Using Parametric Continuity Method for Solving Fredholm Nonlinear Integral Equations

Majied Babaee[a], Mahmoud Paripour[a,*], Nasrin Karamikabir[b]

[a]*Department of Mathematics, College of Basic Sciences, Hamedan Branch, Islamic Azad University,Hamedan, Iran.*

[b]*Department of Computer Engineering and Information Technology, Hamedan University of Technology,Hamedan, Iran.*

Article Info

Keywords

Parametric continuity Method

Fredholm

Nonlinear Integral Equations

Approximate Solution

Article History

Received:2021 March 19

Accepted:2021 December 24

Abstract

This study is based on the article "Parameter Duration" Method for Solving Nonlinear Fredholm Integral Equations of the Second kind and is collected from the writings of Nineh and Vitkha. In this paper, first, the Fredholm nonlinear integral equation of the second type is solved using the parametric continuity method. Next, the parametric continuity method is introduced to solve the turbulent nonlinear integral equation of the second type, which is an extension of the paradoxical mapping method. Also, the parametric continuity method is applied to solve the nonlinear integral equation of the second type. Lastly, sample examples are given to show the effectiveness and convenience of the parametric continuity method.

1 Introduction

The parametric continuity method is employed in Fredholm integral equations to find the answers to some of the problems of change calculus. The main goal is to find the answer to the nonlinear equation that gives rise to the problem of the calculus of variations. It is generally challenging to achieve an analytical approximation of complex nonlinear problems. The expression for nonlinear problems is usually determined by the type of nonlinear equations, and the convergence region of the answer series is heavily dependent on physical parameters. It is well known that nonlinear problem analysis approximations become more ineffective as nonlinear properties become more complex, and turbulence approximations are correct only for nonlinear problems with weak nonlinear properties. In this study, an analytical method for nonlinear problems is proposed named the parametric continuity method in Fredholm nonlinear integral equations and the homotopy analysis technique. It is revealed that even if a nonlinear problem has a unique answer, it may have an infinite number of answers whose convergence region and convergence rate depend on a certain parameter. Unlike all prior analysis methods, the homotopy analysis technique provides a simple approach to control and regulate the convergence region and convergence rate of solution series of nonlinear problems. Wazwaz et al. proposed the method of sequential approximation of the linear and nonlinear integral equations of the function $f(x)$, whose root is to be determined, are written as a$f(x)$ =$g(x) - x$. The roots of the equation $f(x) = 0$ are the cross-sections of the function g(x) and the straight line indicating the x. The iteration technique begins with giving the initial conjecture x_0 to the procedure. The result of the substitution of x_0 in $g(x)$, that is $g(x_0)$, is considered as the next conjecture.

* Corresponding Author's E-mail: paripour@hut.ac.ir(M. Paripour)

This procedure is repeated until two consecutive values are close enough to each other [1]. Ezzati and Najafalizadeh have developed polynomial approximation methods using various basic functions, including Chebyshev and Legendre polynomials for the Bernoulli order functions. In modern decades, the use of continuous orthogonal functions (including Legendre, Chebyshev, Hermit polynomials, etc.) to solve integral equations and integral ?differential equations has gained much attention. The central highlight of this method is that it transforms the assumed equations into a system of algebraic equations. In these systems, the direct placement of the approximation expansion in the given equation along with the completion conditions is adopted [2]. Ninh [3, 4] has examined the parametric continuity method for solving second type operator equations with a sum of two operators (Nineh, 2011). Vetekha [5] suggested the application of the parametric continuity method to solve the boundary value problem for ordinary second-order differential equations (Vetekha, 2000). Nevertheless, the parametric continuity method for the nonlinear integral equation of the second type has not been studied. In this study, the application of the parametric continuity method to solve the Fredholm nonlinear integral equation in the second type, in general, is investigated. Also, a parametric continuity method is introduced to solve the nonlinear Fredholm turbulent equations of the second type. This proposition can be considered as an extension. The point of using the paradoxical mapping method and the parametric continuity method is to solve the Fredholm nonlinear integral equation of the second type, which is obtained by comparing, calculating, and determining the error.

Fredholm's nonlinear integral equation of the second type Here we consider the parametric continuity method for Fredholm's integral equations. Hold the Voltra integral equation below.

$$x(t)=\int_0^t \kappa(t,s,x(s))ds+f(t),t\in[0,T] \tag{1.1}$$

Where k:$S\times\ lR\rightarrow lR$, f: $[0,T]\rightarrow lR$ are the given functions, and the set S is defined as

$$S=\{(t,s):0\leq s\leq t\leq T\}.$$

Now consider Fredholm's nonlinear integral:

$$x(t)=\int_a^b K(t,s,x(s))ds=f(t)\quad a\leq t\leq b) \tag{1.2}$$

Where $K(t,s,x)$ and $f(t)$ are known functions and $x(t)$ is an unknown function which we want to determine.

Theorem 1.1. *Consider that the following terms are sufficient:*

1) $f(t)\in L^2[a,b]$

2) $K(t,s,x)$*receives the terms of a Lipshitz of the following type:*

$$|K(t,s,x)-K(t,s,y)|\leq|\phi(t,s)||x-y|$$

For $a\leq t,s\leq b$*and for every real* x,y*, where ever*

$$\int_a^b\int_a^b|\phi(t,s)|^2dsdt=L^2<\infty$$

3) $K(t,s,x)$ *meets the conditions.*

$$\int_a^b \{\int_a^b [K(t,s,x(s)) - K(t,s,y(s))]ds\}[x(t)-y(t)]dt \geq 0$$

Then, Fredholm's nonlinear integral equation of the second type has a unique solution.

$$x(t) \in L^2[a,b]$$

Theorem 1.2. *Assume that the conditions of Theorem 1 are sufficient. Then, the sequence of approximate solutions* $\{x(n,N)(t)\} \equiv x_n(t)$,$n{=}\ 1,2,...$ *formed by iterative processes converges with the exact solution* $x(t) \in L^2[a,b]$ *of Fredelm's nonlinear integral Equation. Besides, the following estimates are available.*

$$\|x(n,N)(t) - x(t)\| \leq \frac{q^{n+1}}{1-q}\frac{e^{q^N}-1}{e^q-1}\|f(t)\| \tag{1.3}$$

Where N *is the least natural number that* $q{=}\frac{L}{N}{<}\ 1$,$n{=}\ 1,2,...$ *holds.*

Consider the following peripheral problems.

Problem 1.1. *(One step with parameter* ε*). Consider Fredholm's nonlinear integral equation:*

$$x(t) + \varepsilon_0(Fx)(t){=}f(t) \tag{1.4}$$

Since the operator $\varepsilon_0 F$ *is a paradoxical operator with the shrinkage coefficient* $q{<}\ 1$*, it is concluded that equation* (**??**) *is a* $x(\varepsilon_0)(t)$*unique solution for every* $f(t) \in L^2[a,b]$*. Approximate solutions of equation* (**??**) *are obtained as a result of the standard iteration process.*

$$x_{i+1}(t){=}-\varepsilon_0(Fx_i)(t) + f(t). \qquad i{=}\ 0,1,2,...x_0(t){=}f(t) \tag{1.5}$$

According to the paradoxical mapping principle we have:

$$\|x_n(t) - x(\varepsilon_0)(t)\| \leq \frac{q^n}{1-q}\|x_1(t) - x_0(t)\|$$

Since $(F0)(t) = 0$*:*

$$\|x_1(t) - x_0(t)\|{=}\|-\varepsilon_0(Ff)(t) + f(t) - f(t)\|{=}\|\varepsilon_0(Ff)(t) - \varepsilon_0(F0)(t)\| \leq q\|f(t)\|$$

Therefore, the approximate error $x_n(t)$ *of problem* (**??**) *gives*

$$\Delta_1(n) \equiv \partial_1(n) \equiv \|x_n(t) - x(\varepsilon_0)(t)\| \leq \mu(n)$$

Where

$$\mu(n) = \frac{q^{n+1}}{1-q}\|f(t)\| \tag{1.6}$$

Problem 1.2. *(step to of parameter) Consider Fredholm's nonlinear integral equation:*

$$x(t) + 2\varepsilon_0(Fx)(t)=f(t) \tag{1.7}$$

Here, we have a change of the variable:

$$x^{(1)}(t)=x(t) + \varepsilon_0(Fx)(t) \equiv (G_1x)(t) \tag{1.8}$$

This equation has a unique solution for every $x^{(\textbf{??})}(t) \in L^2[a,b]$. For example, the operator G_1^{-1} is determined in the entire $L^2[a,b]$ space.

Proof. Because of the uniformity of operator F, operator G has a continuous Lipshitz coefficient equal to 1. In fact, for every $x^{(\textbf{??})}(t).\overline{x}^{(\textbf{??})}(t) \in L^2[a,b]$we have:

$$\begin{aligned}\|(G_1^{-1}x^{(1)})(t) - (G_1^{-1}\overline{x}^{(1)})(t)\| &= \|x(t) - \overline{x}(t)\| \\ &\leq \|x(t) - \overline{x}(t) + \varepsilon_0[(Fx)(t) - (F\overline{x})(t)]\| \\ &= \|x^{(1)}(t) - \overline{x}^{(1)}(t)\|\end{aligned}$$

After the change of variable in equation (**??**), equation (**??**) is rewritten as:

$$(G_2x^{(1)})(t) \equiv x^{(1)}(t) + \varepsilon_0(FG_1^{-1}x^{(1)})(t)=f(t) \tag{1.9}$$

The operator $\varepsilon_0 FG_1^{-1}$ is a paradoxical operator with a shrinkage coefficient of $q< 1$. Thus, equation (**??**) has a unique solution for every $f(t) \in L^2[a,b]$. Hence, equation (**??**) has a unique solution $x(2\varepsilon_0)(t)$ for every $f(t) \in L^2[a,b]$. The approximate solutions of the integral equation (**??**) are obtained as a result of the standard iteration process.

$$x^{(1)}_{(j+1)}(t)= - \varepsilon_0(FG_1^{-1}x_j^{(1)})(t) + f(t)\; j= 0,1,2,\ldots .x_0^{(1)}(t)=f(t) \tag{1.10}$$

Meanwhile, in every step of a higher iteration process, the standard iteration process will be used again when calculating $G_1^{-1}x_j^{(\textbf{??})}(t)$.

$$x_{i+1}(t)= - \varepsilon_0(Fx_i)(t) + x_j^{(1)}(t). \qquad i= 0,1,2,\ldots .x_0(t)=x_j^{(1)}(t) \tag{1.11}$$

Hence, the approximate solution of the equation (**??**) can be obtained by below steps:

$$\begin{aligned} &x_{i+1}(t)= - \varepsilon_0(Fx_i)(t) + x_j^{(1)}(t). && i= 0,1,2,.. \\ &x^{(1)}_{(j+1)}(t)= - \varepsilon_0(FG_1^{-1}x_j^{(1)})(t) + f(t) && j= 0,1,2,\ldots .x_0^{(1)}(t)=f(t)\end{aligned} \tag{1.12}$$

$G_1^{-1}x_j^{(\textbf{??})})(t)$ values are obtained using the 11-iteration process with error $\mu(n)$. Since $\varepsilon_0 F$ is a paradoxical operator with a shrinkage coefficient of $q< 1$, error $\mu(n)$ when determaining the operator argument $\varepsilon_0 F$equals $q\mu(n)$ when determining the right side of $f(t)$ in equation (**??**). On the other hand, the operator G_2^{-1} has a continuous Lipshitz coefficient equalling 1. Based on theorem **(??)**, for every $f(t),\overline{f}(t) \in L^2[a,b]$ we have:

$$\begin{aligned}&\|(G_2^{-1}f)(t)-(G_2^{-1}\overline{f})(t)\|=\|x^{(1)}(t)-\overline{x}^{(1)}(t)\|\\&=\|x(t)-\overline{x}(t)+\varepsilon_0[(Fx)(t)-(F\overline{x})(t)]\|\\&\leq \|x(t)-\overline{x}(t)+2\varepsilon_0[(Fx)(t)-(F\overline{x}^{(1)})(t)]\|\\&=\|x^{(1)}(t)-\overline{x}^{(1)}(t)+\varepsilon_0[(FG_1^{-1}x^{(1)})(t)-(FG_1^{-1}\overline{x}^{(1)})(t)]\|=\|f(t)-\overline{f}(t)\|\end{aligned}$$

Therefore, the insertion of error $q\mu(n)$ in the right side of the equation (3-10) leads to the production of an error higher than $q\mu(n)$ in the corresponding $x^{(\mathbf{??})}(t)$ solution. The error of an iteration process in the calculation of $x^{(\mathbf{??})}(t)$ is $\frac{q^n}{1-q}\|x_1^{(\mathbf{??})}(t)-\overline{x}_0^{(\mathbf{??})}(t)\|$, since $(F0)(t)=0$, we have $(G_1 0)(t)=0(t)+\varepsilon_0(F0)(t)=0$. Therefore:

$$\begin{aligned}\|x_1^{(1)}(t)-\overline{x}_0^{(1)}(t)\|&=\|-\varepsilon_0(FG_1^{-1}f)(t)+f(t)-f(t)\|\\&=\|\varepsilon_0(FG_1^{-1}f)(t)-\varepsilon_0(FG_1^{-1}0)(t)\|\\&\leq q\|f(t)\|\end{aligned}$$

Then, the error of an iteration process in the calculation of $x^{(\mathbf{??})}(t)$ is $\frac{q^{n+1}}{1-q}\|f(t)\|=\mu(n)$.

Therefore,

$$\partial_2(n)\equiv\|x_n^{(1)}(t)-x^{(1)}(t)\|\leq q\mu(n)+\mu(n)=q\partial_1(n)+\mu(n)$$

Inverse switching, that is the change from variable $x^{(\mathbf{??})}(t)$ to variable $x(t)$, again introduces the error $\mu(n)$. Therefore, the errors of the approximate solutions of $x_n(t)$in problem **(??)** give the following estimation:

$$\Delta_2(n)=\|x_n(t)-x(2\varepsilon_0)(t)\|\leq q\mu(n)+2\mu(n)=\partial_2(n)+\partial_1(n)$$

By using similar arguments for the problem k:$x(t)+k\varepsilon_0(Fx)(t)=f(t)$,$k\in[1,N]$, the estimation can be obtained.

$$\square_K(n)=\|x_n(t)-x(k\varepsilon_0)(t)\|\leq\partial_k(n)+\partial_{k-1}(n)+...+\partial_1(n) \tag{1.13}$$

Where

$$\partial_h(n)\leq q[\partial_{h-1}(n)+...+\partial_1(n)]+\mu(n),\qquad 1\leq h\leq k \tag{1.14}$$

This relation should be rewritten as:

$$\partial_k(n)\leq\mu(n)+q\sum_{h-1}^{k-1}\partial_h(n).\partial_1(n)\leq\mu(n)\quad k=2,3,...N \tag{1.15}$$

By using the discrete analog of the well-known Bellman-Gronwall theorem for inequality 5, we have:

$$\partial_k(n)\leq\mu(n)\prod_{h-1}^{k-1}(1+q)\leq\mu(n)\prod_{h-1}^{k-1}e^q=\mu(n)\,e^{q(k-1)},\ k=1,2,3,...N$$

Hence, the error 14 estimations for the problem (k) can be written as:

$$\Delta_K(n)=\|x_n(t)-x(k\varepsilon_0)(t)\|\leq\sum_{h-1}^{k}\partial_h(n)\leq\mu(n)\sum_{h-1}^{k}e^{q(k-1)}=\mu(n)\,\frac{e^{kq}-1}{e^q-1}$$

By replacing N with K, the proof finishes. □

Fredholm's turbulent nonlinear integral equation of the second type

In this part, the parametric continuity method is proposed for solving non-linear turbulence.

The second type of Fredholm integral equation is as follows.

$$x(t)=\int_a^b K(t,s,x(s))ds+\int_a^b K_1(t,s,x(s))ds=f(t) \qquad a\le t\le b \tag{1.16}$$

Where $k(t,s,x).k_1(t,s,x)$ and $f(t)$ are known functions and $x(t)$ is an unknown function that is to be determined.

Now, Fredholm's turbulence nonlinear integral equation of the second type (**??**) will be examined under the following hypotheses.

$$1.f(t)\in L^2[a,b]$$

2. $k(t,s,x)$ receives the following Lipshitz condition.

$$|K(t,s,x)-K(t,s,y)|\le|\phi(t,s)||x-y|$$

For every $a\le t,s\le b$and every x,y within where

$$\int_a^b\int_a^b|\phi(t,s)|^2dsdt=L^2<\infty$$

3. $k(t,s,x)$ satisfies the following condition.

$$\int_a^b\{\int_a^b[k(t,s,x(s))-k(t,s,y(s))]ds\}[x(t)-y(t)]dbt\ge 0$$

4. $k_1(t,s,x)$ satisfies the following condition.

$$|k_1(t,s,x)-k_1(t,s,y)|\le|\phi(t,x)||x-y|$$

For every $a\le t,s\le b$ and every real x,y within, we have

$$\int_a^b\int_a^b|\phi(t,s)|^2dsdt=B^2,B<1$$

F_1, F_2 need to be defined in $L^2[a,b]$.

$$(Fx)(t)=\int_a^b k(t,s,x(s))ds,(F_1x)(t)=\int_a^b k_1(t,s,x(s))ds$$

It can be concluded from terms (**??**) and (**??**) that the operator F is uniform and is continuous Lipshitz with Lipshitz coefficient equal to L (see the proof of Theorem 3).

Based on the term (**??**), for every $x(t)$,$y(t) \in L^2[a,b]$ we have

$$\begin{aligned}|(F_1x)(t)-(F_1y)(t)|=&|\int_a^b[k_1(t,s,x(s))-k_1(t,s,y(s))]ds|\\ &\leq \int_a^b|k_1(t,s,x(s))-k_1(t,s,y(s))|ds\\ &\leq \int_a^b|\phi(t,s)||x(s)-y(s)|ds\end{aligned}$$

By using the Cauchy-Schwartz inequality we have:

$$\begin{aligned}\|(F_1x)(t)-(F_1y)(t)\|^2=&\int_a^b|(F_1x)(t)-(F_1y)(t)|^2dt\\ &\leq \int_a^b\int_a^b|\phi(t,x)|^2dsdt\int_a^b|x(s)-y(s)|^2ds=B^2\|x-y\|^2\end{aligned}$$

Therefore,

$$\|(F_1x)(t)-(F_1y)(t)\|\leq B\|x-y\|$$

Since $B<1$, F $_1$ is a paradoxical operator with a shrinkage coefficient of $\overline{q}=B<1$. A minimum natural number **N** is obtained so that $q=\varepsilon_0L<1$,$\varepsilon_0=\frac{1}{N}$. Equation (**??**) can be rewritten as:

$$x(t)+N\varepsilon_0(Fx)(t)+(F_1x)(t)=f(t) \tag{1.17}$$

Consider the following peripheral problems.

Problem 3. (N=1) Consider the following Fredholm's turbulent nonlinear integral equation.

$$x(t)+\varepsilon_0(Fx)(t)+(F_1x)(t)=f(t) \tag{1.18}$$

A change of variable should be done:

$$x^{(1)}(t)=x(t)+\varepsilon_0(Fx)(t)=(G_1x)(t) \tag{1.19}$$

For every $x(t)$,$\overline{x}(t)\in L^2[a,b]$ we have:

$$\|\varepsilon_0(Fx)(t)-\varepsilon_0(F\overline{x})(t)\|\leq \varepsilon_0L\|x(t)-\overline{x}(t)\|=q\|x(t)-\overline{x}(t)\|$$

Hence, ε_0F is a paradoxical operation with a shrinkage coefficient $q=\varepsilon_0L<1$.
Then, the equation (??) has a unique solution for every $x^{(\mathbf{??})}(t)\in L^2[a,b]$.
For example, the operator $(G_1^{-1}x^{(1)})(t)$ is determined in the entire $L^2[a,b]$ space and makes the operator F uniform. The operator G_1^{-1} is a continuous Lipshitz with a Lipshitz coefficient of 1. In fact, for every $x^{(\mathbf{??})}(t)$,$\overline{x}^{(\mathbf{??})}(t)\in L^2[a,b]$we have:

$$\begin{aligned}\|(G_1^{-1}x^{(1)})(t)-(G_1^{-1}\overline{x}^{(1)})(t)\|=&\|x(t)-\overline{x}(t)\|\\ &\leq\|x(t)-\overline{x}(t)+\varepsilon_0[(Fx)(t)-\varepsilon_0(F\overline{x})(t)\|\\ &=\|x^{(1)}(t)-\overline{x}^{(1)}(t)\|\end{aligned}$$

After the change of variable in equation (**??**), the equation (**??**) is represented as:

$$(Ax^{(1)})(t)=x^{(1)}(t)+(F_1G_1^{-1}x^{(1)})(t)=f(t) \tag{1.20}$$

For every $x^{(\mathbf{??})}(t)$,$\overline{x}^{(\mathbf{??})}(t) \in L^2[a,b]$ we have:

$$\begin{aligned}\|(F_1G_1^{-1}x^{(1)})(t) - (F_1G_1^{-1}\overline{x}^{(1)})(t)\| &\leq \overline{q}\|(G_1^{-1}x^{(1)})(t) - (G_1^{-1}\overline{x}^{(1)})(t)\| \\ &\leq \overline{q}\|x^{(1)}(t) - \overline{x}^{(1)}(t)\|\end{aligned}$$

Thus, $F_1G_1^{-1}$ is a paradoxical operator with a shrinkage coefficient of $\overline{q}< 1$. Then, the integral equation (4-5) has a unique solution for every $f(t) \in L^2[a,b]$. Therefore, equation (??) has a unique solution $x(\varepsilon_0)(t)$ for every $f(t) \in L^2[a,b]$. The approximate solutions of the equation (**??**) are obtained using the standard iteration process.

$$x^{(1)}_{(j+1)}(t) = -\,(F_1G_1^{-1}x_j^{(1)})(t) + f(t). \quad j{=}\,0,1,2,...x_0^{(1)}(t) = f(t)$$

Meanwhile, in every higher iteration step of calculating the $(G_1^{-1}x_j^{(1)})(t)$value, the standard iteration will be used twice.

$$x_{i+1}(t) = -\,\varepsilon_0(Fx_i)(t) + x_j^{(1)}(t) \quad ,i{=}\,0,1,2,...x_0(t) = x_j^{(1)}(t)$$

As a result, approximate solutions of the integral equation (**??**) can be found by iterative processes.

$$\begin{aligned}&x_{i+1}(t){=}-\varepsilon_0(Fx_i)(t) + x_j^{(1)}(t) \qquad ,i{=}\,0,1,2,... \\ &x^{(1)}_{j+1)}(t){=}-\,(F_1G_1^{-1}x_j^{(1)})(t) + f(t). \;\; j{=}\,0,1,2,...x_0^{(1)}(t){=}f(t)\end{aligned} \tag{1.21}$$

Problem 4. (N=2) considers the following Fredholm's turbulent nonlinear integral equation.

$$x(t) + 2\varepsilon_0(Fx)(t) + (F_1x)(t){=}f(t) \tag{1.22}$$

Two change of variables should be done:

$$\begin{aligned}&x^{(1)}(t) = x(t) + \mathring{a}_0(Fx)(t) \equiv (G_1x)(t) \\ &x^{(2)}(t){=}x^{(1)}(t) + \varepsilon_0(FG_1^{-1}x^{(1)})(t) \equiv (G_2x^{(1)})(t)\end{aligned} \tag{1.23}$$

For every $x^{(\mathbf{??})}(t)H\overline{x}^{(\mathbf{??})}(t) \in L^2[a,b]$ we have

$$\begin{aligned}\|\varepsilon_0(FG_1^{-1}x^{(1)})(t) - \varepsilon_0(FG_1^{-1}\overline{x}^{(1)})(t)\| &\leq \varepsilon_0L\|x^{(1)}(t) - \overline{x}^{(1)}(t)\| \\ &= q\|x^{(1)}(t) - \overline{x}^{(1)}(t)\|\end{aligned}$$

Therefore, $\varepsilon_0F_1G_1^{-1}$ is a Contraction operator with a shrinkage coefficient of $q< 1$. Therefore, the integral equation $x^{(\mathbf{??})}(t) + \varepsilon_0(FG_1^{-1}x^{(1)})(t) = x^{(\mathbf{??})}(t)$ has a unique solution for every $\overline{x}^{(\mathbf{??})}(t) \in L^2[a,b]$. Meaning that the Operator G_2^{-1} is determined in the entire $L^2[a,b]$ space. For every $x^{(\mathbf{??})}(t)\overline{x}^{(\mathbf{??})}(t) \in L^2[a,b]$ we have:

$$\begin{aligned}&\|(G_2^{-1}x^{(2)})(t) - (G_2^{-1}\overline{x}^{(2)})(t)\| \;=\|x^{(1)}(t) - \overline{x}^{(1)}(t)\| \\ &=\|x(t) - \overline{x}(t) + \varepsilon_0[(Fx)(t) - (F\overline{x})(t)]\| \leq \|x(t) - \overline{x}(t) + 2\varepsilon_0[(Fx)(t) - (F\overline{x})(t)\| \\ &=\|x^{(1)}(t) - \overline{x}^{(1)}(t) + \varepsilon_0[(FG_1^{-1}x^{(1)})(t) - (FG_1^{-1}\overline{x}^{(1)})(t)]\|{=}\|x^{(2)}(t) - \overline{x}^{(2)}(t)\|\end{aligned}$$

Therefore, the operator G_2^{-1} is continuous Lipshitz using a continuous Lipshitz coefficient of 1. After the change of variables, equation (??) can be rewritten as:

$$(A_2x^{(2)})(t) \equiv x^{(2)}(t) + (F_1G_1^{-1}G_2^{-1}x^{(2)})(t)$$

$$\equiv f(t) \tag{1.24}$$

For every $x^{(\textbf{??})}(t)H\overline{x}^{(\textbf{??})}(t) \in L^2[a,b]$ we have:

$$\|(F_1G_1^{-1}G_2^{-1}x^{(2)})(t)-(F_1G_1^{-1}G_2^{-1}\overline{x}^{(2)})(t)\| \leq \overline{q}\|x^{(2)}(t)-\overline{x}^{(2)}(t)\|$$

Therefore, $F_1G_1^{-1}G_2^{-1}$ is a paradoxical operator with a shrinkage coefficient $\overline{q}< 1$. Then, the equation (3-9) has a unique solution for every $f(t) \in L^2[a,b]$. Therefore, equation (**??**) has a unique solution $x(2\varepsilon_0)(t)$ for every $f(t) \in L^2[a,b]$. Approximate solutions of equation (**??**) are obtained using the standard iteration process.

$$x_{l+1}^{(2)}(t) = -\left(F_1G_1^{-1}G_2^{-1}x_l^{(2)}\right)(t)+f(t), l= 0,1,2,....x_0^{(2)}(t) = f(t)$$

Meanwhile, we will use " auxiliary function" iterative processes to invert operators. $G_1.G_2$At each stage of this iteration process is the value of a. Hence, approximate solutions of the integral equation (**??**) can be found by iterative processes.

$$\begin{aligned}
&x_{\ i+1}(t)= -\varepsilon_0(Fx_{\ i})(t)+x_j^{(1)}(t). \quad i= 0,1,2,...\\
&x_{j+1}^{(1)}(t)= -\varepsilon_0(FG_1^{-1}x_j^{(1)})(t)+x_l^{(2)}(t). j= 0,1,2,...\\
&x_{l+1}^{(2)}(t) = -\left(F_1G_1^{-1}G_2^{-1}x_l^{(2)}\right)(t)+f(t), l= 0,1,2,....x_0^{(2)}(t) = f(t)
\end{aligned}$$

Problem N (N>2). Consider the following Fredholm's turbulent nonlinear integral equation.

$$x(t)+N\varepsilon_0(Fx)(t)+(F_1x)(t) \equiv x(t)+(Fx)(t)+(F_1x)(t)=f(t) \tag{1.25}$$

N change of variables should be done.

$$\begin{aligned}
x^{(1)}(t) &= x(t)+\varepsilon_0(Fx)(t) \equiv (G_1x)(t)\\
x^{(2)}(t) &= x^{(1)}(t)+\varepsilon_0(FG_1^{-1}x^{(1)})(t) \equiv (G_2x^{(1)}(t)\\
&\cdots.....\\
x^{(N)}(t) &= x^{(N-1)}(t)+\varepsilon_0(FG_1^{-1}...G_{N-1}^{-1}x^{(N-1)})(t) \equiv (G_Nx^{(N-1)}(t)
\end{aligned}$$

Using a similar method, it will be shown that operators $G_3^{-1}....G_N^{-1}$ are determined at the entire space $L^2[a,b]$and are continuous Lipshitz with Lipshitz coefficients of 1. Therefore, after the change of variables, equation (**??**) is rewritten as:

$$(A_Nx^{(N)})(t) \equiv x^{(N)}(t)+(F_1G_1^{-1}...G_N^{-1}x^{(N)})(t)=f(t) \tag{1.26}$$

For every $x^{(N)}(t),\overline{x}^{(N)}(t) \in L^2[a,b]$ we have

$$\|(F_1G_1^{-1}...G_N^{-1}x^{(N)})(t)-(F_1G_1^{-1}...G_N^{-1}\overline{x}^{(N)})(t)\| \leq \overline{q}\|(x^{(N)})(t)-(\overline{x}^{(N)})(t)\|$$

Therefore, $F_1G_1^{-1}...G_N^{-1}$ is a paradoxical operator with a shrinkage coefficient of $\overline{q}< 1$. Then, equation (**??**) has a unique solution for every $f(t) \in L^2[a,b]$. Therefore, equation (**??**) has a unique solution $x(N\varepsilon_0)(t) \equiv x(t) \in L^2[a,b]$ for every $f(t) \in L^2[a,b]$. Approximate solutions of equation (**??**) are obtained using the standard iteration process.

$$x_{P+1}^{(N)}(t)= -\left(F_1G_1^{-1}...G_N^{-1}x_P^{(N)}\right)(t)+f(t). \quad p= 0,1,2,...x_0^{(N)}(t)=f(t) \qquad (29) \tag{1.27}$$

Meanwhile, the iteration processes of the "auxiliary known function" will be used to reverse the operators.

At each step of this process, $G_1.G_2....G_N$is the value of $(G_1^{-1}G_2^{-1}...G_N^{-1}x_P^{(N)})(t)$. Therefore, the approximate solutions of equation (**??**) can be obtained using the iteration process.

$$\begin{array}{l} x_{\ i+1}(t) = -\ \varepsilon_0(Fx_{\ i})(t) + x_j^{(1)}(t).\ i= 0,1,2,... \\ x_{j+1}^{(1)}(t)= -\ \varepsilon_0(FG_1^{-1}x_j^{(1)})(t) + x_l^{(2)}(t). \qquad j= 0,1,2,\dots \\ \qquad\qquad ...... \\ x_{P+1}^{(N)}(t)= -\ (F_1G_1^{-1}...G_N^{-1}x_P^{(N)})(t) + f(t). \qquad p= 0,1,2,...x_0^{(N)}(t) =f(t) \end{array}$$

This equation can be rewritten as:

$$\begin{array}{l} x_{i+1}(t) = -\frac{1}{N}(Fx_i)(t) - \frac{1}{N}(Fx_j)(t) - ... - \frac{1}{N}(Fx_{lt})(t)_{?N\ \ conditions} -(F_1x_p)(t) + f(t), \\ i,j,...,p= 0,1,... \end{array}$$

Based on the results obtained we have the following theorem.

Theorem 3. Provided that terms (**??**) to (**??**) are satisfied, Fredholm's turbulent nonlinear integral equation (4-1) has a unique solution $x(t)\ \in L^2[a,b]$.

Proof. As shown earlier, operators $F_1G_1^{-1}, F_1G_1^{-1}G_2^{-1}, ..., F_1G_1^{-1}...G_N^{-1}$ are paradoxical operators with a shrinkage coefficient $\overline{q}< 1$. Thus, equation (**??**) that equals equation (**??**) has a unique solution for every $f(t)\ \in L^2[a,b]$ and the proof is fulfilled.

Theorem 3. The conditions of theorem (**??**) must be met. In that case, the series of approximate solutions $\{x(n,N)(t)\ \}, n= 1,2,....$ made as a result of the iteration processes (**??**) converges to the exact solution $x(t)\ \in L^2[a,b]$ of Fredholm's turbulent nonlinear integral equation (4-1). Besides, the following estimations hold.

$$\|x(n,N)(t) - x(t)\| \leq \frac{1}{1-\overline{q}}[\frac{q^{n-1}}{1-q}\frac{1-\overline{q}^{n+1}}{1-\overline{q}}\frac{e^{q^N}-1}{e^q-1} + \overline{q}^{n+1}]\|f(t)\| \qquad (31) \tag{1.28}$$

Where N is the minimum natural number such that $q=\frac{1}{N}<1 \qquad \overline{q}=B< 1,n= 1,2,.....$

Proof. For simplicity, we assume that $(F0)(t) = \int_a^b k(t,s,0)ds= 0$ and $(F_10)(t) = \int_a^b k_1(t,s,0)ds= 0$ where $0(t) = 0$ represents the element zero in $L^2[a,b]$. If $(F0)(t) \neq 0$ or $(F_10)(t) \neq 0$, two operators $T, T_1{:}L^2[a,b] \longrightarrow L^2[a,b]$ can be defined:

$$\begin{array}{l} (Tx)(t) = (Fx)(t) - (F0)(t) = \int_a^b [k(t,s,x(s)) - k(t,s,0)]ds, \\ (T_1x)(t) = (F_1x)(t) - (F_10)(t) = \int_a^b [k_1(t,s,x(s)) - k_1(t,s,0)]ds, \end{array}$$

Then, $(T0)(t) = (T_1x)(t) = 0$ and Fredholm's turbulent nonlinear integral equation (4-1) equals

$$x(t) + \int_a^b Q(t,s,x(s))ds + \int_a^b Q_1(t,s,x(s))ds{=}g(t), \qquad a \leq t \leq b,$$

Where

$$\begin{array}{l} Q(t,s,x(s)) =k(t,s,x(s)) - k(t,s,0), \\ Q_1(t,s,x(s)) =k_1(t,s,x(s)) - k_1(t,s,0) \end{array}$$

$$g(t) =f(t) - \int_a^b k(t,s,0)ds - \int_a^b k_1(t,s,0)ds,$$

Plus, for every $a \leq t, s \leq b$ and every x, y available, we have:

$$Q(t,s,x) - \quad Q(t,s,\ y){=}k(t,s,x) - \quad k(t,s,\ y);$$

$$Q_1(t,s,x) - \quad Q_1(t,s,\ y)\ {=}k_1(t,s,x) - \quad k_1(t,s,\ y)$$

Thus, functions $g(t)$,$Q(t,s,x)$. $Q_1(t,s,\ y)$meet the conditions of theorem (**??**). It is necessary to consider that the consequent problems 1, 2, , N are the approximate solutions of problem (??) are obtained by iteration processes. $(G_1^{-1}x_j^{(1)})(t)$ are obtained using the iteration process.

$$x_{\ i+1}(t) = -\ \varepsilon_0(Fx_{\ i})(t) + x_{\ j}^{(1)}(t). \quad i{=}\ 0,1,2,....x_0(t)\ {=}x_{\ j}^{(1)}(t)$$

With error

$$\|x_n(t) - x^*(t)\| \leq \frac{q^{n+1}}{1-q}\|x_{\ j}^{(1)}(t)\|$$

For every $k \in \{1,2,...n\}$ we have

$$\begin{aligned}\|x_{\ k}^{(1)}(t) - x_{\ k-1}^{(1)}(t)\|&{=}\|(F_1G_1^{-1}x_{k-1}^{(1)})(t) - (F_1G_1^{-1}x_{k-2}^{(1)})(t)\|\\ &\leq \overline{q}\|x_{k-1}^{(1)}(t) - x_{k-2}^{(1)}(t)\| \leq ... \leq \overline{q}^{k-1}\|x_l^{(1)}(t) - x_{\ 0}^{(1)}(t)\|.\end{aligned}$$

Such that

$$\begin{aligned}\|x_l^{(1)}(t)\| &\leq \|x_j^{(1)}(t) - x_{j-1}^{(1)}(t)\| + ... + \|x_l^{(1)}(t) - x_{\ 0}^{(1)}(t)\| + \|x_{\ 0}^{(1)}(t)\|\\ &\leq (\overline{q}^{j-1} + \overline{q}^{j-2} + ... + \overline{q} + 1)\|x_l^{(1)}(t) - x_{\ 0}^{(1)}(t)\| + \|x_{\ 0}^{(1)}(t)\|\\ &\leq \tfrac{1-\overline{q}^j}{1-\overline{q}}\|x_l^{(1)}(t) - x_{\ 0}^{(1)}(t)\| + \|x_{\ 0}^{(1)}(t)\|\end{aligned}$$

Since $(F0)(t) = 0$ we have $(G_1 0)(t) = 0(t) + \varepsilon_0(F0)(t) = 0$. Therefore,

$$\begin{aligned}\|x_l^{(1)}(t) - x_{\ 0}^{(1)}(t)\|&{=}\|(F_1G_1^{-1}x_0^{(1)})(t) + f(t) - x_{\ 0}^{(1)}(t)\|\\ &{=}\|(F_1G_1^{-1}f)(t) - (F_1G_1^{-1}0)(t)\|\\ &\leq \overline{q}\|f(t)\|\end{aligned}$$

Then it is concluded from the above inequality:

$$\begin{aligned}\|x_l^{(1)}(t)\| &\leq \tfrac{1-\overline{q}^j}{1-\overline{q}}\|x_l^{(1)}(t) - x_{\ 0}^{(1)}(t)\| + \|x_{\ 0}^{(1)}(t)\| \leq \overline{q}\tfrac{1-\overline{q}^j}{1-\overline{q}}\|f(t)\| + \|f(t)\|\\ &\leq \overline{q}\tfrac{1-\overline{q}^n}{1-\overline{q}}\|f(t)\| + \|f(t)\|{=}\tfrac{1-\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|\end{aligned}$$

Hence, $(G_1^{-1}x_j^{(1)})(t)$ values are obtained with error.

$$\Delta_1(n) \equiv \partial_1(n) \equiv \|x_n(t) - x^*(t)\| \leq \mu(n)$$

Where

$$\mu(n){=}\frac{q^{n+1}}{1-q}\frac{1-\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\| \tag{1.29}$$

Since F1 is a paradoxical operator with a shrinkage coefficient $\overline{q}$< 1, the error $\Delta_1(n)$in determining the argument of the operator F1 is $\overline{q}\Delta_1(n)$by determining the right side of $f(t)$ in the equation (4-4). Plus, the operator A_1^{-1} is continuous Lipshitz and the Lipshitz coefficient is $\frac{1}{1-\overline{q}}$. In fact, for every $f(t).\overline{f}(t) \in L^2[a,b]$ we have:

$$\begin{aligned}
&\|(A_1^{-1}f)(t)-(A_1^{-1}\overline{f})(t)\|=\|x^{(1)}(t)-\overline{x}^{(1)}(t)\|\\
&=\|x^{(1)}(t)-\overline{x}^{(1)}(t)+(F_1G_1^{-1}x^{(1)})(t)-(F_1G_1^{-1}\overline{x}^{(1)})(t)-[(F_1G_1^{-1}x^{(1)})(t)-(F_1G_1^{-1}\overline{x}^{(1)})(t)]\|\\
&\leq \|x^{(1)}(t)-\overline{x}^{(1)}(t)+(F_1G_1^{-1}x^{(1)})(t)-(F_1G_1^{-1}\overline{x}^{(1)})(t)\|+\|(F_1G_1^{-1}x^{(1)})(t)-(F_1G_1^{-1}\overline{x}^{(1)})(t)\|\\
&\leq \|(A_1x^{(1)})(t)-(A_1\overline{x}^{(1)})(t)\|+\overline{q}\|x^{(1)}(t)-\overline{x}^{(1)}(t)\|\\
&=\|(f)(t)-(\overline{f})(t)\|+\overline{q}\|x^{(1)}(t)-\overline{x}^{(1)}(t)\|
\end{aligned}$$

Therefore,

$$\|(A_1^{-1}f)(t)-(A_1^{-1}\overline{f})(t)\| \leq \frac{1}{1-\overline{q}}\|f(t)-\overline{f}(t)\|$$

Hence, the replacement of error $\overline{q}\Delta_1(n)$ in the right side of the equation (4-4) produces an error higher than $\frac{\overline{q}}{1-\overline{q}}$in the corresponding solution $x^{(\textbf{??})}(t)$. The error of a repetition process in the calculation of $x^{(\textbf{??})}(t)$ is $\frac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|$. Therefore,

$$\|x_n^{(1)}(t)-x^{(1)}(t)\| \leq \frac{\overline{q}}{1-\overline{q}}\Delta_1(n)+\frac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|.$$

A reverse multiplication, like a change of variable from $x^{(\textbf{??})}(t)$ to $x(t)$ again, introduces the error $\Delta_1(n)$. Then, the errors of the approximate solutions $x_n(t)$ of problem (**??**) are estimated.

$$\begin{aligned}
\|x_n(t)-x(\varepsilon_0)(t)\| &\leq \tfrac{\overline{q}}{1-\overline{q}}\Delta_1(n)+\Delta_1(n)+\tfrac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|\\
&=\tfrac{1}{1-\overline{q}}\Delta_1(n)+\tfrac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|
\end{aligned}$$

The approximate solution of problem (**??**) is obtained by the repetition process. Values $(G_1^{-1}G_2^{-1}x_l^{(2)})(t)$ are calculated by using the iteration process:

$$x_{\ i+1}(t)=-\,\varepsilon_0(Fx_{\ i})(t)+x_{\ j}^{(1)}(t). \quad i{=}\ 0,1,2,...x_0(t)=x_{\ j}^{(1)}(t).$$

With error

$$\|x_n(t)-x^*(t)\| \leq \frac{q^{n+1}}{1-q}\|x_j^{(1)}(t)\|.$$

We have

$$\|x_j^{(1)}(t)\| \leq \|x_{\ j}^{(1)}(t)-x_{\ j-1}^{(1)}(t)\|+...+\|x_1^{(1)}(t)-x_0^{(1)}(t)\|+\|x_0^{(1)}(t)\|$$

Since the operator G_2^{-1} is continuous Lipshitz with a Lipshitz coefficient of 1,

$$\|x_{\ k}^{(1)}(t)-x_{\ k-1}^{(1)}(t)\|=\|(G_2^{-1}x_k^{(2)})(t)-(G_2^{-1}x_{k-1}^{(2)})(t)\| \leq \|x_k^{(2)}(t)-x_{k-1}^{(2)}(t)\|$$

For every $k \in \{1,2,...,n\}$. Therefore,

$$\|x_j^{(1)}(t)\| \leq \|x_{\ j}^{(2)}(t)-x_{\ j-1}^{(2)}(t).\|+...+\|x_1^{(2)}(t)-x_0^{(2)}(t)\|+\|x_0^{(2)}(t)\|$$

For every $k \in \{1, 2, ..., n\}$ we have:

$$\begin{aligned}\|x_{k}^{(2)}(t) - x_{k-1}^{(2)}(t)\| &= \|(F_1 G_1^{-1} G_2^{-1} x_{k-1}^{(2)})(t) - (F_1 G_1^{-1} G_2^{-1} x_{k-2}^{(2)})(t)\| \\ &\leq \overline{q}\|x_{k-1}^{(2)}(t) - x_{k-2}^{(2)}(t)\| \leq ... \leq \overline{q}^{K-1}\|x_{1}^{(2)}(t) - x_0^{(2)}(t)\|\end{aligned}$$

Therefore,

$$\begin{aligned}\|x_j^{(1)}(t)\| &\leq (\overline{q}^{j-1} + \overline{q}^{j-2} + ... + \overline{q} + 1)\|x_1^{(2)}(t) - x_0^{(2)}(t)\| + \|x_0^{(2)}(t)\| \\ &\leq \tfrac{1-\overline{q}^j}{1-\overline{q}}\|x_1^{(2)}(t) - x_0^{(2)}(t)\| + \|x_0^{(2)}(t)\|\end{aligned}$$

Since $(F0)(t) = 0$ and $(G_1 0)(t) = 0(t) + \varepsilon_0(F0)(t) = 0$

And $(G_2 0)(t) = 0(t) + \varepsilon_0(FG_1^{-1}0)(t) = 0$

Therefore,

$$\begin{aligned}\|x_1^{(2)}(t) - x_0^{(2)}(t)\| &= \|(F_1 G_1^{-1} G_2^{-1} f)(t) + f(t) - f(t)\| \\ &= \|(F_1 G_1^{-1} G_2^{-1} f)(t) - (F_1 G_1^{-1} G_2^{-1} 0)(t)\| \\ &\leq \overline{q}\|f(t)\|\end{aligned}$$

Hence,

$$\begin{aligned}\|x_j^{(1)}(t)\| &\leq \tfrac{1-\overline{q}^j}{1-\overline{q}}\|x_1^{(2)}(t) - x_0^{(2)}(t)\| + \|x_0^{(2)}(t)\| \leq \overline{q}\tfrac{1-\overline{q}^j}{1-\overline{q}}\|f(t)\| + \|f(t)\| \\ &\leq \overline{q}\tfrac{1-\overline{q}^n}{1-\overline{q}}\|f(t)\| + \|f(t)\| = \tfrac{1-\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|\end{aligned}$$

Therefore, the values of $(G_1^{-1} x_j^{(1)})(t)$are estimated with error:

$$\|x_n(t) - x^*(t)\| \leq \frac{q^{n+1}}{1-q}\frac{1-\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\| = \mu(n)$$

Since $\varepsilon_0 F$is a paradoxical operator with a shrinkage coefficient of $q< 1$, the error $\mu(n)$ in the determination of the argument of the operator $\varepsilon_0 F$equals the error $q\mu(n)$. In the determination of the right side of $x^2(t)$in the equation, $x^1(t) + \varepsilon_0(FG_1^{-1}x^1)(t) = x^2(t)$ the operator G_2^{-1} is continuous Lipshitz with a Lipshitz coefficient of 1. The multiplication of the error $q\mu(n)$ from the right side of the equation $x^1(t) + \varepsilon_0(FG_1^{-1}x^1)(t) = x^2(t)$ produces an error $q\mu(n)$ higher than the corresponding solution $x^1(t)$. The iteration process error in the calculation $x^1(t)$is $\frac{q^{n+1}}{1-q}\|x_l^2(t)\|$ for every $l \in \{1, 2, ..., n\}$ we have

$$\begin{aligned}\|x_l^{(2)}(t)\| &\leq \|x_l^{(2)}(t) - x_{l-1}^{(2)}(t)\| + ... + \|x_1^{(2)}(t) - x_0^{(2)}(t)\| + \|x_0^{(2)}(t)\| \\ &\leq (\overline{q}^{l-1} + \overline{q}^{l-2} + ... + \overline{q} + 1)\|x_1^{(2)}(t) - x_0^{(2)}(t)\| + \|x_0^{(2)}(t)\| \\ &\leq \tfrac{1-\overline{q}^l}{1-\overline{q}}\|x_1^{(2)}(t) - x_0^{(2)}(t)\| + \|x_0^{(2)}(t)\| \\ &\leq \overline{q}\tfrac{1-\overline{q}^l}{1-\overline{q}}\|f(t)\| + \|f(t)\| \\ &\leq \overline{q}\tfrac{1-\overline{q}^n}{1-\overline{q}}\|f(t)\| + \|f(t)\| = \tfrac{1-\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|\end{aligned}$$

Then, the error of an iteration process in the calculation of $x^1(t)$ equals $\mu(n)$. Therefore,

$$\partial_2(n) \equiv \|x_n^{(1)}(t) - x^{(1)}(t)\| \leq q\mu(n) + \mu(n) = q\partial_1(n) + \mu(n)$$

The change of variable from $x^1(t)$ to $x(t)$ again produces the error $\mu(n)$. Thus $(G_1^{-1}G_2^{-1}x_l^{(2)})(t)$ is calculated with the error:

$$\Delta_2(n) = \|x_n(t) - x^*(t)\| \leq q\mu(n) + 2\mu(n) = \partial_2(n) + \partial_1(n)$$

Since F1 is a paradoxical operator with a shrinkage coefficient $\overline{q}< 1$, the error $\Delta_2(n)$ in the determination of the argument of the operator F1 equals the error $\overline{q}\Delta_2(n)$ in the determination of the right side $f(t)$ of the equation (3-9). On the other hand, the operator A_2^{-1} continuous Lipshitz with a $\frac{1}{1-\overline{q}}$ correlation coefficient. Because in fact, for every $f(t).\overline{f}(t) \in L^2[a, b]$we have

$$\begin{aligned}
\|(A_2^{-1}f)(t) - (A_2^{-1}\overline{f})(t)\| &= \|x^{\mathbf{(??)}}(t) - \overline{x}^{\mathbf{(??)}}(t)\| \\
&= \|x^{\mathbf{(??)}}(t) - \overline{x}^{\mathbf{(??)}}(t) + (F_1G_1^{-1}G_2^{-1}x^{(2)})(t) \\
&\quad -(F_1G_1^{-1}G_2^{-1}\overline{x}^{(2)})(t) - [(F_1G_1^{-1}G_2^{-1}x^{(2)})(t) - (F_1G_1^{-1}G_2^{-1}\overline{x}^{(2)})(t)]\| \\
&\leq \|x^{\mathbf{(??)}}(t) - \overline{x}^{\mathbf{(??)}}(t) + (F_1G_1^{-1}G_2^{-1}x^{(2)})(t) - (F_1G_1^{-1}G_2^{-1}\overline{x}^{(2)})(t)\| \\
&\quad +\|(F_1G_1^{-1}G_2^{-1}x^{(2)})(t) - (F_1G_1^{-1}G_2^{-1}\overline{x}^{(2)})(t)\| \\
&\leq \|(A_2x^{(2)})(t) - (A_2\overline{x}^{(2)})(t)\| + \overline{q}\|x^{\mathbf{(??)}}(t) \\
-\overline{x}^{\mathbf{(??)}}(t)\| \\
&= \|f(t) - \overline{f}(t)\| + \overline{q}\|x^{\mathbf{(??)}}(t) - \overline{x}^{\mathbf{(??)}}(t)\|
\end{aligned}$$

Therefore,

$$\|(A_2^{-1}f)(t) - (A_2^{-1}\overline{f})(t)\| \leq \frac{1}{1-\overline{q}}\|f(t) - \overline{f}(t)\|$$

Thus, the mistaken replacement of $\overline{q}\Delta_2(n)$in the right side of the equation (4-9) produces an error higher than $\frac{\overline{q}}{1-\overline{q}}\Delta_2(n)$ produced by the corresponding solution $x^2(t)$. The error of an iteration process in the calculation of $x^2(t)$is $\frac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|$. Therefore, we have:

$$\|x_n^{(2)}(t) - x^{(2)}(t)\| \leq \frac{\overline{q}}{1-\overline{q}}\Delta_2(n) + \frac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|$$

A change variations from $x^2(t)$ to $x(t)$ produces the error $\Delta_2(n)$ again. Then, the error of approximate solutions $x\ _n(t)$ of problem (**??**) is estimated.

$$\begin{aligned}
\|x_n(t) - x(2\varepsilon_0)(t)\| &\leq \tfrac{\overline{q}}{1-\overline{q}}\Delta_2(n) + \Delta_2(n) + \tfrac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\| \\
&= \tfrac{1}{1-\overline{q}}\Delta_2(n) + \tfrac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|
\end{aligned}$$

By using a similar argument for the problem k:$x(t)$ + $k\varepsilon_0(Fx)(t)$ + $(F_1x)(t)$ =$f(t)$,$k \in [1,N]$ the estimation is obtained.

$$\|x_n(t) - x(K\varepsilon_0)(t)\| \leq \frac{1}{1-\overline{q}}\square_K(n) + \frac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\| \tag{1.30}$$

Where

$$\square_k(n) \leq \partial_k(n) + \partial_{k-1}(n) + ... + \partial_1(n) \tag{1.31}$$

And

$$\partial_h(n) \leq q[\partial_{h-1}(n) + ... + \partial_1(n)] + \mu(n), \qquad 1 \leq h \leq k \tag{1.32}$$

Which can be rewritten as:

$$\partial_k(n) \leq \mu(n) + q\sum_{h=1}^{k-1}\partial_h(n).\partial_1(n) \leq \mu(n), k= 2, 3, ...,N \tag{1.33}$$

Now see the discrete analog feature of the well-known Bekkman-Gronwall analogy [13, theorem 1]. We obtain from inequality 36:

$$\partial_k(n) \leq \mu(n)\prod_{h=1}^{k-1}(1+q) \leq \mu(n)\prod_{h=1}^{k-1}e^q{=}\mu(n)e^{q(k-1)}.k{=}\ 1,2,...,N$$

Therefore, (**??**) can be rewritten as:

$$\Delta_k(n) \leq \sum_{h=1}^{k}\partial_h(n) \leq \mu(n)\sum_{h=1}^{k}e^{q(h-1)}{=}\mu(n)\frac{e^{kq}-1}{e^q-1}$$

Thus, the error estimation 33 for problem k can be rewritten as:

$$\|x_n(t)-x(k\varepsilon_0)(t)\| \leq \frac{1}{1-\overline{q}}\mu(n)\frac{e^{kq}-1}{e^q-1}+\frac{\overline{q}^{n+1}}{1-\overline{q}}\|f(t)\|$$

By replacing N by k and from equation (**??**) we obtain (**??**). This completes the proof of the theorem.

Numerical examples (practical works) of Fredholm's integral equations

Example 1. Consider the following Fredholm's equation:

$$x(t)+\int_0^{\pi}cos(t)\ \ cos(x)\ \ x(s)ds{=}\ sin(t)\ \ +(1+\frac{\pi}{2})cos(t)\ \ ,0\leq t\leq \pi,$$

Which has the exact solution of $x(t) = sin(t) + cos(t)$.

In this example, we have $k(t,s,x(s) = cos(t)cos(x)x(s)$and $f(t) = sin(t) +\ (1+\frac{\pi}{2})cos(t)$.

Proof.

Table 1. approximate values and If simply $f(t) \in L^2[0{,}\pi]$, for $t,s \in [0{,}\pi]$ for every $x,\ \ y$ we have

$$\begin{aligned}|k(t,s,x(s))-k(t,s,y(s))|&{=}|cos(t)cos(x)x(s)-cos(t)cos(x)y(s)|\\&{=}|cos(t)cos(x)||x(s)-y(s)|\end{aligned}$$

Where

$$\int_a^b\int_a^b|Q(t,s)|^2dsdt{=}\int_0^{\pi}\int_0^{\pi}cos^2(t)cos^2(s)dsdt{=}\frac{\pi^2}{4}{=}L^2{<}\infty;$$

And

$$\begin{aligned}&\textstyle\int_0^{\pi}\{\int_0^{\pi}[k(t,s,x(s)-k(t,s,y(s)]ds\}|x(t)-y(t)|dt\\&=\textstyle\int_0^{\pi}\{\int_0^{\pi}cos(t)cos(s)[x(s)-y(s)]ds\}|x(t)-y(t)|dt\\&=(\textstyle\int_0^{\pi}cos(t)[x(t)-y(t)]dt)^2\geq 0\end{aligned}$$

Therefore, functions $f(t)$,$k(t,s,x)$ satisfy the terms of the third theorem. By executing the iteration process with $N{=}\ 2$ we have:

$$\begin{aligned}x_{i+1}(t)&=-\tfrac{1}{2}\textstyle\int_0^{\pi}cos(t)cos(s)x_i(s)ds+x_j^{(1)}(t).\quad i=\{0,1,2,...\}\\x_{j+1}^{(1)}(t)&=-\tfrac{1}{2}\textstyle\int_0^{\pi}cos(t)cos(s)G_1^{-1}x_j^{(1)}(s)ds+sin(t)+1+\tfrac{\pi}{2}cos(t),j=\{0,1,2,...,\}\\x_0^{(1)}(t)&=sin(t)+1+\tfrac{\pi}{2}cos(t),\end{aligned}$$

Where $(G_1x)(t)$ $=x(t)$ + $\frac{1}{2}\int_0^{\pi}cos(t)cos(s)x(s)ds{=}x^{(\mathbf{??})}(t)$. By choosing $n{=}\ 20$ (the steps of each iteration is the

Homotopy analysis method		Parametric continuity method		Exact solution	T
Absolute error	Approximate solution	Absolute error	Approximate solution		
4.342384×10^{-3}	1.004328417	4.342384×10^{-3}	1.004342384	1	0
3.748526×10^{-3}	1.369773930	3.760616×10^{-3}	1.369786020	1.366025404	$\pi/6$
3.060663×10^{-3}	1.417274225	3.70529×10^{-3}	1.417284091	1.414113562	$\pi/3$
	1	0	1	1	$\pi/2$
2.164213×10^{-3}	0.363861191	2.171192×10^{-3}	0.363854212	0.366025404	$2\pi/3$
3.060661×10^{-3}	-0.003060661	3.070529×10^{-3}	0.003070529-	0	$3\pi/3$
3.748528×10^{-3}	-0.369773932	3.760616×10^{-3}	0.369786020-	0.366025404-	$5\pi/6$
4.328427×10^{-3}	-1.004328427	4.3423848×10^{-3}	1.004342384-	-1	π

Table 1: Absolute errors for some points in example 1

same and equals 20) the approximate values and absolute errors at some points are $t \in [0,\pi]$. The presentation and comparison of the results using the homotopy analysis method considering $n= 20$ is given in table (**??**).

It is worth noting that the table reveals that the results obtained using the Parametric continuity method belong $t \in [0,\pi]$ similar to the homotopy analysis method.

Note 1. The estimation error for each known ∂ can find the number of iteration, for example, such that $\|x(n,N)(t) - x(t)\| \le \partial$. In example (**??**), if $\partial= 1.02 \times 10^{-5}$, the iteration number is 3969 (the number of steps at each iteration plan is the same and equals $n= 63$).

Problem 1.3. *Consider the following Fredholm's turbulent nonlinear integral equation.*

$$x(t) + \frac{9}{2}\int_0^1 ts?x(s)ds + \frac{2}{3}\int_0^1 tcos(x(s))ds=\sqrt{t} + \frac{7 + 20sin(1) + 20cos(1)}{15}t$$

With $0 <t \le 1$ *with the exact solution* $x(t) =\sqrt{t}$.

In this example, we have: $f(t) =\frac{7+20sin(\textbf{??})+20cos(\textbf{??})}{15}t$ *and* $k(t,s,x(s) =\frac{9}{2}ts\,x(s)$, $k_1(t,s,x(s) =\frac{2}{3}tcos(x(s)\,)$.

Proof. To prove that $f(t) \in L^2[0,1]$ is simple. For $t,s \in [0,1]$ for every x,y we have:

$$|k(t,s,x(s) - k(t,s,y(s)|=|\frac{9}{2}ts\;x(s) - \frac{9}{2}ts\;y(s)|=|\frac{9}{2}ts||\;x(s) - y(s)|$$

Where $\int_a^b \int_a^b |\phi(t,s)|^2 dsdt= \int_0^1 \int_0^1 \frac{81}{4}t^2s^2dsdt=\frac{9}{4}=L^2<\infty$,

And

$$\begin{aligned}&\int_0^1 \{\int_0^1 [k(t,s,x(s) - k(t,s,y(s)]ds\}|x(t) - y(t)|dt\\ &= \int_0^1 \{\int_0^1 \tfrac{9}{2}ts[x(s) - y(s)]ds\}|x(t) - y(t)|dt\\ &=\tfrac{9}{2}\int_0^1 t(x(t) - y(t))dt \int_0^1 s(x(s) - y(s)]ds=\tfrac{9}{2}(\int_0^1 t[x(t) - y(t)^2 \ge 0\end{aligned}$$

Plus

$$\begin{aligned}|k_1(t,s,x(s)) - k_1(t,s,y(s))|&=|\tfrac{2}{3}tcos(x(s)) - \tfrac{2}{3}tcos(y(s))|\\ &=|\tfrac{2}{3}t||cos(x(s)) - cos(y(s))|\\ &\le |\tfrac{2}{3}t||x(s) - y(s)|.\end{aligned}$$

The approximate solutions and the corresponding errors of the parametric continuation method are given in table (**??**), □

Error	The approximate solution	N
0.1591747838	$0.2205240321986 \times 10^{-3}t + \sqrt{t}$	20
$0.8963691179 \times 10^{-2}$	$0.124238398349 \times 10^{-3}t + \sqrt{t}$	30
$0.2832576447 \times 10^{-3}$	$0.635113898 \times 10^{-6}t + \sqrt{t}$	50

Table 2: The approximate solutions of the example (**??**)

Where

$$\int_a^b \int_a^b |\phi(t,s)|^2 dsdt = \int_0^1 \int_0^1 \frac{4}{9}t^2 dsdt = \frac{4}{27} = B^2 < \infty, B = \frac{2\sqrt{3}}{9} < 1$$

Therefore, functions $f(t)$,$k(t,s,\square\square)$,$k_1(t,s,x)$ express the assumptions of the fourth chapter. By running the iteration process (14-4) for $N=2$ we have:

$$x_{i+1}(t) = -\frac{9}{4}\int_0^1 ts\, x_i(s)ds + x_j^{(1)}(t), i=0,1,2,\ldots$$
$$x_{j+1}^{(1)}(t) = -\frac{9}{4}\int_0^1 ts\,(G_1^{-1}x_j^{(1)})(s)ds + x_j^{(2)}(t), j=0,1,2,\ldots$$
$$x_{l+1}^{(2)}(t) = -\frac{2}{3}\int_0^1 tcos((G_1^{-1}G_2^{-1}x_l^{(2)})(s))ds + \sqrt{t} + \frac{7+20sin(1)+20cos(1)}{15}t$$
$$l=0,1,2,\ldots x_0^{(2)}(t) = \sqrt{t} + \frac{7+20sin(1)+20cos(1)}{15}t$$

Where

$$(G_1x)(t) = x(t) + \frac{9}{4}\int_0^1 ts\, x(s)ds = x^{(1)}(t)$$

And

$$(G_2x^{(1)})(t) = x^{(1)}(t) + \frac{9}{4}\int_0^1 ts\,(G_1^{-1}x^{(1)})(s)ds = x^{(2)}(t)$$

Table (**??**) shows the available approximate solutions and corresponding errors. For errors obtained using iterative processes and error estimates we have $N=2$, $n=20$, 30, 50.

Note 2. From the error estimation, we infer that

$$\|x(n,N)(t) - x(t)\| \leq C(N)\alpha^{n+1}$$

Where

$$\alpha = max[q,\overline{q}], C[N] = [\frac{1}{(1-q)(1-\overline{q})^2}\frac{e^{q^N}-1}{e^q-1} + 1]\|f(t)\|$$

For each δ given, the iteration number can be obtained where $\|x(n,N)(t) - x(t)\| \leq \partial$. In the second example, if $\delta = 0.009$, the number of iterations is 29791 (the number of steps for each pattern is the same and equals 31).

Conclusion

The convergence problem of the step-by-step parametric continuity method with an arbitrary degree and arbitrary smoothness for Fredelm's integral equations of the second type has been open for several years. In this paper, we prove the convergence and degree of convergence of step-by-step parametric continuity methods for Fredholm integral equations under completely algebraic conditions based solely on the smoothness and order of the parameters. The stability of the parametric continuity method in Fredholm integral equations of the second type was studied by the applied method. If the conditions of convergence theorems are not met, the step-by-step interpolation process is generally unstable. The technique used is based on estimating the expansion coefficient in the proposed functions. Taylor and Lagrange's expansions are successfully studied using step-by-step methods. Excellent degree convergence can be obtained in numerical methods to solve these equations. This operation can be done using the parametric continuity method with special grading. Fredholm integral equations are not dependent on the Volta integral equations.

References

[1] Wazwaz, A.M.: Linear and Nonlinear Integral Equations. Springer, Berlin (2011)

[2] Ezzati, R., Najafalizadeh, S.: Numerical solution of nonlinear Volterra–Fredholm integral equation by using Chebyshev polynomials. Math. Sci.2011; 5: 1-12.

[3] Ninh K.V., Approximate solutions of the equation of a second kind with sum of two operators. Proceedings of Institute of Mathematics and Mechanics of Azerbaijanian Academy of Science. 1999; 5: 97-101.

[4] Ninh K.V., A method of extending by parameter for approximate solutions of operator equations. Acta Math. Vietnam. 2011;36(1):119-127.

[5] Vetekha, V.G., Parameter continuation method for ordinary differential equations. Proc Second ISAAC-Congr. 2000; 1: 737-742.