Theory of Approximation and Applications, 2022, Volume 16, No.2:67-80, DOR:20.1001.1.25382217.2022.16.2.9.7

An Algorithm for Approximating Fixed Points Generalized Double Midpoint Rule

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ARTICLE INFO

KEYWORDS
Fixed point
generalized double midpoint rule
asymptotically nonexpansive mapping
Hilbert space.

ARTICLE HISTORY
RECEIVED: 2022 MAY 23
ACCEPTED: 2022 DECEMBER 20

ABSTRACT

In this paper we introduce a new nonlinear algorithm for approximating fixed points of double midpoint rule on a nonempty, closed and convex subset of real Hilbert spaces. Under some specific assumption imposed on the control parameters, we prove a strong convergence theorem. In this manner, we provide some numerical examples to illustrate our main result and to display the efficiency of the proposed algorithm.

1 Introduction.

There are several methods to face, from a theoretical aspect, to numerous problems which arise from real-world environment. Due to their possible applications, throughout the last years, the fixed point theory becomes the most interesting branch in mathematics. It is well-known that several mathematical and real-word problems are naturally formulated as a fixed point problem, that is, a problem for finding a point x in a domain of an appropriate mapping T such that

$$Tx = x$$

A wide rang of problems of applied sciences and engineering are usually formulated as functional equations. such equations can be written in the form of fixed point equations. Yao et. al [22] proved that the sequence $\{x_n\}$ generated by

$$\frac{1}{h}(x_{n+1} - x_n) = f(\frac{x_{n+1} + x_n}{2}),$$

here $h > 0$ is a stepsize converges to the exact solution of an initial value problem for ODE's of the type

$$\acute{x}(t) = f(x(t)), x(t_0) = x_0.$$

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Numerical reckoning of nonlinear operators is very fascinating research problem of nonlinear analysis. However, it is not a easy task to find the fixed points of some operators. To overcome this kind of problem so many iteration procedures has been evolved over the time. This fact motivated various authors have introduced numerous iterative scheme which have been utilized widely to approximate the fixed points of operators. Convergence analysis for iterative algorithms using the asymptotically nonexpansive and midpoint rule have been introduced by many authors[3, 6, 10, 13, 14, 17, 18, 19, 23] and the references therein.
In 2014, alghamdi et. al [1] presented a recursion sequence for a nonexpansive mapping T on Hilbert space as follow:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T(\frac{x_{n+1}+x_n}{2}). \tag{1.1}$$

They proved the weak convergence of (1.1) under certain condition on $\{\alpha_n\}$.
Still, in Hilbert space, Xu et. al [20] used contractions to recognize the implicit procedure:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(\frac{x_{n+1}+x_n}{2}). \tag{1.2}$$

They proved that a strong convergence for the sequence $\{x_n\}$ which solves a variational inequality. In 2015, Yao et.al [22] introduced

$$x_{n+1} = \alpha_n f(x_n) + b_n x_n + c_n)T(\frac{x_{n+1}+x_n}{2}), \tag{1.3}$$

which gives a faster approximation compared with (1.2).
More precisely, in 2017, Luo et. al [15] extended the work of Xu [21] to uniformly smooth Banach spaces. He et. al [7] considered the generalized viscosity implicit midpoint rule of asymptotically nonexpansive mapping in Hilbert space defined by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T^n(\beta_n x_n + (1-\beta_n)x_{n+1}). \tag{1.4}$$

They showed that the iterative (1.4) converges strongly to a fixed point of $T : H \to H$.
In 2019, Pan et. al [16] introduced the following iteration on Banach space by:

$$x_{n+1} = \alpha_n x_n + \beta_n f(x_n) + \gamma_n T^n(t_n x_n + (1-t_n)x_{n+1}). \tag{1.5}$$

They proved that under suitable conditions their process converges strongly to a fixed point of asymptotically nonexpansive mapping.
The class of asymptotically nonexpansive mapping was introduced as a generalization of the class of nonexpansive mapping in 1974 by Kirk [8]. In recent years many authors have made their contributions for the class of mappings toward the existence of fixed points and asymptotic behavior of $\{T^n x\}$ [8, 9, 11, 12].
Motivated by the above studies, we introduce a generalized double midpoint rule algorithm $(GDMRA)$ for approximating fixed point of a asymptotically nonexpansive mapping where the sequence $\{x_n\}$ is generate iteratively by $x_0 \in C$ and

$$x_{n+1} = \alpha_n f(\frac{x_{n+1}+x_n}{2}) + (1-\alpha_n)T^n(\frac{x_{n+1}+x_n}{2}). \tag{1.6}$$

for each $n \in N$ and $\{\alpha_n\}$ is a sequence in $(0,1)$.
The purpose of this paper is to prove the convergence a newly defined iteration process (1.6) for asymptotically nonexpansive mapping. We also provide some numerical examples to show the genuineness of our study with the help of **Matlab R2018a** software and we show the convergence algorithm by tables and figures.

2 preliminaries.

Throughout this paper, let H is a real Hilbert space with inner product $\langle .,. \rangle$ and $\|.\|$ is a norm induced by $\langle .,. \rangle$. Let C be a nonempty, closed and convex subset of H and T be a self-mapping of H. We denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{z \in H : \ \ Tz = z\}$. Recall that a mapping $T : C \to T$ is said to :

1. $\rho-$contraction if there exists $\rho \in [0,1)$ such that $\|Tx - Ty\| \leq \rho\|x - y\|$,
2. Asymptotically nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1,+\infty)$, $lim_{n\to\infty}k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$.

The nearest point projection from H onto C, P_C, is defined by

$$P_C(x) := argmin_{z\in C}\|x - z\|^2, \ x \in H.$$

Namely, $P_C(x)$ is the only point in C that minimizes the objective $\|x - z\|$ over $z \in C$.

Remark 2.1. *Note that P_C is characterized as follows:*

$$P_C x \in C \ \ and \ \ \langle x - P_C x, z - x - P_C x\rangle \leq 0 \ \ for \ \ all \ \ z \in C.$$

The following lemmas play an important role in our paper.

Lemma 2.1. *([2]). Let H be a real Hilbert space, C be a nonempty closed and convex subset of H, and $T : C \to C$ be a asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that(i): $\{x_n\}$ weakly converges to x and(ii): $(I - T)x_n$ converges strongly to 0, then $x = Tx$.*

Lemma 2.2. *([5])Let H be a real Hilbert space. $x, y \in H$ and $t \in [0,1]$. Then*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$$

It is easy to prove that the following lemma holds:

Lemma 2.3. *Let H be a real Hilbert space.Then for all $u, x, y \in H$ the following inequality holds*

$$\|x - u\|^2 \leq \|y - u\|^2 + 2\langle x - y, x - u\rangle$$

Lemma 2.4. *([20])Let a_n be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

- $\sum_{n=1}^{\infty} \gamma_n = \infty$
- $limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} = 0$ *or* $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $lim_{n\to\infty} a_n = 0$.

3 A Generalized Iterative Algorithm.

Let C be a nonempty closed convex subset a Hilbert space H, $T : C \to C$ be a asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction mapping with the contractive constant $\rho \in [0, 1)$. The generalized double midpoint rule (GDMR) generates a sequence x_n via the following algorithm:

Algorithm 3.1.

$$\begin{cases} x_0 \in C, \\ x_{n+1} = \alpha_n f(\frac{x_{n+1}+x_n}{2}) + (1-\alpha_n)T^n(\frac{x_{n+1}+x_n}{2}), \end{cases} \tag{3.1}$$

where the parameters α_n and k_n satisfies the following conditions:

$(C1) : Lim_{n\to\infty}\alpha_n = 0,$

$(C2) : \sum_{n=1}^{\infty} \alpha_n = \infty,$

$(C3) : Lim_{n\to\infty} \frac{k_n^2-1}{\alpha_n} = 0,$

$(C4) : Lim_{n\to\infty} \|T^n x_n - x_n\| = 0.$

The next remark plays a key role in the proof of our results in subsequence sections.

Remark 3.1. *By condition (C3), for any given positive number* $0 < \varepsilon < 1 - \alpha$*, there exists a sufficient large positive integer* n_0*, such that for any* $n \geq n_0$*, we have*

$$k_n^2 - 1 \leq 2\varepsilon\alpha_n \quad and \quad k_n - 1 \leq \frac{k_n + 1}{2}(k_n - 1) \leq \frac{k_n^2 - 1}{2} \leq \varepsilon\alpha_n,$$

also, we have

$$1 - \frac{(1-\alpha_n)k_n^2+3\rho\alpha_n}{2} = \frac{1-(k_n^2-1)+\alpha_n(k_n^2-3\rho)}{2} \geq \frac{1-2\varepsilon\alpha_n+\alpha_n(1-3\rho)}{2} = \frac{1+(1-3\rho-2\varepsilon)\alpha_n}{2}$$

In this section, we will establish the strong convergence theorems of the (GDMR) under some specific assumptions.

Lemma 3.1. *Let* $T : C \to C$ *with* $F(T) \neq \emptyset$ *be defined on a nonempty closed subset* C *of a Hilbert space,* H*. If the iteration process* $\{x_n\}$ *is defined by algorithm 3.1, then* $\{x_n\}$ *is bounded.*

proof: For any $\eta \in F(T)$, we have

$$\begin{aligned}
\|x_{n+1}-\eta\| &= \|\alpha_n f(\tfrac{x_{n+1}+x_n}{2}) + (1-\alpha_n)T^n(\tfrac{x_{n+1}+x_n}{2}) - \eta\| \\
&\leq \alpha_n\|\alpha_n f(\tfrac{x_{n+1}+x_n}{2}) - f(\eta)\| + \alpha_n\|f(\eta)-\eta\| + (1-\alpha_n)\|T^n(\tfrac{x_{n+1}+x_n}{2}) - T^n(\eta)\| \\
&\leq \alpha_n\rho\|\tfrac{x_{n+1}+x_n}{2} - \eta\| + \alpha_n\|f(\eta)-\eta\| + (1-\alpha_n)k_n\|\tfrac{x_{n+1}+x_n}{2}) - \eta\| \\
&\leq \tfrac{\alpha_n\rho}{2}\|x_n-\eta\| + \tfrac{\alpha_n\rho}{2}\|x_{n+1}-\rho\| + \alpha_n\|f(\eta)-\eta\| + \tfrac{(1-\alpha_n)k_n}{2}\|x_n-\eta\| + \tfrac{(1-\alpha_n)k_n}{2}\|x_{n+1}-\eta\| \\
&= \tfrac{\alpha_n\rho+(1-\alpha_n)k_n}{2}\|x_n-\eta\| + \alpha_n\|f(\eta)-\eta\| + \tfrac{\alpha_n\rho+(1-\alpha_n)k_n}{2}\|x_{n+1}-\eta\|
\end{aligned}$$

which gives

$$(1-\tfrac{\alpha_n\rho+(1-\alpha_n)k_n}{2})\|x_{n+1}-\eta\| \leq \alpha_n\|f(\eta)-\eta\| + \tfrac{\alpha_n\rho+(1-\alpha_n)k_n}{2}\|x_n-\eta\|.$$

As $k_n - 1 \leq \varepsilon\alpha_n$, we can get

$$\begin{aligned}
2-\alpha_n(\rho-k_n)-k_n &= 1-\alpha_n(\rho-k_n)-(k_n-1) \\
&\geq 1-\alpha_n(\rho-k_n)-\varepsilon\alpha_n = 1+\alpha_n(k_n-\varepsilon-\rho) \\
&\geq 1+\alpha_n(1-\varepsilon-\rho)
\end{aligned}$$

Also,

$$\begin{aligned}
\alpha_n(\rho-k_n)+k_n &\leq \alpha_n(\rho-k_n)+\varepsilon\alpha_n+1 \\
&= 1+\alpha_n(\varepsilon+\rho-k_n) \\
&\leq 1-\alpha_n(1-\rho-\varepsilon).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|x_{n+1}-\eta\| &\leq \tfrac{2\alpha_n}{2-\alpha_n(\rho-k_n)-k_n}\|f(\eta)-\eta\| + \tfrac{\alpha_n(\rho-k_n)+k_n}{2-\alpha_n(\rho-k_n)-k_n}\|x_n-\eta\| \\
&\leq \tfrac{2\alpha n}{1+\alpha_n(1-\varepsilon-\rho)}\|f(\eta)-\eta\| + \tfrac{\alpha_n(\rho-k_n)+k_n}{1+\alpha_n(1-\varepsilon-\rho)}\|x_n-\eta\| \\
&\leq \tfrac{2\alpha_n}{1+\alpha_n(1-\varepsilon-\rho)}\|f(\eta)-\eta\| + \tfrac{1-\alpha_n(1-\varepsilon-\rho)}{1+\alpha_n(1-\varepsilon-\rho)}\|x_n-\eta\| \\
&\leq \tfrac{2(1-\varepsilon-\rho)\alpha_n}{1+\alpha_n(1-\varepsilon-\rho)}\{\tfrac{1}{1-\varepsilon-\rho}\}\|f(\eta)-\eta\| + \{1-\tfrac{2\alpha_n(1-\varepsilon-\rho)}{1+\alpha_n(1-\varepsilon-\rho)}\}\|x_n-\eta\|.
\end{aligned}$$

By induction, we readily obtain

$$\|x_n-\eta\| \leq max\{\|x_0-\eta\|, \frac{1}{1-\varepsilon-\rho}\|f(\eta)-\eta\|\}.$$

which shows that the sequence generated by (3.1) is bounded. Therefore, $\{f(\frac{x_{n+1}+x_n}{2})\}$ and $\{T^n(\frac{x_{n+1}+x_n}{2})\}$ are bounded.

Lemma 3.2. *Let C, T and $\{x_n\}$ be described as in lemma 3.1. Then the iterative process $\{x_n\}$ is asymptotically regular, that is, $Lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.*

Proof: For each $n \in N$, we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\le \|x_{n+1} - T^n x_n\| + \|T^n x_n - x_n\| \\
&= \|\alpha_n f(\tfrac{x_{n+1}+x_n}{2}) + (1-\alpha_n)T^n(\tfrac{x_{n+1}+x_n}{2}) - T^n x_n\| + \|T^n x_n - x_n\| \\
&\le (1-\alpha_n)\|T^n(\tfrac{x_{n+1}+x_n}{2}) - T^n x_n\| + \alpha_n\|f(\tfrac{x_{n+1}+x_n}{2}) - T^n x_n\| + \|T^n x_n - x_n\| \\
&\le \tfrac{(1-\alpha_n)k_n}{2}\|x_{n+1} - x_n\| + \alpha_n\|f(\tfrac{x_{n+1}+x_n}{2}) - T^n x_n\| + \|T^n x_n - x_n\| \\
&\le \tfrac{(1-\alpha_n)k_n}{2}\|x_{n+1} - x_n\| + \alpha_n \mathbf{M} + \|T^n x_n - x_n\|
\end{aligned}$$

where $\mathbf{M} := Sup_{n\in N}\{\|f(\frac{x_{n+1}+x_n}{2}) - T^n(\frac{x_{n+1}+x_n}{2})\|\}$.

Since $k_n \subset [1, +\infty)$ and $k_n - 1 \le \varepsilon\alpha_n$, then we have

$$\|x_{n+1} - x_n\| \le \tfrac{2\alpha_n}{1+(1-\varepsilon)\alpha_n}\mathbf{M} + \tfrac{1}{1+(1-\varepsilon)\alpha_n}\|T^n x_n - x_n\|.$$

By using conditions $(C1), (C2)$, we have $Lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Lemma 3.3. *Let C, T and $\{x_n\}$ be described as in lemma 3.1. Then $Lim_{n\to\infty}\|x_n - Tx_n\| = 0$.*

Proof: We have

$$\begin{aligned}
\|x_n - T^{n-1}x_n\| &= |\alpha_{n-1} f(\tfrac{x_n+x_{n-1}}{2}) + (1-\alpha_{n-1})T^{n-1}(\tfrac{x_n+x_{n-1}}{2}) - T^{n-1}x_n\| \\
&\le \alpha_{n-1}\|f(\tfrac{x_n+x_{n-1}}{2}) - T^{n-1}x_n\| + (1-\alpha_{n-1})T^{n-1}(\tfrac{x_n+x_{n-1}}{2}) - T^{n-1}x_n\| \\
&\le \alpha_{n-1}\|f(\tfrac{x_n+x_{n-1}}{2}) - T^{n-1}x_n\| + (1-\alpha_{n-1})k_{n-1}\|\tfrac{x_n+x_{n-1}}{2} - x_n\| \\
&\le \alpha_{n-1}\|f(\tfrac{x_n+x_{n-1}}{2}) - T^{n-1}x_n\| + \tfrac{(1-\alpha_{n-1})k_{n-1}}{2}\|x_{n-1} - x_n\|,
\end{aligned}$$

The conditions $(C1)$ and Lemma 3.2 implies that $Lim_{n\to\infty}\|x_n - T^{n-1}x_n\| = 0$. Therefore,

$$\|x_n - Tx_n\| \le \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\|$$

$$\le \|x_n - T^n x_n\| + k_1\|x_n - T^{n-1}x_n\| \to 0.$$

Lemma 3.4. *Let C, T and $\{x_n\}$ be described as in lemma 3.1. Then the sequence $\{x_n\}$ converges weakly to an element of $F(T)$ and $limsup\langle \zeta - f(\zeta), \zeta - x_n\rangle \leq 0$, where $\zeta \in F(T)$ is the unique fixed point of the contraction $P_{F(T)}$, that is $\zeta = P_{F(T)}f(\zeta)$.*

Proof: First of all, we will show that $\omega_{weak}(x_n) \subset F(T)$ where

$$\omega_{weak}(x_n) = \{x \in H : \ there\ exist\ a\ subsequence\ of\ x_n\ converges\ weakly\ to\ x\}.$$

Suppose that $x \in \omega_{weak}(x_n)$. Then, there exist a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x$. using lemma 2.1, we have $\|(I-T)x_{n_i}\| = \|x_{n_i} - Tx_{n_i}\| \to 0$, which implies that $\{(I-T)x_{n_i}\}$ converges strongly to 0. Consequently, $Tx = x$ and so $x \in F(T)$. Then , we get $\omega_{weak}(x_n) \subset F(T)$.

Now, we prove that $limsup\langle \zeta - f(\zeta), \zeta - x_n\rangle \leq 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup \overline{\zeta}$ for some $\overline{\zeta} \in H$ and

$$Limsup\langle \zeta - f(\zeta), \zeta - x_n\rangle = lim_{j\to\infty}\langle \zeta - f(\zeta), \zeta - x_{n_j}\rangle.$$

Since $\omega_{weak}(x_n) \subset F(T)$, we get $\overline{\zeta} \in F(T)$.From remark 2.1, we obtain

$$limsup\langle \zeta - f(\zeta), \zeta - x_n\rangle = lim_{j\to\infty}\langle \zeta - f(\zeta), \zeta - x_{n_j}\rangle = \langle \zeta - f(\zeta), \zeta - \overline{\zeta}\rangle \leq 0.$$

Afterward, in the next section we prove a strong convergence theorem.

4 Convergence of Algorithm.

We begin this section by proving a strong convergence theorem for a asymptotically nonexpansive mapping.

Theorem 4.1. *Let C, T and $\{x_n\}$ be described as in lemma 3.1. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

proof: Suppose that $\zeta \in F(T)$ and then ζ is also the unique fixed point of the contraction mapping $P_{F(T)}of$. For each $n \in N$, we compute

$$\begin{aligned}
\|x_{n+1}-\zeta\|^2 &= \|\alpha_n(f(\tfrac{x_{n+1}+x_n}{2})-\zeta)+(1-\alpha_n)(T^n(\tfrac{x_{n+1}+x_n}{2})-\zeta)\|^2\\
&\le \|(1-\alpha_n)(T^n(\tfrac{x_{n+1}+x_n}{2})-\zeta)\|^2+2\alpha_n\langle f(\tfrac{x_{n+1}+x_n}{2})-\zeta, x_{n+1}-\zeta\rangle\\
&\le (1-\alpha_n)^2\|T^n(\tfrac{x_{n+1}+x_n}{2})-\zeta)\|^2+2\alpha_n\langle f(\tfrac{x_{n+1}+x_n}{2})-\zeta, x_{n+1}-\zeta\rangle\\
&\le (1-\alpha_n)^2\|T^n(\tfrac{x_{n+1}+x_n}{2})-\zeta)\|^2+2\alpha_n\langle f(\tfrac{x_{n+1}+x_n}{2})-f(\zeta), x_{n+1}-\zeta\rangle\\
&+2\alpha_n\langle f(\zeta)-\zeta, x_{n+1}-\zeta\rangle\\
&\le (1-\alpha_n)^2k_n^2\|\tfrac{x_{n+1}+x_n}{2}-\zeta\|^2+2\alpha_n\rho\|\tfrac{x_{n+1}+x_n}{2}-\zeta\|^2\|x_{n+1}-x_n\|\\
&+2\alpha_n\langle f(\zeta)-\zeta, x_{n+1}-\zeta\rangle\\
&\le \tfrac{(1-\alpha_n)k_n^2}{2}\|x_{n+1}-\zeta\|^2+\tfrac{(1-\alpha_n)k_n^2}{2}\|x_n-\zeta\|^2-\tfrac{(1-\alpha_n)k_n^2}{4}\|x_n-x_{n+1}\|^2\\
&+2\alpha_n\rho(\tfrac{\|x_n-\zeta\|\|x_{n+1}-\zeta\|}{2}+\tfrac{\|x_{n+1}-\zeta\|^2}{2})+2\alpha_n\langle f(\zeta)-\zeta, x_{n+1}-\zeta\rangle\\
&= \tfrac{(1-\alpha_n)k_n^2}{2}\|x_{n+1}-\zeta\|^2+\tfrac{(1-\alpha_n)k_n^2}{2}\|x_n-\zeta\|^2-\tfrac{(1-\alpha_n)k_n^2}{4}\|x_n-x_{n+1}\|^2\\
&+\alpha_n\rho\|x_n-\zeta\|\|x_{n+1}-\zeta\|+\alpha\rho\|x_{n+1}-\zeta\|^2+2\alpha_n\langle f(\zeta)-\zeta, x_{n+1}-\zeta\rangle\\
&\le \tfrac{(1-\alpha_n)k_n^2}{2}\|x_{n+1}-\zeta\|^2+\tfrac{(1-\alpha_n)k_n^2}{2}\|x_n-\zeta\|^2+\tfrac{\alpha_n\rho}{2}\|x_n-\zeta\|^2+\tfrac{\alpha_n\rho}{2}\|x_{n+1}-\zeta\|^2\\
&+\alpha_n\rho\|x_{n+1}-\zeta\|^2+2\alpha_n\langle f(\zeta)-\zeta, x_{n+1}-\zeta\rangle\\
&= \tfrac{(1-\alpha_n)k_n^2+3\rho\alpha_n}{2}\|x_{n+1}-\zeta\|^2+\tfrac{(1-\alpha_n)k_n^2+\alpha_n\rho}{2}\|x_n-\zeta\|^2+2\alpha_n\langle f(\zeta)-\zeta, x_{n+1}-\zeta\rangle
\end{aligned}$$

Then, it follows that

$$[1-\tfrac{(1-\alpha_n)k_n^2+3\rho\alpha_n}{2}]\|x_{n+1}-\zeta\|^2 \le \tfrac{(1-\alpha_n)k_n^2+\alpha_n\rho}{2}\|x_n-\zeta\|^2+2\alpha_n\langle f(\zeta)-\zeta, x_{n+1}-\zeta\rangle$$

that is $\|x_{n+1}-\zeta\|^2 \le \frac{(1-\alpha_n)k_n^2+\alpha_n\rho}{2-(1-\alpha_n)k_n^2-3\rho\alpha_n}\|x_n-\zeta\|^2+\frac{2\alpha_n}{2-(1-\alpha_n)k_n^2-3\rho\alpha_n}\langle f(\zeta)-\zeta, x_{n+1}-\zeta\rangle,$

It follows from remark 3.1, that

$$\begin{aligned}
\|x_{n+1}-\zeta\|^2 &\leq \tfrac{1+(k_n^2-1)-\alpha_n(k_n^2-\rho)}{2-(1-\alpha_n)k_n^2-3\rho\alpha_n}\|x_n-\zeta\|^2+\tfrac{2\alpha_n}{2-(1-\alpha_n)k_n^2-3\rho\alpha_n}\langle f(\zeta)-\zeta,x_{n+1}-\zeta\rangle \\
&\leq \tfrac{1+2\varepsilon\alpha_n-\alpha_n(1-\rho)}{1+\alpha_n(1-3\rho-2\varepsilon)}\|x_n-\zeta\|^2+\tfrac{2\alpha_n}{1+\alpha_n(1-3\rho-2\varepsilon)}\langle f(\zeta)-\zeta,x_{n+1}-\zeta\rangle \\
&\leq \tfrac{1-\alpha_n(1-\rho-2\varepsilon)}{1+\alpha_n(1-3\rho-2\varepsilon)}\|x_n-\zeta\|^2+\tfrac{2\alpha_n}{1+\alpha_n(1-3\rho-2\varepsilon)}\langle f(\zeta)-\zeta,x_{n+1}-\zeta\rangle. \\
&= (1-\tfrac{2\alpha_n(1-2\rho-2\varepsilon)}{1+\alpha_n(1-3\rho-2\varepsilon)})\|x_n-\zeta\|^2+\tfrac{2\alpha_n}{1+\alpha_n(1-3\rho-2\varepsilon)}\langle f(\zeta)-\zeta,x_{n+1}-\zeta\rangle.
\end{aligned}$$

Now, take $\gamma_n = \frac{2\alpha_n(1-2\rho-2\varepsilon)}{1+\alpha_n(1-3\rho-2\varepsilon)}$ and $\delta_n = \frac{2\alpha_n}{1+\alpha_n(1-3\rho-2\varepsilon)}\langle f(\zeta)-\zeta,x_{n+1}-\zeta\rangle$. The conditions $(C1)$, $(C2)$ and lemma 3.1 implies that

$$limsup\frac{\delta_n}{\gamma_n} = limsup\frac{1}{1-2\rho-2\varepsilon}\langle f(\zeta)-\zeta,x_{n+1}-\zeta\rangle \leq 0.$$

lemma 2.4 implies that $x_n \to \zeta$.

5 Numerical Examples.

In this section we provide some numerical examples to show the genuineness of algorithm 3.1. With the help of MATLAB R2018a, we show the convergence by tables and figures. The computing environment was Windows 8.1 run on an Intel Core i7-4790 (3.6 GHz, 4 cores) and 16 GB of memory space.

Example 5.1. *Let $C = [1,10]$ be a subset of real Hilbert space R with the usual inner product and define the mapping $T, f : C \to C$ as follows*

$$T(x) = \sqrt[3]{x^2+4}, \quad f(x) = \frac{1}{2}cos(cosx),$$

for all $x \in C$. Further, f is contraction mapping with constant $\rho = \frac{1}{2}$ and T is a asymptotically nonexpansive mapping with $k_n = 1$. The function $g(x) = \sqrt[3]{x^2+4} - x$ for any $x \in C = [1,10]$ is decreasing. In fact, we have

$$g\acute{(x)} = \frac{1}{3}(2x)(\frac{1}{\sqrt[3]{(x^2+4)^2}}) \leq 1,$$

then, $g\acute{(x)} \leq 0$.
Let $x, y \in C = [1,10]$ with $x \leq y$, then we obtain

$$\begin{aligned}
\sqrt[3]{y^2+4} - y &\leq \sqrt[3]{x^2+4} - x \\
\sqrt[3]{y^2+4} - \sqrt[3]{x^2+4} &\leq y - x \\
|\sqrt[3]{y^2+4} - \sqrt[3]{x^2+4}| &\leq |y-x| \\
|\sqrt[3]{y^2+4} - \sqrt[3]{x^2+4}| &\leq |x-y| \\
|T(x)-T(y)| &\leq |x-y|.
\end{aligned}$$

It is easy to observe that $F(T) = \{2\}$. We can choose $\alpha_n = \frac{1}{n}$ which satisfies the conditions $(C1)-(C4)$ in *algorithm* 3.1. *With the help of MATLAB R2018a, we show the convergence by table and figure.*

$Iterate$	$x_1 = 1$	$x_1 = 5$	$x_1 = 10$
x_2	0.142857142857143	0.714285714285714	0.124997760122152
x_3	0.873995661837622	0.939983750444318	0.872310213289880
x_4	1.236484416136780	1.245624445375212	1.186138164552595
x_5	1.428973610675769	1.430398457897879	1.365276575402056
x_6	1.544884181586953	1.545120708328902	1.480704759551207
x_7	1.621731338223615	1.621772107096926	1.560984260262761
x_8	1.676333575769409	1.676340780057918	1.619924844666935
x_9	1.717128841818699	1.717130137628625	1.664986103804991
x_{10}	1.748773355270465	1.748773591450231	1.700532318356288
x_{11}	1.774040144430866	1.774040187923104	1.729279303793315
x_{12}	1.794683299811890	1.794683307887089	1.753002487038575
x_{13}	1.811866932807423	1.811866934316869	1.772910228093960
x_{14}	1.826394224855754	1.826394225139493	1.789852915711087
x_{15}	1.838837378088730	1.838837378142320	1.804446121619310
x_{16}	1.849615144992518	1.849615145002681	1.817146284404758
$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{96}	1.976554614105112	1.9765546143567112	1.976554614105112
x_{97}	1.976796402196547	1.976796402196547	1.976796402196547
x_{98}	1.977033254105798	1.977033254105798	1.977033254105798
x_{99}	1.977033254105798	1.977033254105798	1.977033254105798
x_{100}	1.977492741929410	1.977492741929410	1.977492741929410

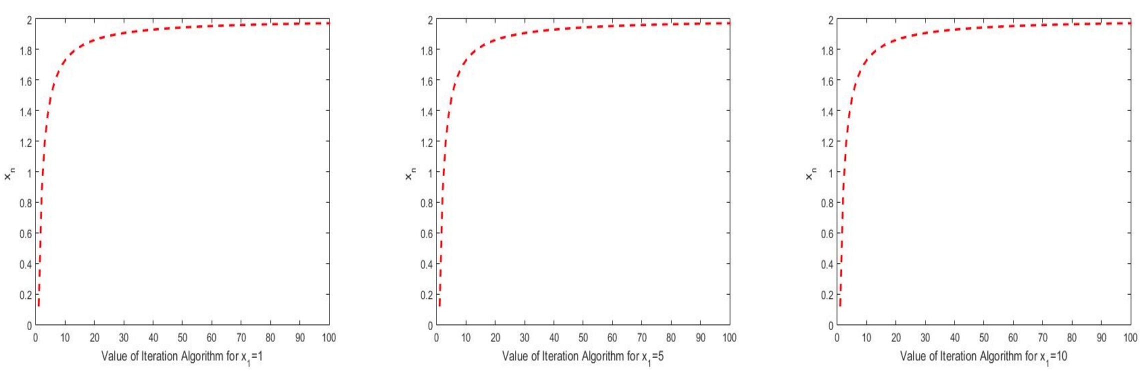

Figure 1: The graph and table of $\{x_n\}$ with different initial value .

Example 5.2. *Let $f : C \to C$ be a contraction mapping defined by $f(x) = x - 1 + \frac{1}{e^x}$ for all $x \in C$. The C, T and α_n be described as in example 5.1. Figure 1 shows that our iteration reaches fixed point at the 35$^t h$ step.*

$Iterate$	$x_1 = 3$	$x_1 = 5$	$x_1 = 10$
x_2	1.240056477975282	3.035978205657063	8.000246789153852
x_3	1.120585128352988	1.836868264895283	4.317597686883846
x_4	1.355010885207149	1.562694821784070	2.472752464969403
x_5	1.540856061768960	1.595554719680772	1.863115949711970
x_6	1.657665616022218	1.671553239001246	1.741723030659494
x_7	1.730476676834236	1.733914969299496	1.751426104001809
x_8	1.778175093274885	1.779008446773090	1.783260555044720
x_9	1.811383620670047	1.811581954777809	1.812594366476930
x_{10}	1.835785725863641	1.835832200041403	1.836069454084732
x_{11}	1.854499703777869	1.854510450354877	1.854565313616684
x_{12}	1.869330934476699	1.869333391459781	1.869345934874736
x_{13}	1.881388597297889	1.881389153571834	1.881391993470375
x_{14}	1.891391927950045	1.891392052828745	1.891392690361614
x_{15}	1.899828885716883	1.899828913543565	1.899829055604828
x_{16}	1.907042982024361	1.907042988184493	1.907043019633308
...	...	...	...
...	...	...	...
x_{96}	1.986347367976860	1.986347367976860	1.986347367976860
x_{97}	1.986489594012439	1.986489594012439	1.986489594012439
x_{98}	1.986628887104372	1.986628887104372	1.977033254105798
x_{99}	1.986765337056956	1.986765337056956	1.986765337056956
x_{100}	1.986899030044945	1.986899030044945	1.986899030044945

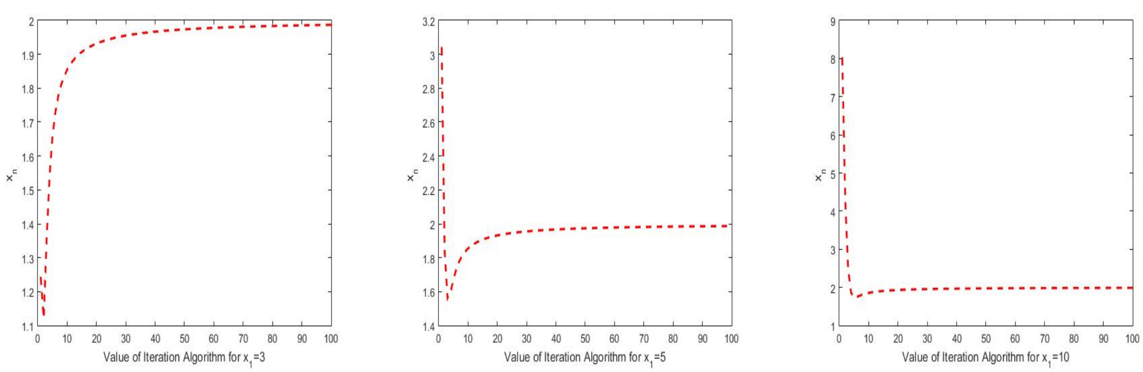

Figure 2: The graph and table of $\{x_n\}$ with different initial value .

Example 5.3. *Let R be the real line with usual norm $\|.\|$ and $C = [-1, 1]$. Define the map $T : C \to C$ as follows:*

$$Tx = \begin{cases} -2sin\frac{x}{2}, & x \in [0, 1] \\ 2sin\frac{x}{2}, & x \in [-1, 0) \end{cases}$$

For any $x, y \in [0, 1]$ or $x, y \in [-1, 0)$, we have

$$\|Tx - Ty\| = 2|sin\tfrac{x}{2} - sin\tfrac{y}{2}| \leq |x - y|.$$

Also, if $x \in [0, 1]$ and $y \in [-1, 0)$ or $x \in [-1, 0)$ and $y \in [0, 1]$, then we have

$$\|Tx - Ty\| = 2|sin\tfrac{x}{2} + sin\tfrac{y}{2}| = 4|sin\tfrac{x+y}{4}cos\tfrac{x-y}{4}| \leq |x + y| \leq |x - y|$$

This implies that T is a asymptotically nonexpansive mapping with $k_n = 1$. It is easy to observe that F$(T) = \{0\}$. We can choose $f(x) = \frac{1}{4}x$ as a contraction mapping with constant $\rho = \frac{1}{4}$ and $\alpha_n = \frac{1}{n}$ which satisfies the conditions $(C1) - (C4)$ in algorithm 3.1.

$Iterate$	$x_1 = 0.2$	$x_1 = -.03$
x_2	0.028571428571429	−0.042857142857143
x_3	0.000065572997730	−0.003070901681365
x_4	0.000001661017372	−0.000153089380610
x_5	0.000000030478528	−0.000006013282726
x_6	0.000000000419005	−0.000000198666311
x_7	0.000000000004394	−0.000000005745093
x_8	0.000000000000035	−0.000000000149348
x_9	0.000000000000000	−0.000000000003557
x_{10}	0.000000000000000	−0.000000000000079
x_{11}	0.000000000000000	−0.000000000000002
x_{12}	0.000000000000000	0.0000000000000000

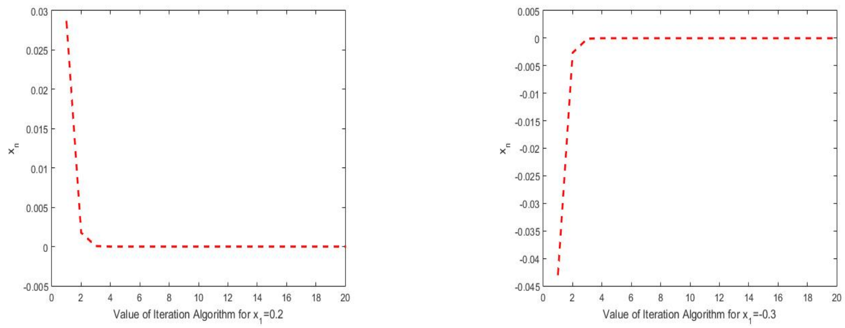

Figure 3: The graph and table of $\{x_n\}$ with different initial value .

Acknowledgement. Authors are very thankful to both the learned referees for pointing out many omissions and valuable comments.

References

[1] M. A. Alghamdi, N. Shahzad, and H. K. Xu, *the implicit midpoint rule for nonexpansive mappings*, Fixed Point :eory and Applications, (2014).

[2] S. S. Chang, Y. J. Cho and H. Zhou, *Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings*, J. Korean Math. Soc., 38 (2001), 1245- 1260.

[3] C.E. Chidume and M.O. Nnakwe, *A strong convergence theorem for an inertial algorithm for a countable family of generalized nonexpansive maps*, Fixed Point Theory, 21(2)(2020) 441- 452.

[4] A. Genel and J. Lindenstraus,*An example concering fixed points*. Israel J. Math. 22, 1 (1975), 81 - 86.

[5] K. Geobel, W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, (1990).

[6] J. Gornicki, *Weak convergence theorems for asymptotically nonexpansive map- pings in formly convex Banach spaces*, Comment Math. Univ. Carolin. 30 (1989),249- 485.

[7] S.J. He, Y.D. Mao, Z.Zhou and J.Q.Zhang, *The generalized viscosity implicit rules of asymptotically nonexpansive mappings in Hilbert spaces*, Appl. Math. Sci. 12(2017),549- 560.

[8] W.A. Kirk, *Fixed point theorems for non-lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math.17(1974)

[9] W.A. Kirk, C.Martinez Yanes and S. S. Shin, *Asymptotically nonexpansive mappings*, Nonlinear Anal.33(1988)1-12

[10] T. F. Li and H. Y. Lan, *On New Picard-Mann Iterative Approximations with Mixed Errors for Implicit Midpoint Rule and Applications*,Journal of Function Spaces(2019), Article ID 4042965, 13 pages

[11] T.C. Lim and H. K. Xu, *fixed point theorems for asymptotically nonexpansive mappings*, Nonlinear Anal.22(1994)1345-1355

[12] P. K. lin, K.K.Tan and H. K.Xu, *demiclosedness principle and asymptotic behaviour for a Asymptotically nonexpansive mappings*, Nonlinear Anal. 24(1995) 926-946.

[13] Z. Liu,R. P. Agarwal, C. Feng and S. M. Kang, *Weak and strong convergence theorems of common fixed points for a pair of nonexpansive and asymptotically nonexpansive mappings* Acta Univ. Palacki. Olomuc., Fac. rer.nat. Math. 44 (2005), 83 - 96.

[14] Z. Liu,R. P. Agarwal, C. Feng and S. M. Kang,*Weak and strong convergence for common fixed points of a pair of nonexpansive and asymptotically nonexpansive mappings* Taiwanese J. Math. 11, 1 (2007), 27 -42.

[15] P. Luo, G. Cai and Y. Shehu, *The viscosity iterative algorithms for the implicit midpoint rule of nonexpansive mappings in uniformly smooth Banach spaces*, J. Inequal Appl. 2017;2017(1):154.

[16] C. Pan and Y. Wang, *Generalized Viscosity Implicit Iterative Process for Asymptotically Non-Expansive Mappings in Banach Spaces*, Mathematics 7(2019), 1-13.

[17] S. H. Rizvi, *General Viscosity Implicit Midpoint Rule For Nonexpansive Mapping*,Filomat 35:1 (2021), 225- 237.

[18] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. 158 (1991), 407- 413.

[19] J. Schu, *Weak and strong convergence to fixed points of asymptotically non- expansive mappings*, Bull. Austral. Math. Soc. 43, 1 (1991), 153-159.

[20] H. K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., 66 (2002), 240–256.

[21] H. K. Xu, M. A. Alghamdi, and N. Shahzad, *the viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces*, Fixed Point Theory and Applications, (2015).

[22] Y. Yao, N. Shahzad, and Y.C. Liou, *Modified semi-implicit midpoint rule for nonexpansive mappings*, Fixed Point :eory and Applications, 166(2015).

[23] L.C. Zhao, S.S. Chang and C.F. Wen, *Viscosity approximation methods for the implicit midpoint rule of asymptotically nonexpansive mappings in Hilbert spaces*, J. Nonlinear Sci. Appl., 9(1) (2016) 4478- 4488.