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An Upper Bound for Restrained Double Roman Domination Number in Honeycomb Networks

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Abstract

Honeycomb networks are built recursively using hexagonal tessellations. Wireless networks such as satellite networks, radio networks, sensor networks, cellular networks, ad hoc networks and other mobile network where honeycomb networks is used extensively. In this paper we study upper bound for restrained double Roman domination number for honeycomb networks.

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1 Introduction

Throughout this paper, we consider G as a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. We use [7] as a reference for terminology and notation which are not explicitly defined here. the concepts of dominating set, restrained dominating set, Roman dominating set and restrained Roman dominating set are defined in [1], [2], [5] and [6]. A restrained double Roman dominating function ($RDRD$ function for short) is a function $f : V \rightarrow \{0,1,2,3\}$ having the property that if $f(v) = 0$, then vertex v must have at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(v) = 1$, then vertex v must have at least one neighbor w with $f(w) \geqslant 2$, and at the same time, the subgraph $G[V_0]$ ($V_i = \{v \in V | f(v) = i\}$) has no isolated vertex. The restrained double Roman domination number ($RDRD$ number) $\gamma_{rdR}(G)$ is the minimum weight $\sum_{v\in V(G)}(f(v))$ of an $RDRD$ function f of G. Mojdeh et al. [4] proved that the RDRD problem is NP-complete for general graphs.

2 Preliminary

The honeycomb network $HC(1)$ is a hexagon. The honeycomb network $HC(2)$ is obtained by adding six hexagons to the boundary edges of $HC(1)$. Inductively, honeycomb network $HC(n)$ is obtained from $HC(n-1)$ by adding a layer of hexagons around the boundary of $HC(n-1)$. The number of vertices and edges of $HC(n)$ are $6n^2$ and $9n^2 - 3n$ respectively. A honeycomb network $HC(3)$ is shown in Figure 1. In Graph Theory to study the honeycomb network we use brick structure of the honeycomb networks. Brick structure is obtained by shrinking one of the upper and lower vertices in the straight lines. Thus in brick representation also there are equal number of vertices and edges. Brick representations of $HC(1)$, $HC(2)$ and $HC(3)$ are shown in Figure 2, Figure 3 and Figure 4 respectively. The application of Honeycomb network are very vast.

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It is applied in different networking's such as all-to-all broadcasting, cellular services, computer networking and etc.It is also used in chemistry to represent the structures of different compounds. The following results are required.

Lemma 2.1. *([3]) The boundary of $HC(n)$ is the cycle $C_{6(2n-1)}$.*

Lemma 2.2. *([3]) For $n \geq 2$, $|V(HC(n))| - |V(HC(n-1))| = 6(2n-1)$.*

Lemma 2.3. *([4]) $\gamma_{rdR}(P_n) = n+2, (n \geq 4)$.*

Lemma 2.4. *([4]) For a cycle C_n, $(n \geq 3)$, $\gamma_{rdR}(C_n) = n$, if $n \equiv 0(mod 3)$, and otherwise $\gamma_{rdR}(C_n) = n+2$.*

3 Main results

Lemma 3.1. *For $k = 1, 2, 3, \cdots$ and $n = 6(2k-1)$, the equality $\gamma_{rdR}(C_n) = n$ holds and the following labeling for vertices is optimal:*

$$3, 0, 0, 3, 0, 0, 3, 0, 0, 3, 0, 0, \cdots.$$

Proof. This lemma holds by the Lemma 2.4. But as a new proof, is used mathematical induction. At first, it is true for C_6.Suppose the Lemma is correct for $n = 6(2k-1)$ and let the cycle $C_{6(2(k+1)-1} = C_{6(2k+1)}$. Let w be an arbitrary vertex of $C_{6(2k-1)}$ such that its label is 3 and z be adjacent of w. Its obvious that the label of z is 0. Since $|V(C_{6(2k+1)})| - |V(C_{6(2k-1)})| = 12$, by joining the path $P_{12} = \{u_1, u_2, \cdots, u_{12}\}$ between w and z, the cycle $C_{6(2k-1)}$ become to $C_{6(2k+1)}$. Finally it is enough to do the labels of $u_1, u_2, \cdots, u_{12}$ in one of the following two ways:

$$3, 0, 0, 3, 0, 0, 3, 0, 0, 3, 0, 0$$

or

$$0, 0, 3, 0, 0, 3, 0, 0, 3, 0, 0, 3$$

□

Lemma 3.2. $\gamma_{rdR}(HC(1)) = 6$ *and for $n \geq 3$, the following inequality is true*

$$\gamma_{rdR}(HC(n)) \leq \gamma_{rdR}(HC(n-1)) + 6(2n-1)$$

.

Proof. $\gamma_{rdR}(HC(1)) = \gamma_{rdR}(C_6) = 6$ and for $HC(n), n \geq 2$, the labels of vertices of central hexagonal are 0 and for the other layers, we label the vertices as follows:

$$3, 0, 0, 3, 0, 0, 3, 0, 0, 3, 0, 0, \cdots.$$

Now proof is done by mathematical induction. At first, it is true for $n = 3$, in fact $\gamma_{rdR}(HC(2)) \leq 18$ (Figure 5.).

Now the boundary of $HC(3)$ is the cycle $C_{6(2*3-1)} = C_{30}$ and $\gamma_{rdR}(C_{30}) = 30$, and by removing this boundary, $HC(2)$ is obtained. In $HC(2)$ and C_{30}, according to the above labeling, $V_1 = V_2 = \emptyset$ and all vertices with label 0 are adjacent exactly to one vertex of label 3. So the maximum of $\gamma_{rdR}(HC(3))$ is $\gamma_{rdR}(HC(2)) + 30$ or $\gamma_{rdR}(HC(2)) + 6(2*3-1)$.

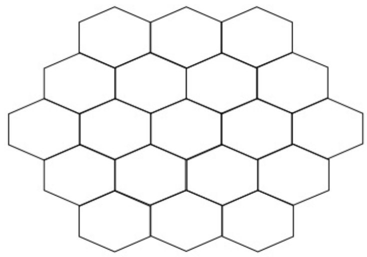

Figure 1: Honeycomb network of dimension 3

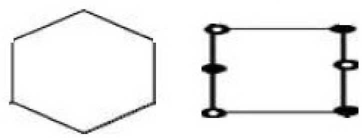

Figure 2: Honeycomb network of dimension 1 and its brick structure

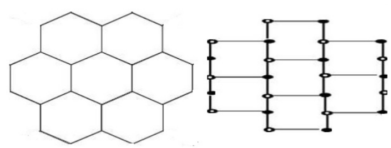

Figure 3: Honeycomb network of dimension 2 and its brick structure

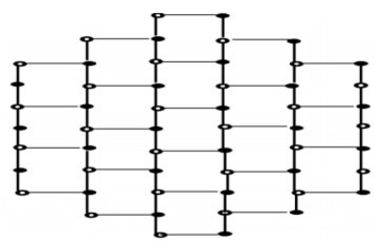

Figure 4: Brick structure of HC(3)

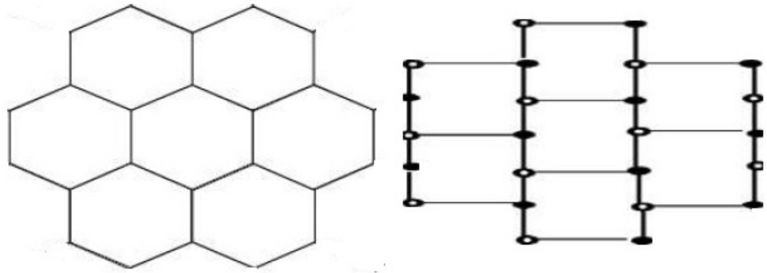

Figure 5: $\gamma_{rdR}(HC(2)) = 18$

Similarly, the boundary of $HC(n)$ is the cycle $C_{6(2*n-1)}$ and by removing this boundary, $HC(n-1)$ is obtained and $\gamma_{rdR}(C_{6(2*n-1)}) = 6(2*n-1)$. In $HC(n-1)$ and $C_{6(2*n-1)}$, according to the above labeling and induction assumption, $V_1 = V_2 = \emptyset$ and all vertices with label 0 are adjacent to one vertex of label 3. So the maximum of $\gamma_{rdR}(HC(n))$ is $\gamma_{rdR}(HC(n-1)) + 6(2*n-1)$. □

Theorem 3.1. $\gamma_{rdR}(HC(n)) \leq 6(n^2-1)$ *for* $n \geq 2$.

Proof. According the Lemma 3.2, $\gamma_{rdR}(HC(2)) \leq 18$ and for $n \geq 3$, the following inequality holds,

$$\gamma_{rdR}(HC(n)) \leq \gamma_{rdR}(HC(n-1)) + 6(2n-1)$$

. Let $\gamma_{rdR}(HC(n)) = a_n$ then the above inequality become to $a_n - a_{n-1} \leq 6(2n-1)$, then:

$$a_3 - a_2 \leq 6(2*3-1) = 30$$

$$a_4 - a_3 \leq 6(2*4-1) = 42$$

$$a_5 - a_4 \leq 6(2*5-1) = 54$$

$$\vdots$$

$$a_n - a_{n-1} \leq 6(2*n-1)$$

By summing the sides of the above inequalities, we have:

$$a_n - a_2 \leq 30 + 42 + 54 + \cdots + 6(2*n-1) = 6(5+7+9+\cdots+(2n-1)) =$$

$$6(1+3+5+7+9+\cdots+(2n-1)-1-3) = 6(n^2-1) - 18.$$

Now since $a_2 = 18$, so $a_n \leq 6(n^2-1)$

□

Conjecture: $\gamma_{rdR}(HC(n)) = 6(n^2-1)$ for $n \geq 2$.

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