dot{u})$ be a first integral and $\mu(x, u, \dot{u})$ be an integrating factor of (3.1), then $\mu(x, u, \dot{u})(\ddot{u} - F(x, u, \dot{u})) = D_x I(x, u, \dot{u})$, and $\mu(x, u, \dot{u}) = I_{\dot{u}}(x, u, \dot{u})$ [8].

When $v = \partial_u$ is a λ -symmetry of (3.1) if and only if

$$(A(\lambda) + \lambda^2 - F_u - \lambda F_{\dot{u}})\partial_{\dot{u}} = 0,$$

where $v^{[\lambda, (1)]} = \partial_u + \lambda \partial_{\dot{u}}$ and $A = \partial_x + \dot{u} \partial_u + F \partial_{\dot{u}}$ by using (2.3).

A procedure to find a λ -symmetry, an integrating factor $\mu(x, u, \dot{u})$ and consequently a first integral $I(x, u, \dot{u})$ of (3.1) is as follows(see[8]):

step1. The vector field $v = \partial_u$ is a λ -symmetry of (3.1), if the function $\lambda(x, u, \dot{u})$ is any particular solution of the equation

$$A(\lambda) = F_u + \lambda F_{\dot{u}} - \lambda^2.$$

step2. If $\lambda(x, u, \dot{u})$ is such that $v = \partial_u$ is a λ -symmetry of (3.1), then any solution $\mu(x, u, \dot{u})$ of the first-order linear system

$$\begin{cases} A(\mu) = -\mu(F_{\dot{u}} - \lambda), \\ \mu_u = -\lambda_{\dot{u}}\mu - \lambda\mu_{\dot{u}}. \end{cases}$$

is an integrating factor of (3.1).

step3. A system of the form

$$\begin{cases} I_x = \mu(\lambda \dot{u} - F), \\ I_u = -\mu\lambda, \\ I_{\dot{u}} = \mu. \end{cases}$$

is compatible for some functions $\mu(x, u, \dot{u})$ and $\lambda(x, u, \dot{u})$ if and only if $v = \partial_u$ is a λ -symmetry of (3.1) and $\mu(x, u, \dot{u})$ an integrating factor of (3.1). In this case $I(x, u, \dot{u})$ is a first integral of (3.1).

A other method for find a λ -symmetry, an integrating factor $\mu(x, u, \dot{u})$ and a first integral $I(x, u, \dot{u})$ of (3.1) is as follows:

step1. The vector field $v = \partial_u$ is a λ -symmetry of (3.1), if the function $\lambda(x, u, \dot{u})$ is any particular solution of the equation

$$A(\lambda) = F_u + \lambda F_{\dot{u}} - \lambda^2. \quad (3.2)$$

step2. If $\lambda(x, u, \dot{u})$ is such that $v = \partial_u$ is a λ -symmetry of (3.1), then any solution $I(x, u, \dot{u})$ of the system

$$\begin{cases} I_u + \lambda I_{\dot{u}} = 0, \\ I_x + \dot{u} I_u + F I_{\dot{u}} = 0. \end{cases} \quad (3.3)$$

is a first integral of (3.1)[8].

step3. If $I(x, u, \dot{u})$ is a first integral of (3.1), then

$$\mu(x, u, \dot{u}) = I_{\dot{u}}(x, u, \dot{u}), \quad (3.4)$$

is an integrating factor of (3.1).

Example 3.1 We consider the second-order differential equation of Lienard type nonlinear oscillators of the form

$$\ddot{u} + ku\dot{u} + \frac{k^2}{9}u^3 + au = 0, \quad (3.5)$$

where $F(x, u, \dot{u}) = -(ku\dot{u} + (k^2/9)u^3 + au)$ is an analytic function of its arguments, k and a are arbitrary constants. Generalizations of this equation are widely used in applications in the content of nonlinear oscillators [3]. We know that (3.5), that is,

$$\ddot{u} = -ku\dot{u} - \frac{k^2}{9}u^3 - au = F.$$

By (3.2), the vector field $v = \partial_u$ is a λ -symmetry of (3.5), if the function $\lambda(x, u, \dot{u})$ is any particular solution of the equation

$$\lambda_x + \dot{u}\lambda_u + (-ku\dot{u} - \frac{k^2}{9}u^3 - au)\lambda_{\dot{u}} + \lambda^2 + k\lambda u + k\dot{u} + \frac{k^2}{3}u^2 + a = 0.$$

For the sake of simplicity, suppose $\lambda(x, u, \dot{u}) = \alpha(x, u)\dot{u} + \beta(x, u)$, therefore, we obtain the following system:

$$\begin{cases} \alpha_u + \alpha^2 = 0, \\ \alpha_x + \beta_u + 2\alpha\beta + k = 0, \\ \beta_x - \frac{k^2}{9}\alpha u^3 - a\alpha u + \beta^2 + k\beta u + \frac{k^2}{3}u^2 + a = 0. \end{cases}$$

A particular solution of the first equation is given by $\alpha = 1/u$. The second and third equations become

$$\begin{cases} \beta_u + \frac{2}{u}\beta + k = 0, \\ \beta_x - \frac{2k^2}{9}u^2 + \beta^2 + k\beta u = 0. \end{cases}$$

A general solution of the first equation is given by $\beta = -ku/3 + \gamma(x)/u^2$. Since the last equation become $\gamma'/u + \gamma^2/u^3 + (1/3)k\gamma = 0$, we can choose $\gamma(x) = 0$. in consequence, the vector field $v = \partial_u$ is a λ -symmetry for $\lambda = (1/u)\dot{u} - ku/3$. Substituting $F(x, u, \dot{u}) = -(ku\dot{u} + (k^2/9)u^3 + au)$ and $\lambda = (1/u)\dot{u} - ku/3$ into system (3.3) and solving them, we obtain

$$I(x, u, \dot{u}) = -x - (1/\sqrt{a})\text{Arctan}((ku^2 + 3\dot{u})/3\sqrt{au}).$$

By (3.4), $\mu(x, u, \dot{u}) = -1/(au + (\dot{u}/u + ku/3)^2u)$, is an integrating factor of (3.5).

4 λ -Symmetry, integrating factor and first integral for vector second-order ODEs

In this section, we present a procedure to find a λ -symmetry, an integrating factor and consequently a first integral for a vector ODEs of second order of the form $\ddot{\mathbf{r}} = \hat{\mathbf{f}}$. We consider a vector ODEs of second order

$$\ddot{\mathbf{r}} = \hat{\mathbf{f}}, \quad (4.1)$$

where $\hat{\mathbf{r}} = u\hat{\mathbf{i}} + w\hat{\mathbf{j}}$ and $\hat{\mathbf{f}} = F_1(x, u, w, \dot{u}, \dot{w})\hat{\mathbf{i}} + F_2(x, u, w, \dot{u}, \dot{w})\hat{\mathbf{j}}$ also u and w are arbitrary functions, $\dot{u}$ denote the derivative of u with respect to x and $\dot{w}$ denote the derivative of w with respect to x . Therefore we have coupled second order ODEs of the form

$$\ddot{u} = F_1(x, u, w, \dot{u}, \dot{w}), \quad (4.2)$$

$$\ddot{w} = F_2(x, u, w, \dot{u}, \dot{w}), \quad (4.3)$$

where F_1 and F_2 are analytic functions of their arguments.

We denote by $A = \partial_x + \dot{u}\partial_u + \dot{w}\partial_w + F_1\partial_{\dot{u}} + F_2\partial_{\dot{w}}$ the vector field associated with a vector ODEs of second order of (4.1). Function $I(x, u, w, \dot{u}, \dot{w})$ is a first integral for (4.1), such that $A(I) = 0$. An integrating factor of (4.2), is any function $\mu_1(x, u, w, \dot{u}, \dot{w})$ such that

$$\mu_1(x, u, w, \dot{u}, \dot{w})(\ddot{u} - F_1(x, u, w, \dot{u}, \dot{w})) = D_x I(x, u, w, \dot{u}, \dot{w})$$

and an integrating factor of (4.3), is any function $\mu_2(x, u, w, \dot{u}, \dot{w})$ such that

$$\mu_2(x, u, w, \dot{u}, \dot{w})(\ddot{w} - F_2(x, u, w, \dot{u}, \dot{w})) = D_x I(x, u, w, \dot{u}, \dot{w}).$$

Using (2.3), the vector field $v_1 = \partial_u$ is a λ_1 -symmetry of (4.2) if and only if

$$(A(\lambda_1) + \lambda_1^2 - F_{1u} - \lambda_1 F_{1\dot{u}})\partial_{\dot{u}} - (F_{2u} + \lambda_1 F_{2\dot{u}})\partial_{\dot{w}} = 0, \quad (4.4)$$

where $v_1^{[\lambda_1, (1)]} = \partial_u + \lambda_1 \partial_{\dot{u}}$ and the vector field $v_2 = \partial_w$ is a λ_2 -symmetry of (4.3) if and only if

$$(A(\lambda_2) + \lambda_2^2 - F_{2w} - \lambda_2 F_{2\dot{w}})\partial_{\dot{w}} - (F_{1w} + \lambda_2 F_{1\dot{w}})\partial_{\dot{u}} = 0, \quad (4.5)$$

where $v_2^{[\lambda_2, (1)]} = \partial_w + \lambda_2 \partial_{\dot{w}}$.

Theorem 4.1 *A system of the form*

$$\begin{cases} I_x = \mu_1(\lambda_1 \dot{u} - F_1) + \mu_2(\lambda_2 \dot{w} - F_2), \\ I_u = -\mu_1 \lambda_1, \\ I_w = -\mu_2 \lambda_2, \\ I_{\dot{u}} = \mu_1, \\ I_{\dot{w}} = \mu_2, \end{cases} \quad (4.6)$$

is compatible for some functions $\lambda_1(x, u, w, \dot{u}, \dot{w}), \lambda_2(x, u, w, \dot{u}, \dot{w}), \mu_1(x, u, w, \dot{u}, \dot{w})$ and $\mu_2(x, u, w, \dot{u}, \dot{w})$, if and only if μ_1 is an integrating factor and $v_1 = \partial_u$ is a λ_1 -symmetry of (4.2) also μ_2 is an integrating factor and $v_2 = \partial_w$ is a λ_2 -symmetry of (4.3). In this case $I(x, u, w, \dot{u}, \dot{w})$ is a first integral for a vector ODEs of second order of (4.1).

Proof. Let $I(x, u, w, \dot{u}, \dot{w})$ be a first integral for a vector ODEs of second order of (4.1), then $\mu_1 = I_{\dot{u}}$ is an integrating factor of (4.2) and $\mu_2 = I_{\dot{w}}$ is an integrating factor of (4.3). Let $v_1 = \partial_u$ be a λ_1 -symmetry of (4.2) then $A(I) = 0$ and $v^{[\lambda_1, (1)]}(I) = 0$ also, Let $v_2 = \partial_w$ be a λ_2 -symmetry of (11) then $A(I) = 0$ and $v^{[\lambda_2, (1)]}(I) = 0$, hence we get

$$\begin{aligned} A(I) = 0 &\Rightarrow I_x = -\dot{u}I_u - \dot{w}I_w - F_1\mu_1 - F_2\mu_2, \\ v^{[\lambda_1, (2)]}(I) = 0 &\Rightarrow I_u = -\lambda_1 I_{\dot{u}} = -\lambda_1 \mu_1 \\ v^{[\lambda_2, (2)]}(I) = 0 &\Rightarrow I_w = -\lambda_2 I_{\dot{w}} = -\lambda_2 \mu_2. \end{aligned}$$

Therefore we have the system of the form (4.6). We prove that, when (4.6) is compatible, necessarily $v_1 = \partial_u$ is a λ_1 -symmetry and necessarily $v_2 = \partial_w$ is a λ_2 -symmetry. The compatibility conditions between the equations (4.6), provide the following conditions

$$A(\lambda_1) = F_{1u} + \lambda_1 F_{1\dot{u}} - \lambda_1^2 + \frac{\mu_2}{\mu_1}(F_{2u} + \lambda_1 F_{2\dot{u}}), \quad (4.7)$$

$$A(\lambda_2) = F_{2w} + \lambda_2 F_{2\dot{w}} - \lambda_2^2 + \frac{\mu_1}{\mu_2}(F_{1w} + \lambda_2 F_{1\dot{w}}), \quad (4.8)$$

$$A(\mu_1) = -\mu_1(F_{1\dot{u}} - \lambda_1) + \mu_2 F_{2\dot{u}}, \quad (4.9)$$

$$A(\mu_2) = -\mu_2(F_{2\dot{w}} - \lambda_2) + \mu_1 F_{1\dot{w}}, \quad (4.10)$$

$$\lambda_1 \mu_{1w} = -\lambda_{1w} \mu_1 + \lambda_2 \mu_{2u} + \lambda_{2u} \mu_2, \quad (4.11)$$

$$\mu_{1u} = -\lambda_{1\dot{u}} \mu_1 - \lambda_1 \mu_{1\dot{u}}, \quad (4.12)$$

$$\mu_{2w} = -\lambda_{2\dot{w}} \mu_2 - \lambda_2 \mu_{2\dot{w}}, \quad (4.13)$$

$$\mu_{1w} = -\lambda_{2\dot{u}} \mu_2 - \lambda_2 \mu_{2\dot{u}}, \quad (4.14)$$

$$\mu_{2u} = -\lambda_{1\dot{w}} \mu_1 - \lambda_1 \mu_{1\dot{w}}, \quad (4.15)$$

$$\mu_{1\dot{w}} = \mu_{2\dot{u}}. \quad (4.16)$$

According to (4.7), we get $(A(\lambda_1) + \lambda_1^2 - F_{1u} - \lambda_1 F_{1\dot{u}})\mu_1 - (F_{2u} + \lambda_1 F_{2\dot{u}})\mu_2 = 0$, since $I_{\dot{u}} = \mu_1 \neq 0$ and $I_{\dot{w}} = \mu_2 \neq 0$, hence

$$(A(\lambda_1) + \lambda_1^2 - F_{1u} - \lambda_1 F_{1\dot{u}})I_{\dot{u}} - (F_{2u} + \lambda_1 F_{2\dot{u}})I_{\dot{w}} = 0,$$

or correspond

$$(A(\lambda_1) + \lambda_1^2 - F_{1u} - \lambda_1 F_{1\dot{u}})\partial_{\dot{u}} - (F_{2u} + \lambda_1 F_{2\dot{u}})\partial_{\dot{w}} = 0$$

and by comparing of (4.4), implies that $v_1 = \partial_u$ is a λ_1 -symmetry of (4.2) also by using (4.8) and by comparing of (4.5), implies that $v_2 = \partial_w$ is a λ_2 -symmetry of (4.3).

According to (4.9) and (4.10), we get

$$\begin{cases} \lambda_1 = \frac{A(\mu_1)}{\mu_1} - \frac{\mu_2}{\mu_1} F_{2\dot{u}} + F_{1\dot{u}}, \\ \lambda_2 = \frac{A(\mu_2)}{\mu_2} - \frac{\mu_1}{\mu_2} F_{1\dot{w}} + F_{2\dot{w}}. \end{cases} \quad (4.17)$$

In summary, a procedure to find integrating factors μ_1 , μ_2 and consequently a first integral $I(x, u, w, \dot{u}, \dot{w})$ for a vector ODEs of second order of (4.1) is as follows:

step1. Substituting λ_1 and λ_2 of (4.17) into the system

$$\begin{cases} A(\lambda_1) = F_{1u} + \lambda_1 F_{1\dot{u}} - \lambda_1^2 + \frac{\mu_2}{\mu_1} (F_{2u} + \lambda_1 F_{2\dot{u}}), \\ A(\lambda_2) = F_{2w} + \lambda_2 F_{2\dot{w}} - \lambda_2^2 + \frac{\mu_1}{\mu_2} (F_{1w} + \lambda_2 F_{1\dot{w}}), \\ \lambda_1 \mu_{1w} = -\lambda_{1w} \mu_1 + \lambda_2 \mu_{2u} + \lambda_{2u} \mu_2, \\ \mu_{1u} = -\lambda_{1\dot{u}} \mu_1 - \lambda_1 \mu_{1\dot{u}}, \\ \mu_{2w} = -\lambda_{2\dot{w}} \mu_2 - \lambda_2 \mu_{2\dot{w}}, \\ \mu_{1w} = -\lambda_{2\dot{u}} \mu_2 - \lambda_2 \mu_{2\dot{u}}, \\ \mu_{2u} = -\lambda_{1\dot{w}} \mu_1 - \lambda_1 \mu_{1\dot{w}}, \\ \mu_{1\dot{w}} = \mu_{2\dot{u}}. \end{cases} \quad (4.18)$$

and solving them, we get, integrating factors μ_1 of (4.2) and μ_2 of (4.3).

step2. Substituting μ_1 and μ_2 into the system (4.17), we obtain, the functions λ_1 and λ_2 , therefore the vector field $v_1 = \partial_u$ is a λ_1 -symmetry of (4.2) and the vector field $v_2 = \partial_w$ is a λ_2 -symmetry of (4.3).

step3. Substituting the functions μ_1 , μ_2 , λ_1 and λ_2 into the system (4.6) and solving them, we find a first integral $I(x, u, w, \dot{u}, \dot{w})$ for a vector ODEs of second order of (4.1).

Example 4.1 We consider the Kepler problem in the $u - w$ plane, that is,

$$\ddot{\hat{r}} + \frac{\hat{r}}{r^3} = 0, \quad (4.19)$$

where $\hat{r} = u\hat{i} + w\hat{j}$ and $r = |\hat{r}|$. The respective equations of motions are

$$\ddot{u} = -\frac{u}{(u^2 + w^2)^{\frac{3}{2}}} = F_1(x, u, w, \dot{u}, \dot{w}), \quad (4.20)$$

$$\ddot{w} = -\frac{w}{(u^2 + w^2)^{\frac{3}{2}}} = F_2(x, u, w, \dot{u}, \dot{w}), \quad (4.21)$$

Substituting $\lambda_1 = \frac{A(\mu_1)}{\mu_1}$ and $\lambda_2 = \frac{A(\mu_2)}{\mu_2}$ of (4.17) into the system (4.18) and solving them, we get, $\mu_1 = w$ and $\mu_2 = u$ that are particular solutions of this equations.

Substituting $\mu_1 = w$ and $\mu_2 = u$ into the system (4.17), we obtain, the functions

$$\lambda_1 = \frac{A(\mu_1)}{\mu_1} = \frac{\dot{w}}{w}, \quad \lambda_2 = \frac{A(\mu_2)}{\mu_2} = \frac{\dot{u}}{u}.$$

Therefore the vector field $v_1 = \partial_u$ is a λ_1 -symmetry of (4.20) and the vector field $v_2 = \partial_w$ is a λ_2 -symmetry of (4.21).

Substituting the functions $\mu_1 = w$, $\mu_2 = u$, $\lambda_1 = \frac{\dot{w}}{w}$ and $\lambda_2 = \frac{\dot{u}}{u}$ into the system (4.6) and solving them, we find a first integral

$$I(x, u, w, \dot{u}, \dot{w}) = w\dot{u} - u\dot{w},$$

for a class of second order ODEs of (4.19).

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