Hyper Wiener Index and Connectivity Index of a Fuzzy Graph

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Abstract

Hyper Wiener index of fuzzy graphs and edge deleted fuzzy subgraphs are proposed in this article. A relationship between Connectivity index and Hyper Wiener index of fuzzy graphs is obtained.

1 Introduction

The idea of Fuzzy graphs (FGs) was introduced by Rosenfeld [12] after the landmark work of Zadeh's [16]. The idea of FGs was separately studied by Yeh and Bang [15]. The relevant literature of graphs and FGs can be seen in [6, 10, 13].

Bhutani and Rosenfeld [2, 3] gave an idea of strong arcs and fuzzy end nodes. Mathew and Sunitha [9] classified arcs of a FG as an $\alpha - strong, \beta - strong$ and $\delta-$ edge.

In [14] Wiener index (WI) was first investigated by Harold Wiener when he was studying about the boiling point of paraffin. It is a topological index and important from an application point of view.

Mathew and Jicy [7] introduced different concepts for weighted graphs like Connectivity index (CI) and WI. Binu et al. [4] further investigated CI of FGs. HWI of graph operations discussed by Khalifeh and Ashrafi [8]. It is the distance-dependent graph invariants. HWI is used for the structure descriptor of organic compounds. It is the generalization of WI introduced by Randic [11]. Construction of this paper is as follows: Section 1 contains some basic notations and terminologies of FGs which are required to understand the HWI of FGs. Definition of HWI of FG and its subgraph with an example is in section 2. A relationship between HWI and CI of FG is obtained in section 3. Section 4 describes HWI of edge deleted fuzzy subgraphs (FSs).

Below we give some basic definitions from [4, 5].

Let M be a set. A pair $G = (\kappa, \omega)$ is a FG, where κ and ω are fuzzy subsets of M and $M \times M$ respectively such that $\omega(l, m) \leq \kappa(l) \wedge \kappa(m)$. The underlying graph of $G = (\kappa, \omega)$ is expressed by $G^* = (\kappa^*, \omega^*)$, where $\kappa^* = \{v \in M | \kappa(v) > 0\}$ and $\omega^* = \{(u, v) \in M \times M | \omega(u, v) > 0\}$. Here minimum is represented by $\wedge$. We denote an element (l, m) of ω called an edge of G by lm. Vertices of G are represented by κ.

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A graph H is called a FS of G if $\rho(l) = \kappa(l)\,\forall\, l \in \rho$ and $\tau(lm) = \omega(lm)\,\forall\, lm \in \tau$. A path P of length t in a graph G is a sequence of distinct nodes $p_0, p_1, p_2, ..., p_t$, where $\omega(p_{i-1}p_i) > 0$ for $i = 1, 2, ..., t$. If $\omega(xy) = \wedge\{\omega(xy)|\omega(xy) > 0\}$ then this type of edge is weakest edge and the membership value of the weakest edge is called the strength of the path P. A FG is connected if strength of connectedness $(u, v) > 0$ for all $u, v \in \kappa^*$. A connected FG, $G = (\kappa, \omega)$ is said to be a fuzzy tree (FT) if it has a spanning fuzzy subgraph $E = (\rho, \tau)$ which is a tree, where $\forall\, lm$ not in E, $\exists$ a path from l to m in E, whose strength is $> \omega(lm)$.

The strength of connectedness between two nodes l and m is the greatest of strengths of all paths between l and m. It is denoted by $CONN_G(l, m)$. P is said to be strongest $l - m$ path if the strength of path $P = CONN_G(l, m)$. An edge lm of FG is said to be $\alpha - strong$ if $\omega(lm) > CONN_{G-lm}(l, m)$. An edge lm of FG is said to be $\beta - strong$ if $\omega(lm) = CONN_{G-lm}(l, m)$. An edge lm of FG is said to be $\delta - edge$ if $\omega(lm) < CONN_{G-lm}(l, m)$. An $\alpha - strong$ or $\beta - strong$ edge is said to be strong edge. If all edges of a path P are strong then this type of path is said to be strong path. A FG, G is said to be complete if $\forall\, l, m \in \kappa^*$, $\omega(lm) = \kappa(l) \wedge \kappa(m)$. A geodesic in a FG, G is a strong path P from l to m if there is no shorter strong path from l to m. The sum of membership values of all arcs in the shortest path is called weight of a geodesic. In a graph G a geodesic is a path of minimum length. A graph G is said to be $r - regular$ if all of its vertices have degree r.

Proposition 1.1. *[4] A complete FG, G does not contains $\delta - edges$.*

Proposition 1.2. *[4] If $H = (\rho, \tau)$ is a FS of G, then $CONN_H(l, m) \leq CONN_G(l, m)$ for any two $l, m \in \rho^*$.*

Connectivity is the most important notion in FGs. We check stability of a FG by finding the strength of connectedness between each pair of its vertices.

Definition 1.1. *[4] Consider G be a FG. The formula to calculate $CI(G)$ is given by*

$$CI(G) = \Sigma_{l,m\in\kappa^*}\kappa(l)\kappa(m)CONN_G(l, m)$$

where $CONN_G(l, m)$ shows connectedness strength of l and m.

Definition 1.2. *[7] Consider G be a graph. The distance $d(l, m)$ between two vertices $l, m \in V(G)$ is the minimum number of edges in a path between l and m in G.*

Example 1.1. *Consider G be FG with $\kappa^* = \{l, m, n\}$ such that $\kappa^*(l) = \kappa^*(m) = \kappa^*(n) = 1$ and having all strong edges. $\omega(lm) = 0.3$, $\omega(ln) = 0.3$ and $\omega(mn) = 0.3$. The membership values of edges are called weights of edges. There are two paths between l and m which are*

1. $l - m = 0.3$
2. $l - n - m = 0.3 + 0.3 = 0.6$

Here, $l-m$ is the shortest path and the minimum sum of weights of shortest path between l and m is $d_s(l, m) = min\{0.3, 0.6\} = 0.3$

2 Hyper Wiener index of fuzzy graph.

Now, in this section we are going to introduce HWI for FGs. HWI is a generalization of WI. To calculate HWI of any FG we need to compute the $d_s(l, m)$ and $d_s^2(l, m)$ between every pair of vertices of FG. The formal definition of HWI of a FG is given below.

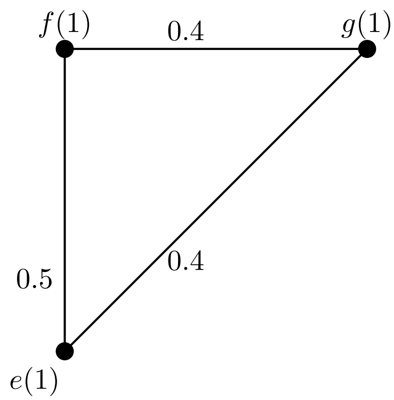

Figure 1: Fuzzy graph with $HWI = 0.935$

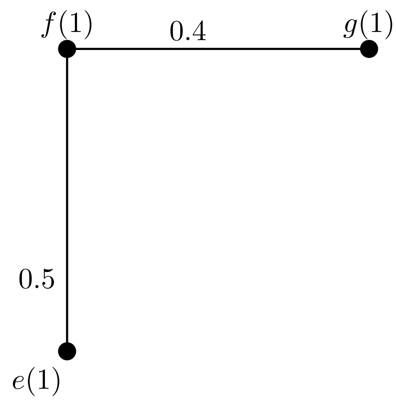

Figure 2: Fuzzy subgraph with $HWI = 1.51$

Definition 2.1. *Let $G = (\kappa, \omega)$ be a FG. To calculate HWI of FG the formula is as follows*

$HWI(G) = \frac{1}{2}\Sigma_{l,m\in\kappa^*}[\kappa(l)\kappa(m)d_s(l,m) + \kappa(l)\kappa(m)d_s^2(l,m)]$

where $d_s(l,m)$ is the minimum sum of weights of shortest path from l to m. Also, $\kappa(l)$ and $\kappa(m)$ are the membership values of the vertices.

Example 2.1. *Consider G be a FG given in Figure 1 having all strong edges and $\kappa^* = \{e, f, g\}$; and $\kappa^*(e) = \kappa^*(f) = \kappa^*(g) = 1$, $\omega(ef) = 0.5$, $\omega(eg) = 0.4$ and $\omega(fg) = 0.4$. The HWI of G is 0.935.*

The HWI of a FS, $H = (\rho, \tau)$ of $G = (\kappa, \omega)$ need not be less than or equal to $HWI(G)$. It can be seen in next example

Example 2.2. *Consider H be a FS of G given in Figure 2 and $\rho^* = \{e, f, g\}$; and $\rho^*(e) = \rho^*(f) = \rho^*(g) = 1$, $\tau(ef) = 0.5, \tau(eg) = 0$ and $\tau(fg) = 0.4$. The HWI of H is $1.51 > HWI(G) = 0.935$.*

Theorem 2.1. *Let $G = (\kappa, \omega)$ and $G' = (\kappa', \omega')$ be isomorphic FGs, then $HWI(G) = HWI(G')$.*

Proof. Let $G = (\kappa, \omega)$ and $G' = (\kappa', \omega')$ be two isomorphic FGs. Then $\exists$ a bijective map t from κ^* to κ'^* such that for any $u \in \kappa^*$ and for any $yz \in \omega^*$, $\kappa(u) = \kappa'(t(u))$ and $\omega(yz) = \omega'(t(y)t(z))$. For $y, z \in \kappa^*$, let $P_{y,z}$ be the path which serves $d_s(y,z) + d_s^2(y,z)$. Corresponding to each edge $pq \in P_{y,z}$, there corresponds an edge $t(p)t(q)$ in G' such that $\omega(pq) = \omega'(t(p)t(q))$. Therefore, we can directly say that corresponding to the path $P_{y,z} \in G$, $\exists$ a path $P'_{t(x),t(y)} \in G'$ such that the sum and sum of squares of membership values of arcs of P' is weakest among all geodesics from $t(p)$ to $t(q)$. Therefore, $d_s(y,z) + d_s^2(y,z) = d_s(t(y), t(z)) + d_s^2(t(y), t(z))$. Hence

$HWI(G) = \frac{1}{2}\Sigma_{y,z\in\kappa^*}[\kappa(y)\kappa(z)d_s(y,z) + \kappa(y)\kappa(z)d_s^2(y,z)]$

$= \frac{1}{2}\Sigma_{t(y),t(z)\in\kappa'^*}[\kappa' t(y)\kappa' t(z)d_s(t(y), t(z)) + \kappa' t(y)\kappa' t(z)d_s^2(t(y), t(z))$

$= HWI(G')$ □

3 Hyper Wiener index and connectivity index of a fuzzy graph.

In this section a connection between HWI and CI is provided. In some FGs, the $HWI(G) < CI(G)$ while in some FGs, the $HWI(G) > CI(G)$ but $HWI(G)$ cannot be equal to $CI(G)$. It can be seen in coming examples.

Example 3.1. *Consider G be a FG given in Figure 3 having all strong edges except $\delta-$ edge $= \omega(gh)$ and $\kappa^* = \{e, f, g, h, i\}$; and $\kappa^*(e) = \kappa^*(f) = \kappa^*(g) = \kappa^*(h) = \kappa^*(i) = 1$ $\omega(ef) = 0.6, \omega(fg) = 0.7, \omega(gh) = 0.2, \omega(hi) = 0.3$ and $\omega(ie) = 0.5$. The HWI of $G - \delta$ edge is 12.47 while $CI - \delta$ edge of G is 4.6 which is less than $HWI(G)$.*

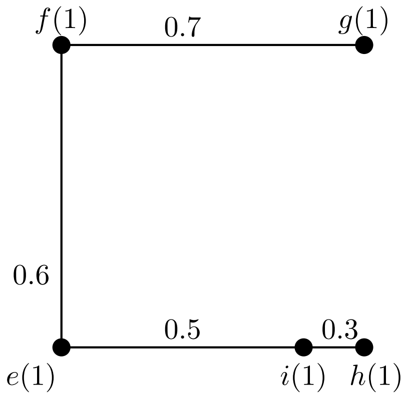

Figure 3: Fuzzy graph with $CI < HWI$

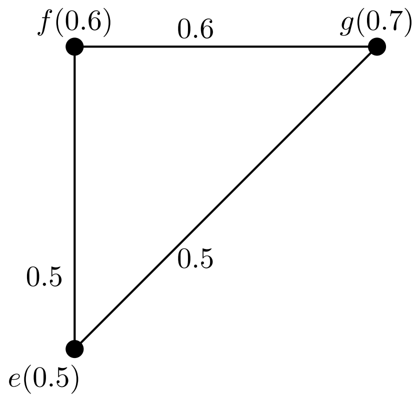

Figure 4: A complete 2-regular fuzzy graph with $CI > HWI$

Proposition 3.1. *Let G be a complete r-regular FG with $\mid \kappa^* \mid \geq 3$ and $\kappa(a) \leq \kappa(b) \leq \kappa(c)$. Then $CI(G) > HWI(G)$.*

Example 3.2. *Consider G be a 2-regular complete FG given in Figure 4 having all strong edges with $\kappa^* = \{e, f, g\}$; and $\kappa^*(e) = 0.5 < \kappa^*(f) = 0.6 < \kappa^*(g) = 0.7$ and $\omega(ef) = 0.5, \omega(fg) = 0.5, \omega(eg) = 0.6$. The HWI of G is 0.445 while CI of G is 0.577 which is greater than $HWI(G)$.*

Theorem 3.1. *Let G be a FT having $\mid \kappa^* \mid \geq 3$. Then $HWI(G) > CI(G)$.*

Proof. Consider a FT with $\mid \kappa^* \mid \geq 3$. There exists a single strong path P joining every pair of nodes which is also the single strongest path in a fuzzy tree. For any $l, m \in \kappa^*$, $\frac{d_s(l,m)+d_s^2(l,m)}{2}$ is the average sum and sum of squares of all membership values in the single strongest path P joining l and m on the other side $CONN_G(l, m)$ is value of the weakest arc of P. This implies $CONN_G(l, m) < \frac{d_s(l,m)+d_s^2(l,m)}{2}$. Thus for
$\frac{1}{2}\Sigma_{l,m\in\kappa^*}[\kappa(l)\kappa(m)d_s(l, m) + \kappa(l)\kappa(m)d_s^2(l, m)] > \Sigma_{l,m\in\kappa^*}\kappa(l)\kappa(m)CONN_G(l, m)$,
we have $HWI(G) > CI(G)$. □

Example 3.3. *Assume G is a fuzzy tree given in Figure 5 having all strong edges with $\kappa^* = \{e, f, g, h\}$; and $\kappa^*(e) = 0.5, \kappa^*(f) = 0.4, \kappa^*(g) = 0.3$ and $\kappa^*(h) = 0.5$; $\omega(ef) = 0.4, \omega(fg) = 0.3, \omega(gh) = 0.3$ The HWI of G is 0.513 while CI of G is 0.341 which is less than $HWI(G)$.*

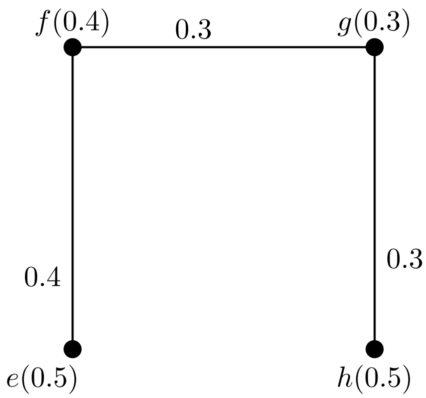

Figure 5: A fuzzy tree with $HWI > CI$

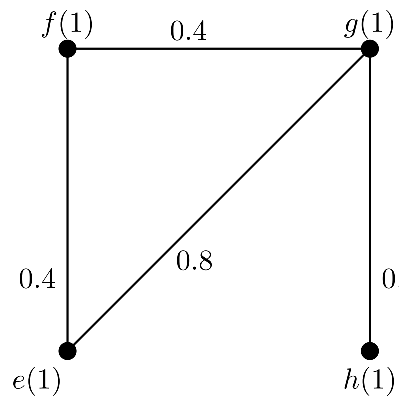

Figure 6: HWI of an α -strong and β - strong arcs deleted FG

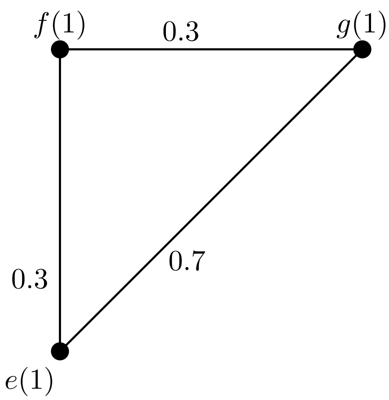

Figure 7: HWI of an α -strong arc deleted FG

4 Hyper Wiener indices of arc deleted fuzzy subgraphs.

Many connectivity parameters have low values for the arc removing subgraphs of a FG. The HWI of a FG can be increased or decreased by deleting of a strong arc.

Example 4.1. *Consider G be a FG given in Figure 6 having $\kappa^* = \{e, f, g, h\}$; with $\omega(eg) = 0.8$ is an α - strong edge; $\omega(ef) = 0.4, \omega(fg) = 0.4, \omega(gh) = 0.4$ are β - strong edges and $\kappa^*(e) = \kappa^*(f) = \kappa^*(g) = \kappa^*(h) = 1$. Then $HWI(G) = HWI(G - eg) = 3.6$, $HWI(G - ef) = 4.64 > HWI(G)$, $HWI(G - fg) = 6 > HWI(G)$ and $HWI(G - ch) = 1.28 < HWI(G)$.*

Proposition 4.1. *Let G be a FG having all edges are α - strong and β - strong with all β - strong edges have equal membership values. Then $HWI(G) = HWI(G - \alpha)$ strong - edge.*

Proof. Let G be a FG having an α - strong edge and remaining edges are β - strong having equal membership values. Therefore, $\omega^*(lm) = d_s(l, m)$. Thus $HWI(G) = HWI(G - lm)$. □

Example 4.2. *Consider G be a FG given in Figure 7 having $\kappa^* = \{e, f, g\}$; with $\omega(eg) = 0.7$ is an α - strong edge;*

$\omega(ef) = 0.3, \omega(fg) = 0.3$ *are* β *- strong edges having equal membership values and* $\kappa^*(e) = \kappa^*(f) = \kappa^*(g) = 1$. *Then* $HWI(G) = (G - eg) = 0.87$.

5 Conclusion.

Connectivity parameters are very useful to measure connectedness of FGs. HWI of FGs and its related results are discussed in this paper. We explained the HWI of vertex and arc deleted subgraphs. A link between CI and HWI is also obtained. For a future work, the different characteristics of topological indices ideas can be transformed from crisp theory to fuzzy theory.

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